On the Determination of Star and Convex Bodies by Section Functions[*]

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Abstract. The ith section function of a star body in $\mathbb{E}^n$ gives the i-dimensional volumes of its sections by i-dimensional subspaces. It is shown that no star body is determined among all star bodies, up to reflection in the origin, by any of its ith section functions. Moreover, the set of star bodies that are determined among all star bodies, up to reflection in the origin, by their ith section functions for all i, is a nowhere dense set. The determination of convex bodies in this sense is also studied. The results complement and contrast with recent results on the determination of convex bodies by ith projection functions. The paper continues the development of the dual Brunn–Minkowski theory initiated by Lutwak.

1. Introduction

An important concern in geometric tomography is the question of whether an object in a given class can be distinguished from others in the class by various measurements of its projections on planes or sections by planes. If this is the case, we say that the object is *determined* in the class by the data. For projections, the natural class of objects to consider is the class $\mathcal{K}^n$ of convex bodies in $\mathbb{E}^n$. One natural type of data is the ith *projection function* of some $K \in \mathcal{K}^n$, which gives the i-dimensional volume of the projection of K on each i-dimensional subspace. Here $i \in \mathbb{N}$ and $1 \leq i \leq n-1$, and the special cases $i = 1$ and $i = n-1$ are usually called the *width function* and *brightness function*, respectively. The methods and results for this sort of inverse problem form part of the powerful Brunn–Minkowski theory. For example, a classical result of Aleksandrov (see

Theorem 3.3.6 of [5]) states:

> (1) For each $i \in \mathbb{N}$, $1 \le i \le n - 1$, any centered convex body in $\mathbb{E}^n$ is determined in $\mathcal{K}_c^n$ by its ith projection function.

Here $\mathcal{K}_c^n$ denotes the class of centered convex bodies; see Section 2 for definitions of unexplained terms. Recently there has been considerable interest in the determination of a convex body in the class $\mathcal{K}^n$ of *all* convex bodies in $\mathbb{E}^n$. The main results are as follows (note that a projection function cannot distinguish a convex body K from any of its translates or its reflection $-K$ in the origin):

> (2) ([9], from a result in [3].) Centered convex bodies in $\mathbb{E}^n$ are determined in $\mathcal{K}^n$ by their ith projection functions for two different values of i, $1 \le i \le n - 1$.
> (3) (See, for example, Chapter 3 of [5].) A convex body in $\mathbb{E}^n$ is determined in $\mathcal{K}^n$, up to translation and reflection in the origin, by its width function if and only if it is irreducible. (A convex body K is *irreducible* if the equation $K = \triangle C$, where $\triangle C = \frac{1}{2}(C + (-C))$ is the *central symmetral* of the convex body C, implies that K is a translate of C; see pp. 123–4 of [5] for more information.) Such bodies must be centrally symmetric, since $\triangle K$ is a centered convex body with the same width function as K.
> (4) ([7]; see also [1].) A convex body in $\mathbb{E}^n$ is determined in $\mathcal{K}^n$, up to translation and reflection in the origin, by its brightness function if and only if it is a parallelotope.
> (5) [19] When $1 \le i < n - 1$, most centered convex bodies in $\mathbb{E}^n$ are determined in $\mathcal{K}^n$ by their ith projection function.
> (6) [2] When $2 \le i \le n - 2$, most convex bodies in $\mathbb{E}^n$ are determined in $\mathcal{K}^n$, up to translation and reflection in the origin, by their ith projection function.
> (7) ([9], a refinement of earlier work of Campi and of Gardner and Volčič; see also [8] and [10].) A dense set of convex bodies in $\mathcal{K}^n$ are not determined in $\mathcal{K}^n$, up to translation and reflection in the origin, by their ith projection functions for *all* i, $1 \le i \le n - 1$.
> (8) [10] Most convex bodies in $\mathbb{E}^n$ are determined in $\mathcal{K}^n$, up to translation and reflection in the origin, by their width and brightness functions.

In these results, "most" means most in the sense of Baire category, that is, all except for a set of first category in the space $\mathcal{K}^n$ endowed with the Hausdorff metric.

It is remarkable that there is a duality in geometric tomography between results on projections of convex bodies and results on sections of star (rather than convex) bodies. This still unexplained fact has led to the creation, due principally to Lutwak, of an extensive dual Brunn–Minkowksi theory; see [5] and [6] for detailed discussions. For example, dual to Aleksandrov's theorem above is a result of Funk (see Theorem 7.2.6 of [5]) which states:

> (1′) For each $i \in \mathbb{N}$, $1 \le i \le n - 1$, any centered star body in $\mathbb{E}^n$ is determined in $\mathcal{S}_c^n$ by its ith section function.

Here $\mathcal{S}_c^n$ denotes the class of centered star bodies in $\mathbb{E}^n$ and the ith *section function* of a star body L gives the i-dimensional volume of the section of L by each i-dimensional subspace. The special case $i = n - 1$ is simply called the *section function*.

For determination of a star body in the class $\mathcal{S}^n$ of *all* star bodies in $\mathbb{E}^n$, the result dual to (2) above is also known:

(2′) (Theorem 5.9 of [8] or Theorem 7.2.15 of [5]; see also Corollary 4.4 below.) Centered star bodies in $\mathbb{E}^n$ are determined in $\mathcal{S}^n$ by their ith section functions for two different values of i, $1 \le i \le n - 1$.

The main purpose of this paper is to continue the attempt to understand this duality. Corresponding to (3)–(6) above for projections, a simple argument yields the following result:

(3′) (Corollary 4.2) When $1 \le i \le n - 1$, no star body in $\mathbb{E}^n$ is determined in $\mathcal{S}^n$, up to reflection in the origin, by its ith section function.

We prove the following result corresponding to (7) and (8) above:

(4′) (Corollary 4.9) The set of star bodies in $\mathbb{E}^n$ that are determined in $\mathcal{S}^n$, up to reflection in the origin, by their ith section functions for *all* i, $1 \le i \le n - 1$, is nowhere dense.

We see from the results obtained in this paper that there is a sharp contrast between the situation for projections and that for sections. This is interesting because it indicates possible limitations on the duality referred to above.

Although the duality makes it more natural to study sections of star bodies, convex bodies retain a wide general interest and we also consider in the last section the determination of convex bodies by section functions. In contrast to (4) and (5) above we have, somewhat surprisingly:

(5′) (Theorem 5.1) No centered convex body in $\mathbb{E}^2$ is determined in $\mathcal{K}_o^2$ by its section function.

