Size estimates for fat inclusions in an isotropic Reissner-Mindlin plate *

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Abstract. In this paper we consider the inverse problem of determining, within an elastic isotropic thick plate modelled by the Reissner-Mindlin theory, the possible presence of an inclusion made of a different elastic material. Under some a priori assumptions on the inclusion, we deduce constructive upper and lower estimates of the area of the inclusion in terms of a scalar quantity related to the work developed in deforming the plate by applying simultaneously a couple field and a transverse force field at the boundary of the plate. The approach allows to consider plates with boundary of Lipschitz class.

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1 Introduction

The inverse problem of damage identification via non-destructive testing has attracted increasing interest in the applied and mathematical literature of the last years. Its applicability is particularly suited to those cases in which a simple visual inspection of the damaged system is not sufficient to conclude whether the defect is present or absent and, in the former case, how extended it is. Non-destructive tests in dynamic regime are rather common for large full-scale structures, such as bridges or buildings. However, in case of simple structural elements such as plates, the mechanical systems that will be considered in this paper, static tests are easily executable and can provide valuable information for solving the diagnostic problem.

In most of applications on plates, an accurate model describing the structural defect, such as diffuse cracking in reinforced concrete plates or yielding phenomena in metallic plates, is not a priori available. Therefore, the defected plate is usually modelled by introducing a variation of the elastic properties of the material in an unknown subregion D (*inclusion*) of the mid-surface Ω of the plate. Under the assumption that the reference undamaged configuration of the plate is known, the inverse problem is reduced to the determination of the inclusion D by comparing the results of boundary static tests executed on the reference specimen (with $D = \varnothing$) and on the possibly defected plate.

This appears to be a difficult inverse problem and a general uniqueness result has not been obtained yet. Partial answers have been given in the last ten years for thin elastic plates described by the Kirchhoff-Love theory by pursuing a relative modest, but realistic goal: to estimate the *area* of the unknown inclusion D from a single static experiment. More precisely, it was supposed to apply a given couple field $\widehat{M}$ at the boundary $\partial\Omega$ of the plate in the reference and in a possibly defected state, and to evaluate the work W_0, W exerted in deforming the undamaged and defected specimen, respectively. Constructive estimates, from above and from below, of $area(D)$ in terms of the difference $|W_0 - W|$ were determined for Kirchhoff-Love elastic plates when the background material is isotropic [MRV07] or belongs to a suitable class of anisotropy [DiCLMRVW13]. Extensions to the limit cases of rigid inclusions and cavities were also established [MRV13]. Analogous results were derived for size estimates of inclusions in shell structures (i.e., curved Kirchhoff-Love plates) [DiCLW13], [DiCLVW13]. For the sake of completeness we recall that the size estimates approach traces back to the paper by Friedman [Fri87] where, assuming that the measure of the possible inclusion in a conducting body is a-priori known, a criterion was given to decide from a single boundary measurement of current and corresponding voltage whether

the inclusion is present of not. Subsequently, the method has been developed in [AR98], [KSS97] and [ARS00], and extended also to the detection of inclusions in elastic bodies [Ik98], [AMR02a]. Finally, we mention an interesting approach to size estimates developed in [KKM12], [KM13] and in [MN12] where the translation method and the splitting method were introduced, respectively.

All the available size estimates results for plate-like systems have been obtained using the Kirchhoff-Love mechanical model of plate, that is assuming that the material fibre initially orthogonal to the mid-surface of the plate remains straight and perpendicular to the mid-surface during deformation. Experiments and numerical simulations show that this mechanical model accurately describes the behavior of thin plates, whereas it definitely looses precision as the thickness of the plate increases. Specifically, when the thickness reaches the order of one tenth the planar dimensions, the plates should be described by means of an extension of the Kirchhoff-Love model, namely the Reissner-Mindlin model [Rei45], [Min51], that takes into account also the shear deformations through the thickness of the plate. Moreover, it should be recalled that size estimates for the Kirchhoff-Love plate model were derived under the a priori condition that the mid-surface Ω is highly regular. This technical assumption obstructs, for example, the application of the size estimates to rectangular plates, in spite of their frequent use in practical applications. In this paper, both the two above mentioned limitations of the existing theory are removed, and the size estimates approach is extended to the Reissner-Mindlin model of plates with boundary $\partial\Omega$ of Lispchitz class.

