Corresponding Continuous Approximations of Gamma Function in [2,12] and of its Inverse in [1,11!] - A Draft

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Abstract

An approximation of the Gamma function in $[2, 12]$ at 1% is given. Its inverse is computed, and it is a continuous approximation in [1,11!] of the inverse of the increasing restriction of Gamma function; 11! is nearly 40 000 000. It is piecewise defined in the 10 intervals [1,2[, [2,6[, [6,24[,...[10!,11![by log and operations -, ×, / and power elevation.

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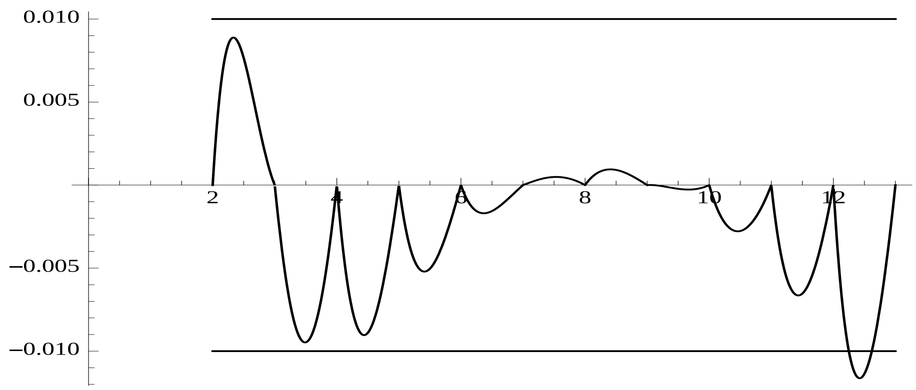
We are going to give 2 approximations, one the inverse of the other, of $\Gamma(x)$ on $[2, 12]$, and (more important) of the inverse of its restriction to $[1, 11!]$. By $\Gamma_{|}^{-1}(y)$ we denote the inverse of $\Gamma(x)$ restricted to $x \geq \bar{x}$, $\Gamma(\bar{x}) = \min \Gamma(x)$.

Theorem 1. This approximation $g(x)$ of $\Gamma(x)$ has relative error $<1\%$ (in absolute value) in $[2, 12]$:

$$\begin{array}{c} 2 \leq x \leq 12 \qquad c := \frac{19}{40} \qquad |\varepsilon_r(x)| < 1\% \\ \Gamma(x) \ \simeq \ g(x) \ := \ \lfloor x-1 \rfloor! + \frac{e^{cx^{1.5}} - e^{c\lfloor x \rfloor^{1.5}}}{e^{c\lfloor x+1 \rfloor^{1.5}} - e^{c\lfloor x \rfloor^{1.5}}} (\lfloor x \rfloor! - \lfloor x-1 \rfloor!) \end{array} \tag{1}$$

being $\lfloor\ .\ \rfloor$ the floor function (giving the largest integer less than or equal to x; for example $\lfloor \pi \rfloor = 3$, and $\lfloor 3 \rfloor = 3$). The approximation is a continuous function, and gives the exact value $(x-1)!$ for integers.

Proof. The absolute value $|\varepsilon_r(x)|$ of the relative error remains in $[-0.01, 0.01]$ for $2 \leq x \leq 12$ (see figure, graph of $|(g(x) - \Gamma(x))/\Gamma(x)|$ in $[2, 13]$).



An immediate computation gives $g(x) = (x-1)! = \Gamma(x)$ for integer $x > 1$.

To prove the continuity of $g(x)$, observe that $\lfloor x \rfloor$ is continuous in all non-integer points, is right-continuous in integer points, and for integer $n > 1$

$$\lim_{x \to n^-} \Gamma_1(x) \; = \; \lim_{x \to n^-} \lfloor x \rfloor! \; = \; (n-1)! \; = \; \Gamma(n)$$

because

$$\lim_{x \to n^-} \frac{e^{cx^{1.5}} - e^{c\lfloor x \rfloor^{1.5}}}{e^{c\lfloor x+1 \rfloor^{1.5}} - e^{c\lfloor x \rfloor^{1.5}}} = 1$$