Here $\mathcal{K}_o^n$ is the subclass of $\mathcal{K}^n$ whose members contain the origin in their interiors. This theorem also shows that the results of [4] on determination of convex polygons in the class of convex polygons do not extend to their determination in the class of all convex bodies.

In Theorem 5.3 we show that a dense set of convex bodies in $\mathcal{K}_o^n$, those with everywhere positive Gaussian curvature, are not determined in $\mathcal{K}_o^n$, up to reflection in the origin, by their ith section functions. Despite this, the question of whether most convex bodies in $\mathbb{E}^n$ are determined in $\mathcal{K}_o^n$, up to reflection in the origin, by their section functions remains open. One has to work quite hard to find any member of $\mathcal{K}_o^n$ that is determined in this sense; in Theorem 5.4 we show that examples are provided by triangles whose centroids lie at the origin. This result and Fig. 2 are due to the second author.

2. Definitions and Preliminaries

If A is a set, we write int A, bd A, and conv A for the interior, boundary, and convex hull of A, respectively. We denote the origin, unit sphere, and closed unit ball in n-dimensional Euclidean space $\mathbb{E}^n$ by o, S^{n-1}, and B, respectively. If $u \in S^{n-1}$, we denote by $u^\perp$ the $(n-1)$-dimensional subspace orthogonal to u. For $1 \le i \le n - 1$, we denote

the Grassmann manifold of i-dimensional subspaces in $\mathbb{E}^n$ by $\mathcal{G}(n, i)$. If $1 \leq i \leq n$, we write λ_i for *Lebesgue i-dimensional measure* in $\mathbb{E}^n$ (which we identify with i-dimensional Hausdorff measure in $\mathbb{E}^n$), and $\kappa_n = \lambda_n(B)$.

We say that a set is *centered* if it is centrally symmetric, with center at the origin. A *convex body* is a compact convex set with nonempty interior. The class of convex bodies in $\mathbb{E}^n$ is denoted by $\mathcal{K}^n$, and $\mathcal{K}_c^n$, $\mathcal{K}_o^n$ signify the subclasses consisting of centered convex bodies and convex bodies containing the origin in their interiors, respectively.

A set L is *star-shaped* at the origin if every line through the origin that meets L does so in a (possibly degenerate) closed line segment. If L is a set which is star-shaped at the origin, its *radial function* ρ_L is defined, for all $u \in S^{n-1}$ such that the line through the origin parallel to u intersects L, by

$$\rho_L(u) = \max\{c : cu \in L\}.$$

We denote by $\mathcal{S}^n$ the set of all nonempty compact sets which are star-shaped at the origin and whose radial functions are defined and continuous on S^{n-1}. Each member of $\mathcal{S}^n$ contains the origin and, with the single exception of the singleton set $\{o\}$, is a body in the sense that it is the closure of its interior. Let $\mathcal{S}_c^n$ be the set of all centered members of $\mathcal{S}^n$ and let $\mathcal{S}_o^n$ denote the set of members of $\mathcal{S}^n$ that contain the origin in their interiors.

In this paper we call members of $\mathcal{S}^n$ *star bodies*. We note, however, that there are different definitions of this term in the recent literature. In the work of Lutwak [16], [17] and others, the term implies membership in the smaller class $\mathcal{S}_o^n$. Another definition, introduced by two of the present authors in [8], allows for certain sets not containing the origin. In addition to these variants, Klain [14] considers the class of sets L star-shaped at the origin in $\mathbb{E}^n$ and such that ρ_L is an L^n function on S^{n-1}; such sets are called L^n-*stars*.

Observe that a convex body is a star body if and only if it contains the origin and its boundary does not contain a line segment containing the origin.

We now define the i-chord functions of a star body. (See Chapter 6 of [5] for extensive comments and history concerning this much-studied concept.) Let $i > 0$ be a real number and let L be a star body in $\mathbb{E}^n$. The i-*chord function* $\rho_{i,L}$ of L is defined by

$$\rho_{i,L}(u) = \rho_L(u)^i + \rho_L(-u)^i,$$

for $u \in S^{n-1}$. When $i = 1$, this gives the length of chords of L through the origin, and for this reason, $\rho_{1,L}$ is sometimes also called the *X-ray of L at the origin*. Another generalization of the latter is the ith section function of L, defined for integer values of i with $1 \leq i \leq n - 1$ by $\lambda_i(L \cap S)$, for $S \in \mathcal{G}(n, i)$. The following known result (see Theorem 7.2.3 and Remark 7.2.5 of [5], and compare Theorem 3.1 of [8]) connects these definitions.

Proposition 2.1. *Let $i \in \mathbb{N}$ and $1 \leq i \leq n - 1$. Two star bodies in $\mathbb{E}^n$ have equal i-chord functions if and only if they have equal ith section functions.*

The following result (see Theorem 5.2 of [8] or Theorem 6.2.16 of [5]) will also be useful in what follows.

Proposition 2.2. *Let i and j be distinct positive real numbers. Two star bodies L and M in $\mathbb{E}^n$ have equal i-chord functions and equal j-chord functions if and only if*

$$\{\rho_L(u), \rho_L(-u)\} = \{\rho_M(u), \rho_M(-u)\},$$

for all $u \in S^{n-1}$.

3. Metrics for Star Bodies

The distance between nonempty compact sets C and D can be measured by means of the Hausdorff metric, defined by

$$\delta(C, D) = \min\{\varepsilon > 0 : C \subset D + \varepsilon B \text{ and } D \subset C + \varepsilon B\}.$$

For star bodies, however, the *radial metric* $\tilde{\delta}$ is in many ways more natural. This is defined by setting

$$\tilde{\delta}(L, M) = \max_{u \in S^{n-1}} |\rho_L(u) - \rho_M(u)|,$$

for star bodies L and M. With this metric, S^n is a complete metric space and hence a Baire space. It is easy to see that S_o^n is an open and dense subset of S^n.

Goodey and Weil [11] prove that convergence in the Hausdorff metric does not imply convergence in the radial metric. It is easy to see that (S^n, δ) is not a complete metric space. Our preference for the radial metric here is a consequence of the following stronger result.