Let us formulate our problem in mathematical terms. Let D, $D \subset\subset \Omega$, be the subdomain of the mid-surface Ω occupied by the inclusion, and denote by h the constant thickness of the plate. A transverse force field $\overline{Q}$ and a couple field $\overline{M}$ are supposed to be acting at the boundary $\partial\Omega$ of the plate. Working in the framework of the Reissner-Mindlin theory (see also [PPGT07]), at any point $x = (x_1, x_2) \in \Omega$, we denote by $w = w(x)$ and by $\omega_\alpha = \omega_\alpha(x)$, $\alpha = 1, 2$, the infinitesimal transverse displacement at x and the infinitesimal rotation of the transverse material fibre through x, respectively. The pair (φ, w), with $\varphi_1 = \omega_2$, $\varphi_2 = -\omega_1$, satisfies the Neumann boundary value problem

$$
\begin{cases}
\operatorname{div}\left((\chi_{\Omega\setminus D}S + \chi_D\widetilde{S})(\varphi + \nabla w)\right) = 0, & \text{in } \Omega, \quad (1.1) \\
\operatorname{div}\left((\chi_{\Omega\setminus D}\mathbb{P} + \chi_D\widetilde{\mathbb{P}})\nabla\varphi\right) - (\chi_{\Omega\setminus D}S + \chi_D\widetilde{S})(\varphi + \nabla w) = 0, & \text{in } \Omega, \quad (1.2) \\
(S(\varphi + \nabla w)) \cdot n = \overline{Q}, & \text{on } \partial\Omega, \quad (1.3) \\
(\mathbb{P}\nabla\varphi)n = \overline{M}, & \text{on } \partial\Omega, \quad (1.4)
\end{cases}
$$

where χ_A denotes the characteristic function of the set A and n is the unit outer normal to $\partial\Omega$. In the above equations, $(S, \mathbb{P})$ and $(\widetilde{S}, \widetilde{\mathbb{P}})$ are the second-

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order shearing tensor and the fourth-order bending tensor of the reference and defected plate, respectively. The work exerted by the boundary loads $(\overline{Q}, \overline{M})$ is denoted by

$$W = \int_{\partial\Omega} \overline{Q}w + \overline{M} \cdot \varphi. \tag{1.5}$$

When the inclusion D is absent, the equilibrium problem (1.1)–(1.4) becomes

$$\begin{cases} \operatorname{div}\left(S(\varphi_0 + \nabla w_0)\right) = 0, & \text{in } \Omega, & (1.6) \\ \operatorname{div}\left(\mathbb{P}\nabla\varphi_0\right) - S(\varphi_0 + \nabla w_0) = 0, & \text{in } \Omega, & (1.7) \\ \left(S(\varphi_0 + \nabla w_0)\right) \cdot n = \overline{Q}, & \text{on } \partial\Omega, & (1.8) \\ \left(\mathbb{P}\nabla\varphi_0\right)n = \overline{M}, & \text{on } \partial\Omega, & (1.9) \end{cases}$$

where (φ_0, w_0) is the deformation of the reference plate. The corresponding work exerted by the boundary loads is given by

$$W_0 = \int_{\partial\Omega} \overline{Q}w_0 + \overline{M} \cdot \varphi_0. \tag{1.10}$$

