Very Simply Explicitly Invertible Corresponding Approximations of sin, cos, arcsin and arccos, Avoiding Trigonometric Functions – A Draft

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On the net: 23 March 2016 – Last updated: 5 April 2016

Abstract

This approximation of sin(x) in [0,π] with relative error $< 0.35\%$ in absolute value is given: exp(1-c(x-π/2)^2)+1-e with c=-4log(1-1/e)/π^2, about 0.185894. Its restriction in $[0, \pi/2]$ is very simply explicitly invertible, with only 1 entry of x, and avoids trigonometric functions, just as its inverse, which approximates in [0,1] arcsin(y) with the same relative error $< 0.35\%$ in absolute value. Substituting x with its absolute value and multiplying all by sign x that approximation of sin holds, with the same relative error, in [-π,π]. Similarly it is done for arcsin y, in $[-1, 1]$.

Mathematics Subject Classification: ??? , 33F05 , 65D20 ???, 97N50

Keywords: Approximation, trigonometric functions, sin, cos, arcsin, arccos

It is quite usual in applied sciences to happen the equation $\sin(x) = k$ or the inequation $\sin(x) \geq k$, or similar, knowing $0 \leq x \leq \frac{\pi}{2}$. Of course the solutions are respectively $x = \arcsin k$, and $\arcsin k \leq x \leq \frac{\pi}{2}$. Sometimes the lost of exactness could be worth, to have solutions expressed by functions as log and square root and usual operations, instead of arcsin. This paper is devoted to give corresponding approximations of sin, cos, arcsin and arccos, all with precision better than 0.6%, avoiding all trigonometric functions. Only exp, log, square root and the usual operations are used. The approximations are all *very simply explicitly invertible*, each having only 1 entry of x.

Theorem 1. This approximations of $\sin(x)$ and $cos(x)$ have relative error <0.35% (in absolute value):

$$c := \frac{4(1-\log(e-1))}{\pi^2} = -\frac{4\log\left(1-\frac{1}{e}\right)}{\pi^2} \simeq 0.185894 \qquad |\varepsilon_r(x)| < 0.35\%$$

$$\cos x \simeq e^{1-cx^2} + 1 - e \qquad =: p(x) \qquad \forall\, x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\sin(x) \simeq \begin{cases} e^{1-c\left(x-\frac{\pi}{2}\right)^2} + 1 - e & =: q(x) \qquad \forall\, x \in [0, \pi] \\ (\operatorname{sign} x)\,(e^{1-c\left(|x|-\frac{\pi}{2}\right)^2} + 1 - e) & =: r(x) \qquad \forall\, x \in [-\pi, \pi] \end{cases}$$

Proof.
Of course it is sufficient to prove the first approximation.
The absolute value $|\varepsilon_r(x)|$ of the relative error of the approximation remains in $[0, 0.0035]$ for $0 \le x < \frac{\pi}{2}$ (see figures, graphs of $0.0035-|(p(x)-\cos(x))/\cos(x)|$ in $[0, \frac{\pi}{2}[$ and in $[1.56, \frac{\pi}{2}[$).

Fig. 1 [p]

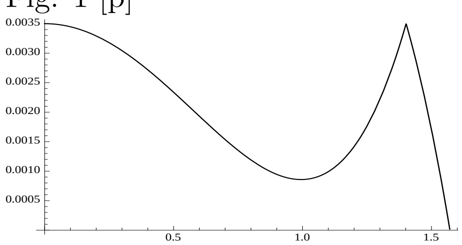
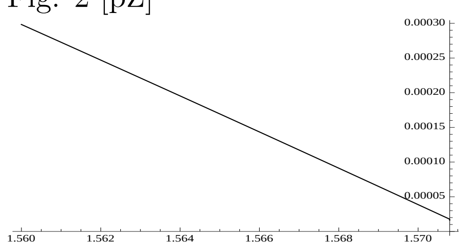


Fig. 2 [pZ]