Theorem 3.1. *The set S^n, when endowed with the Hausdorff metric, is not a Baire space.*

Proof. For each $k \in \mathbb{N}$, let

$$\mathcal{G}_k = \left\{ L \in S^n : \lambda_{n-1}\left(\left\{u \in S^{n-1} : \rho_L(u) \geq \frac{1}{k}\right\}\right) < \frac{1}{k}\right\}.$$

We claim that $\mathcal{G}_k$ is open in (S^n, δ). To see this, suppose that $L \in \mathcal{G}_k$. For $m \in \mathbb{N}$, let $L_m = L + (1/m)B$; it is easy to see that $L_m \in S^n$ and $\bigcap_m L_m = L$. The sets

$$E_m = \left\{u \in S^{n-1} : \rho_{L_m}(u) \geq \frac{1}{k}\right\}$$

decrease with intersection

$$E = \left\{u \in S^{n-1} : \rho_L(u) \geq \frac{1}{k}\right\}.$$

It follows that $\lambda_{n-1}(E_m)$ decreases with limit $\lambda_{n-1}(E)$, so there is an m_0 such that

$\lambda_{n-1}(E_{m_0}) < 1/k$. Let $M \in \mathcal{S}^n$ be such that $\delta(L, M) < 1/m_0$. Then $M \subset L_{m_0}$, so

$$\lambda_{n-1}\left(\left\{u \in S^{n-1} : \rho_M(u) \geq \frac{1}{k}\right\}\right) < \lambda_{n-1}(E_{m_0}) < \frac{1}{k},$$

which shows that $M \in \mathcal{G}_k$, proving the claim.

Let $L \in \mathcal{S}^n$ and let $\varepsilon > 0$. Then there is a finite subset F of S^{n-1} such that if

$$C = \bigcup\{[o, \rho_L(u)u] : u \in F\},$$

then C is a finite union of line segments with $\delta(C, L) < \varepsilon/2$. Clearly, there is also an $M \in \mathcal{G}_k$ such that $\delta(M, C) < \varepsilon/2$. Then $\delta(L, M) < \varepsilon$, so $\mathcal{G}_k$ is dense in $(\mathcal{S}^n, \delta)$.

Let L be any member of $\mathcal{S}^n$ other than $\{o\}$. Then there is a $u_0 \in S^{n-1}$ such that $\rho_L(u_0) > 1/m$ for some $m \in \mathbb{N}$. Choose $k \in \mathbb{N}$ with $k \geq m$ and

$$\lambda_{n-1}\left(\left\{u \in S^{n-1} : \rho_L(u) \geq \frac{1}{m}\right\}\right) \geq \frac{1}{k}.$$

Then $L \notin \mathcal{G}_k$, so $\bigcap_k \mathcal{G}_k = \{o\}$ is not dense in $(\mathcal{S}^n, \delta)$. This proves the theorem. $\quad\square$

4. Determination of Star Bodies

Theorem 4.1. *Let $i > 0$. No star body in $\mathbb{E}^n$ is determined in $\mathcal{S}^n$, up to reflection in the origin, by its i-chord function.*

Proof. Let $L \in \mathcal{S}^n$. Choose $u_0 \in S^{n-1}$ with $\rho_L(u_0) > 0$, and let U be any open cap containing u_0 and contained in a hemisphere in S^{n-1} such that $\rho_L(u) > 0$ for all $u \in U$. Let $f \in C(S^{n-1})$ be such that $0 < f(u) < \rho_L(u)$ for $u \in U$, $f(u) = 0$ for $u \notin U$, and

$$f(u_0) \neq \rho_L(u_0) - \rho_L(-u_0).$$

Let

$$g(u) = (\rho_L(u)^i + \rho_L(-u)^i - (\rho_L(u) - f(u))^i)^{1/i} - \rho_L(-u),$$

$u \in U$. Define $M \in \mathcal{S}^n$ by $\rho_M(u) = \rho_L(u) - f(u)$ if $u \in U$, $\rho_M(u) = \rho_L(u) + g(-u)$ if $u \in -U$, and $\rho_M(u) = \rho_L(u)$ otherwise. It is easy to check that $\rho_M \in C(S^{n-1})$, that L and M have equal i-chord functions, and that $\rho_M(u_0) \neq \rho_L(\pm u_0)$, as required. $\quad\square$

Corollary 4.2. *Let $i \in \mathbb{N}$ and $1 \leq i \leq n - 1$. No star body in $\mathbb{E}^n$ is determined in $\mathcal{S}^n$, up to reflection in the origin, by its ith section function.*

Proof. This follows immediately from Proposition 2.1 and the previous theorem. $\quad\square$

Theorem 4.3. *A star body in $\mathbb{E}^n$ is determined in $\mathcal{S}^n$, up to reflection in the origin, by its i-chord functions for two different values of i if and only if the set*

$$V_L = \{u \in S^{n-1} : \rho_L(u) > \rho_L(-u)\}$$

is connected.

Proof. Suppose that V_L is not connected and let U be one of its components. By Proposition 2.2 the star body M with radial function

$$\rho_M(u) = \begin{cases} \rho_L(-u) & \text{if} \quad u \in U \cup (-U), \\ \rho_L(u) & \text{otherwise,} \end{cases}$$

has all its i-chord functions equal to those of L, but $M \neq \pm L$.

Conversely, suppose that V_L is connected and that M has its i-chord functions equal to those of L for two different values of i, but $M \neq \pm L$. By Proposition 2.2,

$$\{\rho_L(u), \rho_L(-u)\} = \{\rho_M(u), \rho_M(-u)\}.$$

This implies that there exist u_1 and u_2, where we may assume $u_1, u_2 \in V_L$, such that

$$\rho_M(u_1) = \rho_L(u_1) \neq \rho_M(-u_1) = \rho_L(-u_1)$$

and

$$\rho_M(u_2) = \rho_L(-u_2) \neq \rho_M(-u_2) = \rho_L(u_2).$$

Then we have

$$\rho_M(u_1) > \rho_M(-u_1)$$

and

$$\rho_M(u_2) < \rho_M(-u_2).$$

Since V_L is connected and ρ_M is continuous, there exists $u_0 \in V_L$ such that $\rho_M(u_0) = \rho_M(-u_0)$ and therefore $\rho_L(u_0) = \rho_L(-u_0)$, a contradiction. $\qquad\square$

Corollary 4.4. *A centered star body in $\mathbb{E}^n$ is determined in $\mathcal{S}^n$ by its ith section functions for two different values of i.*

Proof. This follows directly from Proposition 2.1 and the previous theorem. $\qquad\square$

Let M be a star body. For the remainder of this section we continue to use the notation of Theorem 4.3, so that

$$V_M = \{u \in S^{n-1} : \rho_M(u) > \rho_M(-u)\}.$$

Denote by $\mathcal{M}$ the set of $M \in \mathcal{S}_o^n$ such that there exist disjoint open components U_1, U_2 of V_M which are separated by $-V_M$, that is, U_1 and U_2 are contained in disjoint components of $S^{n-1} \setminus (-V_M)$.