The first step towards the determination of the size estimates of the area of the inclusion consists in proving that the strain energy of the reference plate stored in the region D is comparable with the difference between the works exerted by the boundary load fields in deforming the plate with and without the inclusion. Under suitable assumptions on the jumps $(\widetilde{\mathbb{P}} - \mathbb{P})$ and $(\widetilde{S} - S)$ of the elastic coefficients between the defected region D and the surrounding background material, and using the ellipticity of the tensors S and $\mathbb{P}$, the above property can be stated as

$$K_1 \int_D |\widehat{\nabla}\varphi_0|^2 + |\varphi_0 + \nabla w_0|^2 \le |W - W_0| \le K_2 \int_D |\widehat{\nabla}\varphi_0|^2 + |\varphi_0 + \nabla w_0|^2, \tag{1.11}$$

for suitable positive constants K_1, K_2 only depending on the data. Here, $\widehat{\nabla}\varphi_0 = \frac{1}{2}(\nabla\varphi_0 + (\nabla\varphi_0)^T)$. We refer to Lemma 5.1 for the precise statement.

The lower bound for $area(D)$ follows from the right hand side of (1.11) and from regularity estimates for the solution (φ_0, w_0) to (1.6)–(1.9). It should be noticed that such regularity estimates hold true also for anisotropic background material, provided that the tensors $\mathbb{P}$ and S have suitable regularity.

In order to obtain the upper bound for $area(D)$, an estimate from below of the strain energy expression appearing on the left hand side of (1.11) is needed. This issue is rather technical and involves the determination of quantitative estimates of unique continuation for the strain energy of the solution (φ_0, w_0) to the reference plate problem.

In this paper we assume that the inclusion D satisfies the *fatness condition*

$$area\left(\{x \in D \mid dist(x, \partial D) > h_1\}\right) \geq \frac{1}{2} area(D), \qquad (1.12)$$

for a given positive number h_1. Under the assumption of isotropic material, and requiring suitable regularity of the tensors $\mathbb{P}$ and S, we shall prove a three spheres inequality for the strain energy density $(|\widehat{\nabla}\varphi_0|^2 + |\varphi_0 + \nabla w_0|^2)$ of the solution (φ_0, w_0) to (1.1)–(1.4), see Theorem 4.2. This three spheres inequality for the energy strongly relies on a three spheres inequality for $(|\varphi_0|^2 + |w_0|^2)$, with optimal exponent, and on a generalized Korn inequality, both derived in [MRV17]. Our main result (see Theorem 3.3) states that if, for a given $h_1 > 0$, the fatness-condition (1.12) holds, and some a priori assumptions on the unknown inclusion are satisfied, then

$$C_1 \left|\frac{W - W_0}{W_0}\right| \leq area(D) \leq C_2 \left|\frac{W - W_0}{W_0}\right|, \qquad (1.13)$$

where the constants C_1, C_2 only depend on the a priori data. Clearly, the lower bound for $area(D)$ in (1.13) continues to hold even if the inclusion D does not satisfy the fatness condition (1.12).

The paper is organized as follows. Section 2 collects some notation. The formulation of the inverse problem is provided in Section 3, together with our main result (Theorem 3.3). Section 4 contains quantitative estimates of unique continuation in the form of three spheres inequality (Theorem 4.2) and Lipschitz propagation of smallness property (Theorem 4.5) for the strain energy density of solutions to the Neumann problem for the reference plate. The proof of Theorem 3.3 is presented in Section 5, whereas Section 6 is devoted to the proof of Theorem 4.2.

2 Notation

Let $P = (x_1(P), x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius r and center P and by $R_{a,b}(P)$ the rectangle $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a, \ |x_2 - x_2(P)| < b\}$. To simplify the notation, we shall denote $B_r = B_r(O)$, $R_{a,b} = R_{a,b}(O)$.