Theorem 2. This approximations of $\arcsin(y)$ have relative error <0.35% (in absolute value):

$$c := \frac{4(1-\log(e-1))}{\pi^2} = -\frac{4\log\left(1-\frac{1}{e}\right)}{\pi^2} \simeq 0.185894 \qquad |\varepsilon_r(x)| < 0.35\%$$

$$\arcsin(y) \simeq \begin{cases} \frac{\pi}{2} - \sqrt{-\frac{1}{c}\log\left(1+\frac{y-1}{e}\right)} & =: f(y) \qquad \forall\, y \in [0,\, 1] \\ (\operatorname{sign} y)\left(\frac{\pi}{2} - \sqrt{-\frac{1}{c}\log\left(1+\frac{y-1}{e}\right)}\right) & =: g(y) \qquad \forall\, y \in [-1,\, 1] \end{cases}$$

Proof.
The absolute value $|\varepsilon_r(y)|$ of the relative error remains in $[0, 0.0035]$ for $0 <$

$y \leq 1$ (see figures, graphs of $0.0035 - |(f(y) - \arcsin y)/ \arcsin y|$ in $]0, 1]$ and in $]0, 0.01]$).

Fig. 3 [f]

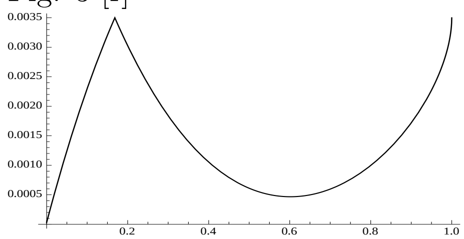
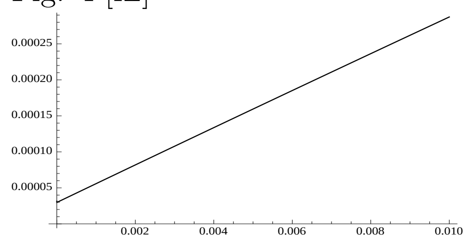


Fig. 4 [fZ]



Theorem 3. This approximations of $\arcsin(y)$ have relative error <0.53% (in absolute value):

$$c := \frac{4(1-\log(e-1))}{\pi^2} = -\frac{4\log\left(1-\frac{1}{e}\right)}{\pi^2} \simeq 0.185894 \qquad |\varepsilon_r(x)| < 0.53\%$$

$$\arccos(y) \simeq \sqrt{-\frac{1}{c}\log\left(1+\frac{y-1}{e}\right)} \qquad =: h(y) \qquad \forall\, y \in [0,\, 1]$$

Proof.
The absolute value $|\varepsilon_r(y)|$ of the relative error remains in $[0, 0.0035]$ for $0 \leq y < 1$ (see figures, graphs of $0.0053 - |(h(y) - \arccos(y))/ \arccos y|$ in $[0, 1[$ and in $[0.95, 1[$).

Fig. 5 [h]

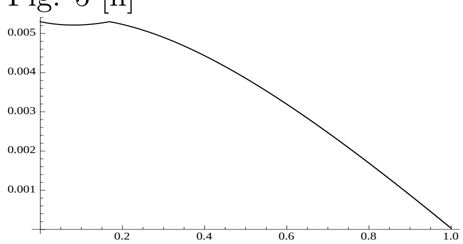
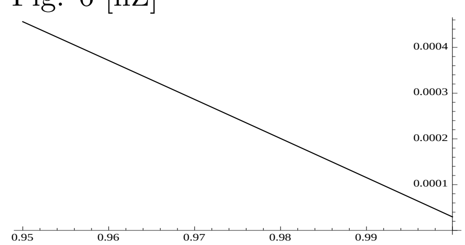


Fig. 6 [hZ]



Remarks. $\cos(x)$ is approximated in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ by Taylor-Mclaurin expansion $1-\frac{x^2}{2}$ with unbounded relative error; by $\frac{4}{\pi^2}\left(\frac{\pi^2}{4} - x^2\right)$ with relative error $> 27\%$ (in absolute value).

The relative error is not defined in 0 for sin and arcsin, and in $\frac{\pi}{2}$ for cos and arccos, but notice that in all those cases the absolute error is 0.

It is $h = \left(p_{|[0,\frac{\pi}{2}]}\right)^{-1}$ and $f = \left(q_{|[0,\frac{\pi}{2}]}\right)^{-1}$.