The following lemma provides a useful characterization of $\mathcal{M}$.

Lemma 4.5. *A star body $M \in \mathcal{S}_o^n$ belongs to $\mathcal{M}$ if and only if there exist open components U_1 and U_2 of V_M and an $a > 0$ such that*

$$W_M(a) = \{u \in S^{n-1} : \rho_M(u) < \rho_M(-u) - a\}$$

separates U_1 from U_2.

Proof. Let $M \in \mathcal{S}_o^n$. It is clear that if the stated condition holds, then $M \in \mathcal{M}$.

Conversely, suppose that $M \in \mathcal{M}$. Let U_1 and U_2 be open components of V_M contained in disjoint components of $S^{n-1} \setminus (-V_M)$. For $k \in \mathbb{N}$, the sets $W_M(1/k)$ are open in S^{n-1} and increase with union $-V_M$. Then $S^{n-1} \setminus W_M(1/k)$ is closed and contains U_1 and U_2, for each $k \in \mathbb{N}$. Suppose that, for all k, both U_1 and U_2 are contained in the same (compact) component C_k of $S^{n-1} \setminus W_M(1/k)$. Then $C = \bigcap_k C_k$ is a component of $S^{n-1} \setminus (-V_M)$ (see, for example, Theorem 5 of Section 47 in [15]) that contains both U_1 and U_2, a contradiction. Therefore there is a k_0 such that $W_M(1/k_0)$ separates U_1 from U_2. Letting $a = 1/k_0$, we have proved the lemma.   □

Lemma 4.6.   *The set $\mathcal{M}$ is open in $\mathcal{S}_o^n$.*

Proof.   Let $M \in \mathcal{M}$ and let U_1, U_2, $a > 0$, and $W_M(a)$ be as in Lemma 4.5. Let $a_j = \max_{u \in U_j}(\rho_M(u) - \rho_M(-u))$, $j = 1, 2$, and let $0 < \varepsilon < \min\{a_1, a_2, a\}/2$. Suppose that $L \in \mathcal{S}_o^n$ and $\tilde{\delta}(L, M) < \varepsilon$, which implies that

$$\rho_M(u) - \varepsilon < \rho_L(u) < \rho_M(u) + \varepsilon,$$

for all $u \in S^{n-1}$. Let $u_j \in U_j$ be such that $\rho_M(u_j) = a_j$, $j = 1, 2$. Then

$$\rho_L(u_j) - \rho_L(-u_j) > \rho_M(u_j) - \varepsilon - \rho_M(-u_j) - \varepsilon = a_j - 2\varepsilon > 0,$$

so $u_j \in V_L$, $j = 1, 2$. Let U_j' be the component of V_L that contains u_j, $j = 1, 2$, and let $a' = a - 2\varepsilon$. If $u \in W_M(a)$, then

$$\rho_L(u) - \rho_L(-u) < \rho_M(u) + \varepsilon - \rho_M(-u) + \varepsilon < -a + 2\varepsilon = -a',$$

so $u \in W_L(a')$. Therefore $W_M(a) \subset W_L(a')$ and since $U_j' \cap W_L(a') = \emptyset$ and $U_j \cap U_j' \neq \emptyset$, $j = 1, 2$, $W_L(a')$ separates U_1' and U_2'. So $L \in \mathcal{M}$, as required.   □

Lemma 4.7.   *The set $\mathcal{M}$ is dense in $\mathcal{S}_o^n$.*

Proof.   Suppose that $L \in \mathcal{S}_o^n$, and let $0 < \varepsilon < \min_{u \in S^{n-1}} \rho_L(u)$. From the continuity of ρ_L if follows that for each $\varepsilon > 0$ there is an open cap C contained in a hemisphere of S^{n-1} such that

$$\rho_L(-u) - \frac{\varepsilon}{2} < \rho_L(u) < \rho_L(-u) + \frac{\varepsilon}{2}$$

for all $u \in C$. Choose distinct points u_1 and u_2 in C. Let C_1 and C_2 be closed caps centered at u_1 and contained in C, with the radius of C_1 greater than that of C_2, and such that $u_2 \notin C_1$. Let $A = \text{relint } C_1 \setminus C_2$; then A is an annular region in S^{n-1} centered at u_1.

Let $f(u)$ be a continuous function on S^{n-1} such that $|f(u)| \leq \varepsilon$ for all $u \in S^{n-1}$, $f(u_j) = \varepsilon$, $j = 1, 2$, $f(u) = -\varepsilon$ for $u \in A$, and $f(u) = 0$ if $u \notin C$. Let M be the star body defined by

$$\rho_M(u) = \rho_L(u) + f(u).$$

Then

$$\rho_M(u_j) = \rho_L(u_j) + \varepsilon > \rho_L(-u_j) + \frac{\varepsilon}{2} = \rho_M(-u_j) + \frac{\varepsilon}{2} > \rho_M(-u_j),$$

so $u_j \in V_M$, for $j = 1, 2$. On the other hand, for $u \in A$ we have

$$\rho_M(u) = \rho_L(u) - \varepsilon < \rho_L(-u) - \frac{\varepsilon}{2} = \rho_M(-u) - \frac{\varepsilon}{2},$$

so $A \subset W_M(\varepsilon/2)$. If U_j is the component of V_M containing u_j, $j = 1, 2$, then U_1 and U_2 are separated by A and so also $W_M(\varepsilon/2)$. Therefore $M \in \mathcal{M}$ and $\tilde{\delta}(L, M) = \varepsilon$. $\square$

Lemma 4.8. *If $M \in \mathcal{M}$, there exists an $M' \in \mathcal{M}$ such that $M' \neq \pm M$ and yet M and M' have equal i-chord functions for all $i > 0$.*

Proof. Let $M \in \mathcal{M}$ and let U_1, U_2, $a > 0$, and $W_M(a)$ be as in Lemma 4.5. Define M' by

$$\rho_{M'}(u) = \begin{cases} \rho_M(-u) & \text{if} \quad u \in U_1 \cup (-U_1), \\ \rho_M(u) & \text{if} \quad u \in S^{n-1} \backslash (U_1 \cup (-U_1)). \end{cases}$$

We claim that $\rho_{M'}$ is continuous. It is obviously continuous at each $u \in U_1 \cup (-U_1)$ and at each point in the relative interior of $S^{n-1} \backslash (U_1 \cup (-U_1))$. If u_0 belongs to the common relative boundary of these sets, then

$$\rho_{M'}(u_0) = \rho_M(u_0) = \rho_M(-u_0).$$

For each $\varepsilon > 0$ there exists a neighborhood U of u_0 such that $|\rho_M(u_0) - \rho_M(u)| < \varepsilon$ and $|\rho_M(-u_0) - \rho_M(-u)| < \varepsilon$ for each $u \in U$. If $u \in U \cap (U_1 \cup (-U_1))$, we have

$$|\rho_{M'}(u_0) - \rho_{M'}(u)| = |\rho_M(-u_0) - \rho_M(-u)| < \varepsilon,$$

while if $u \in U \backslash (U_1 \cup (-U_1))$, then

$$|\rho_{M'}(u_0) - \rho_{M'}(u)| = |\rho_M(u_0) - \rho_M(u)| < \varepsilon,$$

and this proves the claim.