Definition 2.1. ($C^{k,1}$ regularity) Let Ω be a bounded domain in $\mathbb{R}^2$. Given $k \in \mathbb{N}$, we say that a portion Σ of $\partial\Omega$ is of *class $C^{k,1}$ with constants ρ_0, $M_0 > 0$*, if, for any $P \in \Sigma$, there exists a rigid transformation of coordinates under which we have $P = O$ and

$$\Omega \cap R_{\rho_0, M_0\rho_0} = \{x = (x_1, x_2) \in R_{\rho_0, M_0\rho_0} \mid x_2 > \psi(x_1)\},$$

where ψ is a $C^{k,1}$ function on $(-\rho_0, \rho_0)$ satisfying

$$\psi(0) = 0,$$

$$\psi'(0) = 0, \quad \text{when } k \geq 1,$$

$$\|\psi\|_{C^{k,1}\left(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}\right)} \leq M_0 \rho_0.$$

When $k = 0$ we also say that Σ is of *Lipschitz class with constants* ρ_0, M_0.

Remark 2.2. We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous with the L^∞ norm and coincide with the standard definition when the dimensional parameter equals one, see [MRV07] for details.

For any $t > 0$ we denote

$$\Omega_t = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > t\}. \tag{2.1}$$

Given a bounded domain Ω in $\mathbb{R}^2$ such that $\partial\Omega$ is of class $C^{k,1}$, with $k \geq 0$, we consider as positive the orientation of the boundary induced by the outer unit normal n in the following sense. Given a point $P \in \partial\Omega$, let us denote by $\tau = \tau(P)$ the unit tangent at the boundary in P obtained by applying to n a counterclockwise rotation of angle $\frac{\pi}{2}$, that is $\tau = e_3 \times n$, where $\times$ denotes the vector product in $\mathbb{R}^3$, $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$ and $e_3 = e_1 \times e_2$.

We denote by $\mathbb{M}^2$ the space of 2×2 real valued matrices and by $\mathcal{L}(X, Y)$ the space of bounded linear operators between Banach spaces X and Y.

For every 2×2 matrices A, B and for every $\mathbb{L} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$, we use the following notation:

$$(\mathbb{L}A)_{ij} = L_{ijkl}A_{kl}, \tag{2.2}$$

$$A \cdot B = A_{ij}B_{ij}, \quad |A| = (A \cdot A)^{\frac{1}{2}}, \quad tr(A) = A_{ii}, \tag{2.3}$$

$$(A^T)_{ij} = A_{ji}, \quad \widehat{A} = \frac{1}{2}(A + A^T). \tag{2.4}$$

Notice that here and in the sequel summation over repeated indexes is implied.

3 The inverse problem

Let us consider a plate, with constant thickness h, represented by a bounded domain Ω in $\mathbb{R}^2$ having boundary of Lipschitz class, with constants ρ_0 and M_0, and satisfying

$$\text{diam}(\Omega) \leq M_1 \rho_0, \tag{3.1}$$

$$B_{s_0\rho_0}(x_0) \subset \Omega, \tag{3.2}$$

for some $M_1 > 0$, $s_0 > 0$ and $x_0 \in \Omega$. Moreover, we assume that for $r < h_0\rho_0$, where $h_0 > 0$ only depends on M_0, the domain

$$\Omega_r \text{ is of Lipschitz class with constants } \rho_0, M_0. \tag{3.3}$$

Condition (3.3) has been introduced to simplify the arguments. However, it should be noticed that it is a rather natural assumption, for instance trivially satisfied for polygonal plates.

The *reference* plate is assumed to be made by linearly elastic isotropic material with Lamé moduli λ and μ satisfying the ellipticity conditions

$$\mu(x) \geq \alpha_0, \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0, \quad \text{in } \overline{\Omega}, \tag{3.4}$$

for given positive constants α_0, γ_0, and the regularity condition

$$\|\lambda\|_{C^{0,1}(\overline{\Omega})} + \|\mu\|_{C^{0,1}(\overline{\Omega})} \leq \alpha_1, \tag{3.5}$$

where α_1 is a given constant. Therefore, the shearing and bending plate tensors take the form

$$SI_2, \quad S = h\mu, \quad S \in C^{0,1}(\overline{\Omega}), \tag{3.6}$$

$$\mathbb{P}A = B\left[(1-\nu)\widehat{A} + \nu tr(A)I_2\right], \quad \mathbb{P} \in C^{0,1}(\overline{\Omega}), \tag{3.7}$$

where I_2 is the two-dimensional unit matrix, A denotes a 2×2 matrix and

$$B = \frac{Eh^3}{12(1-\nu^2)}, \tag{3.8}$$

with Young's modulus E and Poisson's coefficient ν given by

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}, \quad \nu = \frac{\lambda}{2(\mu + \lambda)}. \tag{3.9}$$