Theorem 2. This piecewise-defined continuous function $f(y)$ gives the inverse of $g(x)$ (so approximating $\Gamma_{|}^{-1}(y)$) in $[1,\ 11!]$:

$$\boxed{\begin{array}{c} n := 2, 3, ..., 11 \qquad c := \frac{19}{40} \\ a_n := \frac{n! - (n-1)!}{e^{c(n+1)^{1.5}} - e^{cn^{1.5}}} \qquad b_n := (n-1)! - a_n e^{cn^{1.5}} \\ \forall y \in [(n-1)!, n![\qquad g^{-1}(y) \; = \; f(y) \; := \; \left(\frac{1}{c} \log \frac{y - b_n}{a_n} \right)^{\frac{2}{3}} \; \simeq \; \Gamma_{|}^{-1}(y) \end{array}} \tag{2}$$

Proof. With $c := \frac{19}{40}$, for integer $n > 1$ it is

$$\forall x \in [n, n+1[\qquad g(x) \; = \; (n-1)! + \frac{e^{cx^{1.5}} - e^{cn^{1.5}}}{e^{cx(n+1)^{1.5}} - e^{cn^{1.5}}} (n! - (n-1)!)$$

or

$$\forall x \in [n, n+1[\qquad g(x) = a_n \, e^{c\, x^{1.5}} + b_n$$

with a_n and b_n above defined. Then, observing that $\frac{1}{1.5} = \frac{2}{3}$, we conclude.

Notice that $x^{1.5}$ is $x\sqrt{x}$ and $t^{\frac{2}{3}}$ is $t/\sqrt[3]{t}$. And that $11! = 39\,916\,800 \simeq 4 \cdot 10^7$.

The function $\Gamma_{|}^{-1}(y)$ is required sometimes in Mathematics, though rarely (see for example [3] [4] [5]), and the topic recurrently arises in mathematical forums.

(Closing last interval) this is an explicit version of the approximation (2):

$$c := \tfrac{19}{40} \qquad a_n := \tfrac{n!-(n-1)!}{e^{c(n+1)^{1.5}} - e^{cn^{1.5}}} \qquad b_n := (n-1)! - a_n e^{cn^{1.5}}$$

$$f_n(y) := \left(\tfrac{1}{c}\log\tfrac{y-b_n}{a_n}\right)^{\frac{2}{3}}$$

$$\Gamma_{|}^{-1}(y) \;\simeq\; g^{-1}(y) \;=\; f(y) \;:=\; \begin{cases} f_2(y) & 1 \le y < 2 \\ f_3(y) & 2 \le y < 6 \\ f_4(y) & 6 \le y < 24 \\ f_5(y) & 24 \le y < 120 \\ f_6(y) & 120 \le y < 720 \\ f_7(y) & 720 \le y < 5040 \\ f_8(y) & 5040 \le y < 40320 \\ f_9(y) & 40320 \le y < 362880 \\ f_{10}(y) & 362880 \le y < 3628800 \\ f_{11}(y) & 3628800 \le y \le 39916800 \end{cases} \tag{3}$$

Considering 2 terms in classical [1] asymptotic expansion of $\Gamma(x)$ we have

$$\Gamma(x) \;\simeq\; e^{-x}\, x^{x-\frac{1}{2}}\, \sqrt{2\pi}\,\Big(1 + \frac{1}{12\,x}\Big) \tag{4}$$

which has, in the considered interval $[2, 12]$, a precision about 0.05%. So, if one is interested only in a simple approximation of $\Gamma(x)$, (4) is better than (1). But the function at the right side of Formula (4) is not explicitly invertible by means of elementary functions, even if we accept a piecewise definition.
If one is interested in $\Gamma(x)$ together with its inverse, or only in the inverse, (1), (2), (3) become meaningful, at least for $1 \le \Gamma(x) \le 11!$.

[2] gives this really good approximation of $\Gamma_{|}^{-1}(y)$ for $y \ge \Gamma(k) = \min \Gamma_{|x>0}$:

$$AIG(y) := \frac{L(y)}{W(L(y)/e)} + \frac{1}{2} \quad L(y) := \log\frac{y+c}{\sqrt{2\pi}} \quad c := \frac{\sqrt{2\pi}}{e} - \Gamma(k) \;\simeq\; 0.036534$$

being $W(y)$ the Lambert function, non-elementary function avoided by (3).

A further development could be a good determination of the precision of the approximation (2), (3), which at moment may be roughly estimated 0.7%.

References

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