Since

$$\{\rho_M(u), \rho_M(-u)\} = \{\rho_{M'}(u), \rho_{M'}(-u)\}$$

for all $u \in S^{n-1}$, Proposition 2.2 implies that M and M' have equal i-chord functions for all $i > 0$. Moreover, since $\rho_M(u) > \rho_M(-u) = \rho_{M'}(u)$ when $u \in U_1$, we have $M \neq M'$, and since $\rho_M(u) > \rho_M(-u) = \rho_{M'}(-u)$ when $u \in U_2$, we also have $M \neq -M'$. $\square$

Corollary 4.9. *The set of star bodies in $\mathbb{E}^n$ that are determined in $\mathcal{S}^n$, up to reflection in the origin, by their ith section functions for all i, $1 \leq i \leq n - 1$, is nowhere dense.*

Proof. By Lemmas 4.6, 4.7, and 4.8, there is an open dense subset of $\mathcal{S}_o^n$, and hence an open dense subset of $\mathcal{S}^n$, consisting of star bodies not determined, up to reflection in the origin, by their i-chord functions for all $i > 0$. The result now follows from Proposition 2.2. $\square$

We know from Lemma 4.8 that the set of star bodies in $\mathcal{S}_o^n$ that are not determined in $\mathcal{S}^n$, up to reflection in the origin, by their ith section functions for two different values of i, contains the open set $\mathcal{M}$, dense in $\mathcal{S}_o^n$. Despite this, we have the following result.

Theorem 4.10. *The set of all star bodies in $\mathcal{S}_o^n$ that are not determined in $\mathcal{S}^n$, up to reflection in the origin, by their ith section functions for two different values of i, is not open.*

Proof. Let L' be the planar star body such that

$$\rho_{L'}(\theta) = \begin{cases} 1 + \sin^2 4\theta & \text{if} \quad 0 \leq \theta \leq \pi, \\ 1 & \text{if} \quad \pi \leq \theta \leq 2\pi. \end{cases}$$

Since L' is symmetric with respect to the y-axis, we can rotate a copy of L' in the (x_1, x_n)-plane in $\mathbb{E}^n$ about the x_n-axis to obtain a star body L in $\mathbb{E}^n$.

By Theorem 4.3, L is not determined, up to reflection in the origin, by its ith section functions for two different values of i, since V_L has components $\{u \in S^{n-1} : 0 < u \cdot u_0 < 1/\sqrt{2}\}$ and $\{u \in S^{n-1} : 1/\sqrt{2} < u \cdot u_0 < 1\}$, where $u_0 = (0, \ldots, 0, 1)$. Let $\varepsilon > 0$, and define a planar star body M' by

$$\rho_{M'}(\theta) = \begin{cases} 1 + \sin^2 4\theta + \varepsilon(2/\pi)^2 \theta(\pi - \theta) & \text{if} \quad 0 \leq \theta \leq \pi, \\ 1 & \text{if} \quad \pi \leq \theta \leq 2\pi. \end{cases}$$

Note that $\tilde{\delta}(L', M') = \varepsilon$. Since M' is symmetric with respect to the y-axis, we can rotate a copy of M' in the (x_1, x_n)-plane in $\mathbb{E}^n$ about the x_n-axis to obtain a star body M in $\mathbb{E}^n$ such that $\tilde{\delta}(L, M) = \varepsilon$. Since

$$V_M = \{u \in S^{n-1} : u \cdot u_0 > 0\},$$

Theorem 4.3 implies that M is determined, up to reflection in the origin, by its ith section functions for two different values of i. $\qquad\square$

5. Determination of Convex Bodies

Theorem 5.1. *Let $i > 0$ and suppose that K is a centered convex body in $\mathbb{E}^2$. Then K is not determined in $\mathcal{K}_o^2$ by its i-chord function (and hence not by its section function or X-ray at the origin).*

Proof. By rotating K, if necessary, we may suppose that it is strictly convex at the point v where its boundary bd K meets the positive x-axis; since K is centered, it is also strictly convex at $-v$. Let w be a point on the positive x-axis with $w \notin K$, and let p_j, $j = 1, 2$, be the points where the tangents to K through w meet bd K. We assume that w is sufficiently close to bd K to ensure that if u is a unit vector in the direction of a point q on $[p_j, w]$, $j = 1, 2$, then

$$2^{1/i} \rho_K(u) > \|q\|. \tag{1}$$

Let $L \in \mathcal{S}_o^2$ be the star body with the same i-chord functions as K, such that bd L is obtained from bd K by replacing the arc in bd K from p_1 to p_2 containing v by the line segments $[p_j, w]$, for $j = 1, 2$, and the arc in bd K from $-p_1$ to $-p_2$ containing $-v$ by suitable arcs P_j joining $-p_j$ and a point $z \in K$ on the negative x-axis. The body L is defined by

$$\rho_L(u)^i + \rho_L(-u)^i = \rho_K(u)^i + \rho_K(-u)^i = 2\rho_K(u)^i, \tag{2}$$

for all $u \in S^1$.

The body L is in general not convex (see Fig. 1 for a special case when it is), but we claim that the arcs P_j, $j = 1, 2$, are convex with respect to the origin. To see this, fix j and let u_1 and u_2 be unit vectors in the direction of points on $[p_j, w]$. The convexity of K implies (see Lemma 5.1.4 of [5]) that

$$\rho_K(u_1 + u_2)^{-1} \leq \rho_K(u_1)^{-1} + \rho_K(u_2)^{-1} \tag{3}$$

and the linearity of bd L along $[p_j, w]$ yields

$$\rho_L(u_1 + u_2)^{-1} = \rho_L(u_1)^{-1} + \rho_L(u_2)^{-1}. \tag{4}$$

For $p < 0$, Minkowski's inequality states that

$$((a_1 + b_1)^p + (a_2 + b_2)^p)^{1/p} \geq (a_1^p + a_2^p)^{1/p} + (b_1^p + b_2^p)^{1/p},$$

for $a_k > 0$ and $b_k > 0$, $k = 1, 2$; see (B.4), p. 367 of [5] or Section 2.11 of [13]. Letting $c_k = a_k + b_k$, $k = 1, 2$, and $p = -1/i$, we obtain

$$(c_1^{-1/i} + c_2^{-1/i})^{-i} - (b_1^{-1/i} + b_2^{-1/i})^{-i} \geq ((c_1 - b_1)^{-1/i} + (c_2 - b_2)^{-1/i})^{-i}, \tag{5}$$

for $c_k > b_k > 0$, $k = 1, 2$.