By (3.4) and (3.5), we have

$$h\sigma_0 \leq S \leq h\sigma_1, \quad \text{in } \overline{\Omega}, \tag{3.10}$$

and

$$\frac{h^3}{12}\xi_0|\widehat{A}|^2 \leq \mathbb{P}A \cdot A \leq \frac{h^3}{12}\xi_1|\widehat{A}|^2, \quad \text{in } \overline{\Omega}, \tag{3.11}$$

for every 2×2 matrix A, where

$$\sigma_0 = \alpha_0, \quad \sigma_1 = \alpha_1, \quad \xi_0 = \min\{2\alpha_0, \gamma_0\}, \quad \xi_1 = 2\alpha_1. \tag{3.12}$$

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Moreover,

$$\|S\|_{C^{0,1}(\overline{\Omega})} \le h\alpha_1, \qquad \|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})} \le Ch^3, \tag{3.13}$$

with $C > 0$ only depending on α_0, α_1, γ_0.

Let the plate be subject to a transverse force field $\overline{Q}$ and a couple field $\overline{M}$ acting on the boundary $\partial\Omega$, and such that

$$\int_{\partial\Omega} \overline{Q} = 0, \quad \int_{\partial\Omega} (\overline{Q}x - \overline{M}) = 0, \tag{3.14}$$

$$\overline{Q} \in H^{-\frac{1}{2}}(\partial\Omega), \quad \overline{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2). \tag{3.15}$$

Under the above assumptions, the static equilibrium of the reference plate is described within the Reissner-Mindlin theory by the following Neumann boundary value problem

$$\begin{cases} \operatorname{div}(S(\varphi_0 + \nabla w_0)) = 0 & \text{in } \Omega, & (3.16) \\ \operatorname{div}(\mathbb{P}\nabla\varphi_0) - S(\varphi_0 + \nabla w_0) = 0, & \text{in } \Omega, & (3.17) \\ (S(\varphi_0 + \nabla w_0)) \cdot n = \overline{Q}, & \text{on } \partial\Omega, & (3.18) \\ (\mathbb{P}\nabla\varphi_0)n = \overline{M}, & \text{on } \partial\Omega. & (3.19) \end{cases}$$

Remark 3.1. It should be noticed that Reissner [Rei45] and Mindlin [Min51] theories are in fact similar, but different ones. The former was originally formulated within the static context only, whereas the latter was proposed to improve the dynamic response of the classical Kirchhoff-Love plate theory for sharp transients and for the eigenfrequencies of modes of vibration of high order. Interestingly, both the Reissner and Mindlin theories lead to the conclusion that three scalar boundary conditions are to be satisfied (e.g. equations (3.18)-(3.19) above) rather than the two of the Kirchhoff-Love plate theory.

Concerning the well-posedness of the Neumann problem for the Reissner-Mindlin plate model, it was proved in [MRV17] (Proposition 5.2) that the problem (3.16)–(3.19) admits a weak solution $(\varphi_0, w_0) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$, that is *for every $\psi \in H^1(\Omega, \mathbb{R}^2)$ and for every $v \in H^1(\Omega)$,*

$$\int_\Omega \mathbb{P}\nabla\varphi_0 \cdot \nabla\psi + \int_\Omega S(\varphi_0 + \nabla w_0) \cdot (\psi + \nabla v) = \int_{\partial\Omega} \overline{Q}v + \overline{M} \cdot \psi, \tag{3.20}$$