By (1), we have $2^{1/i} \rho_K(u_k) > \rho_L(u_k)$, $k = 1, 2$. Then we can apply (5) with $b_k = \rho_L(u_k)^i$ and $c_k = 2\rho_K(u_k)^i$, $k = 1, 2$, together with (2)–(4), to obtain

$$
\begin{aligned}
\rho_L(-u_1 - u_2)^{-1} &= (2\rho_K(u_1 + u_2)^i - \rho_L(u_1 + u_2)^i)^{-1/i} \\
&\leq ((2^{-1/i}\rho_K(u_1)^{-1} + 2^{-1/i}\rho_K(u_2)^{-1})^{-i} \\
&\quad - (\rho_L(u_1)^{-1} + \rho_L(u_2)^{-1})^{-i})^{-1/i} \\
&= (((2\rho_K(u_1)^i)^{-1/i} + (2\rho_K(u_2)^i)^{-1/i})^{-i} \\
&\quad - ((\rho_L(u_1)^i)^{-1/i} + (\rho_L(u_2)^i)^{-1/i})^{-i})^{-1/i} \\
&\leq (2\rho_K(u_1)^i - \rho_L(u_1)^i)^{-1/i} + (2\rho_K(u_2)^i - \rho_L(u_2)^i)^{-1/i} \\
&= \rho_L(-u_1)^{-1} + \rho_L(-u_2)^{-1}.
\end{aligned}
$$

This shows that the arc P_j is convex and proves the claim.

If L is convex, the proof is finished. Otherwise, let $H = \operatorname{conv} L$. Then there are points $q_j \in P_j$, $j = 1, 2$, such that bd H contains the line segment $[q_1, q_2]$ but otherwise agrees with bd L.

Let $M \in \mathcal{S}_o^2$ be the star body defined by

$$\rho_M(u)^i + \rho_H(-u)^i = \rho_L(u)^i + \rho_L(-u)^i = 2\rho_K(u)^i,$$

for all $u \in S^1$, where the last equality is a consequence of (2). The points in bd M in the direction of $u \in S^1$ such that $-u$ is in the direction of a point in $[q_1, q_2]$ form an arc R joining points r_1 and r_2, say, with $r_j \in [p_j, w]$.

The arc R is convex with respect to the origin. The argument that proves this is the same as that for the convexity of the arc P_j, since if u_1 and u_2 are unit vectors in the direction of points on $[q_1, q_2]$, we have

$$\rho_M(-u_1 - u_2)^{-1} = (2\rho_K(u_1 + u_2)^i - \rho_H(u_1 + u_2)^i)^{-1/i},$$

and bd H is linear along $[q_1, q_2]$.

Finally, we let K' be the body whose boundary contains the arc R but otherwise agrees with bd H. By construction, $K' \in \mathcal{K}_o^2$ and K' has the same i-chord function as K, so the proof is complete. □

Figure 1 depicts a body K' obtained from a centered square K by the construction of the previous theorem with $i = 1$. For this example it can be shown that when w is sufficiently close to v, as it is in Figure 1, the body L in the proof of Theorem 5.1 is already convex. Figure 1 was generated with Mathematica and therefore gives the precise shape of K' up to the tolerances of that package. The existence of such examples is surprising and of independent interest since it would be natural to conjecture that a convex body containing the origin in its interior and having the same X-ray at the origin as a convex polygon must itself be a convex polygon. The example also shows that the results of [4] on determination of convex polygons in the class of convex polygons do not extend to their determination in the class of all convex bodies.
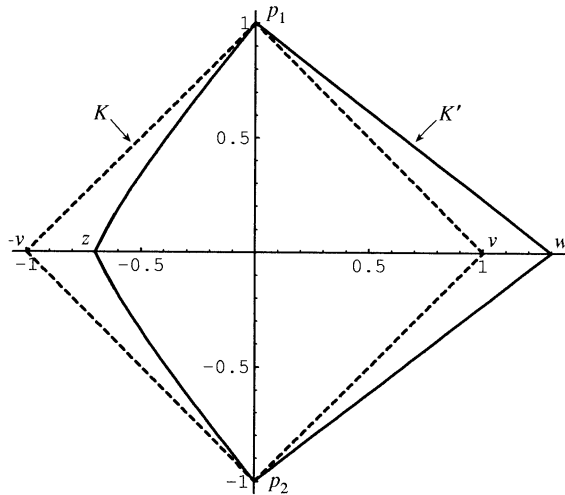

Fig. 1. Convex bodies with the same section function.

We conjecture that Theorem 5.1 holds in higher dimensions. The following question is also open.

Question 5.2. Let $i \in \mathbb{N}$ and $1 \le i \le n - 1$. Are most convex bodies in $\mathcal{K}_o^n$ determined in $\mathcal{K}_o^n$ by their ith section function?

The next theorem provides an important class of convex bodies in $\mathbb{E}^n$ that are not determined in the sense of the previous question.

Theorem 5.3. *Let $i > 0$ and suppose that $K \in \mathcal{K}_o^n$ has everywhere positive Gaussian curvature. Then K is not determined in $\mathcal{K}_o^n$, up to reflection in the origin, by its i-chord function, or (if $i \in \mathbb{N}$ and $1 \le i \le n - 1$) by its ith section function.*

Proof. Let U be an open cap contained in a hemisphere in S^{n-1} and let $f \in C^2(S^{n-1})$ be such that $f(u) > 0$ if $u \in U$ and $f(u) = 0$ if $u \notin U$. Let $c > 0$ be such that $cf(u) < \rho_K(u)$ if $u \in U$. Define

$$g_c(u) = (\rho_K(u)^i + \rho_K(-u)^i - (\rho_K(u) - cf(u))^i)^{1/i} - \rho_K(-u),$$

for $u \in U$. Define $L \in \mathcal{S}_o^n$ by

$$\rho_L(u) = \begin{cases} \rho_K(u) - cf(u) & \text{if} \quad u \in U, \\ \rho_K(u) + g_c(-u) & \text{if} \quad u \in -U, \\ \rho_K(u) & \text{otherwise.} \end{cases}$$

It is easy to check that $\rho_L \in C^2(S^{n-1})$ and that K and L have equal i-chord functions (and by Proposition 2.1, if $i \in \mathbb{N}$ and $1 \le i \le n - 1$, equal ith section functions). We can also choose $u_0 \in U$ and $c_0 > 0$ such that

$$cf(u_0) \ne \rho_K(u_0) - \rho_K(-u_0),$$

for all c with $0 < c \le c_0$; note that this ensures that $\rho_L(u_0) \ne \rho_K(\pm u_0)$ for the corresponding bodies.

It remains to show that we can also choose c so that L is convex. To see this, note that it suffices to show that $L \cap S$ is convex for each two-dimensional subspace, so we may assume that $n = 2$ and use polar coordinates. By the formula for curvature in polar coordinates, we have

$$2\rho_K'(\theta)^2 - \rho_K(\theta)\rho_K''(\theta) + \rho_K(\theta)^2 > 0,$$

for $0 \le \theta \le 2\pi$. Now for $\theta \in U$, we have $\rho_L(\theta) = \rho_K(\theta) - cf(\theta)$ and hence

$$2\rho_L'(\theta)^2 - \rho_L(\theta)\rho_L''(\theta) + \rho_L(\theta)^2 = 2\rho_K'(\theta)^2 - \rho_K(\theta)\rho_K''(\theta) + \rho_K(\theta)^2 + cF(\theta),$$

where $F = F(f, f', f'', \rho_K, \rho_K', \rho_K'')$ is a polynomial. Therefore we can choose $c \le c_0$ sufficiently small to ensure that

$$2\rho_L'(\theta)^2 - \rho_L(\theta)\rho_L''(\theta) + \rho_L(\theta)^2 \ge 0,$$

for $\theta \in U$. For $\theta \in -U$, we have $\rho_L(\theta) = \rho_K(\theta) + g_c(-\theta)$ and

$$2\rho_L'(\theta)^2 - \rho_L(\theta)\rho_L''(\theta) + \rho_L(\theta)^2 = 2\rho_K'(\theta)^2 - \rho_K(\theta)\rho_K''(\theta) + \rho_K(\theta)^2 + G(\theta),$$

where $G = G(g_c, g_c', g_c'', \rho_K, \rho_K', \rho_K'')$ is a polynomial. It is easy to check that as $c \to 0$, g_c and its first two derivatives converge to zero uniformly in θ and that this implies that G also converges to zero uniformly in θ. Therefore we can choose c sufficiently small to ensure that also

$$2\rho_L'(\theta)^2 - \rho_L(\theta)\rho_L''(\theta) + \rho_L(\theta)^2 \geq 0,$$

for $\theta \in -U$, and hence that L is convex. $\square$

The class of convex bodies in $\mathcal{K}_o^n$ with everywhere positive Gaussian curvature is dense in $\mathcal{K}_o^n$ (this follows from Theorem 3.3.1 of [18]). On the other hand, most members of $\mathcal{K}_o^n$ have zero Gaussian curvature at almost every point on their boundaries. This is a result of Zamfirescu (see p. 1332 of [12] and the references given there).

Theorem 5.4. *Let T be a triangle in $\mathbb{E}^2$ with centroid at the origin. Then T is determined in $\mathcal{K}_o^2$, up to reflection in the origin, by its section function (X-ray at the origin).*

Proof.   By applying a linear transformation, if necessary, we may assume that T has vertices $(1, 0)$, $(-\frac{1}{2}, \sqrt{3}/2)$, and $(-\frac{1}{2}, -\sqrt{3}/2)$. Suppose that $K \in \mathcal{K}_o^2$ has the same section function as T.

We work with the radial function of K in polar coordinates. Let $x_n = \rho_K(n\pi/3)$, so that

$$x_{n+6} = x_n, \tag{6}$$

for all $n \in \mathbb{N}$. Since K has the same section function as T, we also have

$$x_n + x_{n+3} = \tfrac{3}{2} \tag{7}$$

and

$$0 < x_n < \tfrac{3}{2} \tag{8}$$

for all $n \in \mathbb{N}$. Our goal is to prove that either (i) $x_1 = \frac{1}{2}$, $x_2 = 1$, and $x_3 = \frac{1}{2}$, or (ii) $x_1 = 1$, $x_2 = \frac{1}{2}$, and $x_3 = 1$. If (i) holds, then by (7) the vertices of T belong to K, so $T \subset K$, and the equality of the section functions then implies $K = T$. If (ii) holds, the vertices of $-T$ belong to K, so we conclude similarly that $K = -T$.

We first establish two more relations among the x_n's. The convexity of K yields (see Lemma 5.1.4 of [5])

$$\frac{1}{x_{n+1}} \leq \frac{1}{x_n} + \frac{1}{x_{n+2}} \tag{9}$$

and

$$\frac{\sqrt{3}}{\rho_K((2n+1)\pi/6)} \leq \frac{1}{x_n} + \frac{1}{x_{n+1}},$$

for all $n \in \mathbb{N}$. Since K has the same section function as T, we have

$$\rho_K \left(\frac{(2n+1)\pi}{6} \right) + \rho_K \left(\frac{(2n+7)\pi}{6} \right) = \frac{2\sqrt{3}}{3},$$

so the previous inequality gives

$$\frac{\sqrt{3}}{1/x_n + 1/x_{n+1}} + \frac{\sqrt{3}}{1/x_{n+3} + 1/x_{n+4}} \leq \frac{2\sqrt{3}}{3}.$$

Combining this with (7) we have

$$\frac{\sqrt{3}(3x_n + 3x_{n+1} + 16x_n x_{n+1} - 10x_n^2 - 10x_{n+1}^2)}{12(x_n + x_{n+1})(3 - (x_n + x_{n+1}))} \leq 0,$$

so, by (8),

$$3x_n + 3x_{n+1} + 16x_n x_{n+1} - 10x_n^2 - 10x_{n+1}^2 \leq 0, \tag{10}$$

for all $n \in \mathbb{N}$. Note that (10) is satisfied unless (x_n, x_{n+1}) lies in the interior of an ellipse E in $\mathbb{E}^2$ that is symmetric with respect to the line $y = x$. The ellipse E contains $(0, 0)$, $(0, \frac{3}{10})$, $(\frac{1}{10}, \frac{1}{2})$, $(\frac{1}{2}, 1)$, $(1, \frac{7}{5})$, and $(\frac{3}{2}, \frac{3}{2})$ in its boundary, and therefore also contains the squares $S_1 = \{(x, y) : 0 \leq x, y \leq \frac{3}{10}\}$, $S_2 = \{(x, y) : \frac{1}{10} \leq x, y \leq \frac{1}{2}\}$, and $S_3 = \{(x, y) : \frac{1}{2} \leq x, y \leq 1\}$.