where $\int_{\partial\Omega} \overline{Q}v + \overline{M} \cdot \psi$ stays for the duality pairing $< \overline{Q}, v_{|\partial\Omega} >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} + < \overline{M}, \psi_{|\partial\Omega} >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$. The solution (φ_0, w_0) can be uniquely identified provided it satisfies the normalization conditions

$$\int_\Omega \varphi_0 = 0, \quad \int_\Omega w_0 = 0. \tag{3.21}$$

For this normalized solution, the following stability estimate holds

$$\|\varphi_0\|_{H^1(\Omega)} + \frac{1}{\rho_0}\|w_0\|_{H^1(\Omega)} \leq \frac{C}{\rho_0^2}\left(\|\overline{M}\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right), \quad (3.22)$$

where the constant $C > 0$ only depends on M_0, M_1, s_0, α_0, α_1, γ_0 and $\frac{\rho_0}{h}$.

Remark 3.2. Existence, uniqueness and H^1-stability for the Neumann problem (3.16)–(3.19) can be proved for generic anisotropic linearly elastic material with bounded shearing and bending plate tensors satisfying suitable ellipticity conditions, see Proposition 5.2 in [MRV17] for details. In fact, the additional hypotheses of isotropy and regularity we have required on the elastic coefficients are needed to obtain the key quantitative estimate of unique continuation of the solution (φ_0, w_0) in the form of the three spheres inequality (4.1).

The inclusion D is assumed to be a measurable, possibly disconnected subset of Ω satisfying

$$\text{dist}(D, \partial\Omega) \geq d_0\rho_0, \quad (3.23)$$

where d_0 is a positive constant. The shearing and bending tensors of the plate with the inclusion are denoted by $(\chi_{\Omega\setminus D}S+\chi_D\widetilde{S})$, $(\chi_{\Omega\setminus D}\mathbb{P}+\chi_D\widetilde{\mathbb{P}})$, where χ_D is the characteristic function of D and $\widetilde{S} \in L^\infty(\Omega, \mathbb{M}^2)$, $\widetilde{\mathbb{P}} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$. Differently from the surrounding material, no isotropy condition is introduced on the inclusion D, and the tensors $\widetilde{S}$, $\widetilde{\mathbb{P}}$ are requested to satisfy the following properties:

i) *Minor and major symmetry conditions*

$$\widetilde{S}_{\alpha\beta} = \widetilde{S}_{\beta\alpha}, \quad \alpha, \beta = 1, 2, \quad \text{a.e. in } \Omega, \quad (3.24)$$

$$\widetilde{P}_{\alpha\beta\gamma\delta} = \widetilde{P}_{\beta\alpha\gamma\delta} = \widetilde{P}_{\alpha\beta\delta\gamma} = \widetilde{P}_{\gamma\delta\alpha\beta}, \quad \alpha, \beta, \gamma, \delta = 1, 2, \quad \text{a.e. in } \Omega. \quad (3.25)$$

ii) *Bounds on the jumps $\widetilde{S} - S$, $\widetilde{\mathbb{P}} - \mathbb{P}$ and uniform strong convexity for $\widetilde{S}$ and $\widetilde{\mathbb{P}}$*

Either there exist $\eta > 0$ and $\delta > 1$ such that

$$\eta S \leq \widetilde{S} - S \leq (\delta - 1)S, \quad \text{a.e. in } \Omega, \quad (3.26)$$

$$\eta\mathbb{P} \leq \widetilde{\mathbb{P}} - \mathbb{P} \leq (\delta - 1)\mathbb{P}, \quad \text{a.e. in } \Omega, \quad (3.27)$$

or there exist $\eta > 0$ and $0 < \delta < 1$ such that

$$-(1 - \delta)S \leq \widetilde{S} - S \leq -\eta S, \quad \text{a.e. in } \Omega, \quad (3.28)$$

$$-(1 - \delta)\mathbb{P} \leq \widetilde{\mathbb{P}} - \mathbb{P} \leq -\eta\mathbb{P}, \quad \text{a.e. in } \Omega. \quad (3.29)$$