We prove that either (i) or (ii) holds by a series of claims.

Claim 1. *If $\frac{1}{2} \leq x_n, x_{n+1} \leq 1$ for any $n \in \mathbb{N}$, then $x_n = \frac{1}{2}$ and $x_{n+1} = 1$ or vice versa.*

This is an immediate consequence of the fact that $S_3 \subset E$.

Claim 2. *It is not possible that $x_n, x_{n+1} \leq \frac{1}{2}$ for any $n \in \mathbb{N}$.*

Observe first that since E contains the squares S_1 and S_2, we have $\frac{3}{10} < x_n \leq \frac{1}{2}$ and $x_{n+1} < \frac{1}{10}$ or vice versa. If the former possibility holds, then, by (4),

$$\frac{1}{x_{n+2}} \geq \frac{1}{x_{n+1}} - \frac{1}{x_n} > \frac{1}{\frac{1}{10}} - \frac{1}{\frac{3}{10}} = \frac{20}{3},$$

so $x_{n+2} < \frac{3}{20}$. Since $x_{n+1} < \frac{1}{10}$, we must have $\frac{3}{10} < x_{n+2} \leq \frac{1}{2}$, by the above argument with n replaced by $(n+1)$, and this is a contradiction. The assumption that $0 < x_n < \frac{1}{10}$ and $\frac{3}{10} < x_{n+1} \leq \frac{1}{2}$ similarly leads to a contradiction, so Claim 2 is proved.

Claim 3. *If $1 \leq x_n, x_{n+2}$ and $x_{n+1} \leq \frac{1}{2}$ for any $n \in \mathbb{N}$, then $x_n = 1$, $x_{n+1} = \frac{1}{2}$, and $x_{n+2} = 1$.*

This follows immediately from (9).

Claim 4. *If $\frac{1}{2} \leq x_n \leq 1$, $x_{n+1} \leq \frac{1}{2}$, and $1 \leq x_{n+2} \leq \frac{3}{2}$ for any $n \in \mathbb{N}$, then $x_n = 1$, $x_{n+1} = \frac{1}{2}$, and $x_{n+2} = 1$.*

Note that $x_n \le 2x_{n+1}$, for otherwise, by (9),

$$\frac{1}{x_{n+1}} \le \frac{1}{x_n} + \frac{1}{x_{n+2}} \le \frac{1}{x_n} + 1 < \frac{1}{2x_{n+1}} + 1 \le \frac{1}{2x_{n+1}} + \frac{1}{2x_{n+1}} = \frac{1}{x_{n+1}}.$$

The line $y = x/2$ meets bd E at the origin and $(1, \frac{1}{2})$, so the triangle determined by this line and the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$ is contained in E and intersects bd E only at $(1, \frac{1}{2})$. It follows that if $\frac{1}{2} \le x_n \le 1$, $x_{n+1} \le \frac{1}{2}$, and $x_n \le 2x_{n+1}$, then (10) holds only if $x_n = 1$ and $x_{n+1} = \frac{1}{2}$, in which case $x_{n+2} = 1$ by (9).

We now represent any sequence $\{x_n\}$ by a sequence of letters S, M, and L, each letter being used at the nth place if $x_n \le \frac{1}{2}$, $\frac{1}{2} \le x_n \le 1$, or $1 \le x_n$, respectively. Each sequence is determined by the first three letters; thus if the sequence begins S, L, M, it must continue L, S, M, by (7), and then repeat by (6). Thus we have 27 different possible sequences. Claims 1–4 above each exclude sequences containing some concatenation of letters. Specifically, the appearance of M, M, or S, S, or L, S, L, or M, S, L, is forbidden by Claims 1–4, respectively, with certain exceptional values of x_n allowed as in the statements of the claims. Each of the 27 sequences can now be examined individually. For example, the sequence above beginning S, L, M continues with L, S, M, S, L, M and therefore contains M, S, L. By Claim 4 this is only possible when $x_6 = 1$, $x_7 = \frac{1}{2}$, and $x_8 = 1$, so using (6) and (7), we see that (i) holds . A routine check shows that in each case (i) or (ii) holds, completing the proof. $\qquad\square$

If it is true that a set of convex polytopes dense in $\mathcal{K}_o^n$ are determined in $\mathcal{K}_o^n$, up to reflection in the origin, by their section functions, then a standard argument shows that the answer to Question 5.2 is affirmative. In view of Theorem 5.3, however, this may be false or difficult to prove. There are certainly convex polygons whose vertices are in general position with respect to the origin that are not determined in this sense. An example is the pentagon P depicted in Fig. 2.
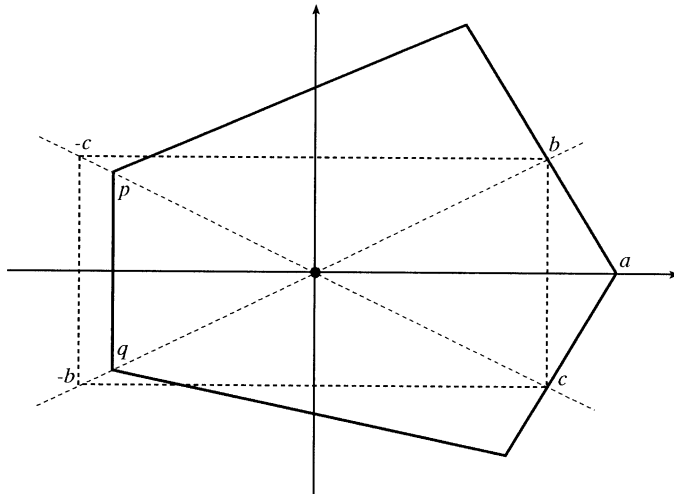

Fig. 2.   The pentagon P.

To see that P is not determined in $\mathcal{K}_o^2$, up to reflection in the origin, by its section function, Minkowski's inequality can be used as in the proof of Theorem 5.1 to show that there is a convex body H, whose intersection with P is precisely the segment $[p, q]$, such that H has the same section function as the reflection in the origin of the triangle T with vertices at a, b, and c. The body K formed by removing T from P and adding H then has the same section function as P. Moreover, the interior angles of P at p and q are sufficiently small to ensure that K is convex.

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