As a further a priori information, let $\mathcal{F} > 0$ be the following ratio of norms of the boundary data

$$\mathcal{F} = \frac{\|\overline{M}\|_{H^{-1/2}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-1/2}(\partial\Omega)}}{\|\overline{M}\|_{H^{-1}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-1}(\partial\Omega)}}. \tag{3.30}$$

Under the above assumptions, the equilibrium problem for the plate with the inclusion D is as follows

$$\begin{cases} \operatorname{div}\left((\chi_{\Omega\setminus D}S + \chi_D\widetilde{S})(\varphi + \nabla w)\right) = 0, & \text{in } \Omega, \text{(3.31)} \\ \operatorname{div}\left((\chi_{\Omega\setminus D}\mathbb{P} + \chi_D\widetilde{\mathbb{P}})\nabla\varphi\right) - (\chi_{\Omega\setminus D}S + \chi_D\widetilde{S})(\varphi + \nabla w) = 0, & \text{in } \Omega, \text{(3.32)} \\ (S(\varphi + \nabla w)) \cdot n = \overline{Q}, & \text{on } \partial\Omega, \text{(3.33)} \\ (\mathbb{P}\nabla\varphi)n = \overline{M}, & \text{on } \partial\Omega. \text{(3.34)} \end{cases}$$

Problem (3.31)–(3.34) has a unique solution $(\varphi, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ satisfying the normalization conditions (3.21).

Finally, we introduce the works exerted by the boundary loads when the inclusion is present or absent, respectively:

$$W = \int_{\partial\Omega} \overline{Q}w + \overline{M} \cdot \varphi, \tag{3.35}$$

$$W_0 = \int_{\partial\Omega} \overline{Q}w_0 + \overline{M} \cdot \varphi_0. \tag{3.36}$$

Let us recall that, according to (2.1),

$$D_t = \{x \in D \mid \operatorname{dist}(x, \partial D) > t\}$$

Our main theorem is as follows.

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^2$, such that $\partial\Omega$ is of $C^{0,1}$ class with constants ρ_0, M_0 and satisfying (3.1)–(3.3). Let D be a measurable subset of Ω satisfying (3.23) and*

$$|D_{h_1\rho_0}| \geq \frac{1}{2}|D|, \tag{3.37}$$

for a given positive constant h_1. Let the reference plate be made by linearly elastic isotropic material with Lamé moduli λ, μ satisfying (3.4), (3.5), and denote by S, $\mathbb{P}$ the corresponding shearing and bending tensors given in (3.6), (3.7), respectively. The shearing tensor $\widetilde{S} \in L^\infty(\Omega, \mathbb{M}^2)$ and the bending tensor $\widetilde{\mathbb{P}} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$ of the inclusion D are assumed to satisfy the symmetry conditions (3.24), (3.25).

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If (3.26) *and* (3.27) *hold, then we have*

$$\frac{1}{\delta - 1}C_1^+ \rho_0^2 \frac{W_0 - W}{W_0} \leq |D| \leq \frac{\delta}{\eta}C_2^+ \rho_0^2 \frac{W_0 - W}{W_0}. \tag{3.38}$$

If, conversely, (3.28) *and* (3.29) *hold, then we have*

$$\frac{\delta}{1 - \delta}C_1^- \rho_0^2 \frac{W - W_0}{W_0} \leq |D| \leq \frac{1}{\eta}C_2^- \rho_0^2 \frac{W - W_0}{W_0}, \tag{3.39}$$

where C_1^+, C_1^- *only depend on* M_0, M_1, s_0, $\frac{\rho_0}{h}$, d_0, α_0, α_1, γ_0, *whereas* C_2^+, C_2^- *only depend on* M_0, M_1, s_0, $\frac{\rho_0}{h}$, α_0, α_1, γ_0, h_1 *and* $\mathcal{F}$.

Remark 3.4. Let us highlight that the upper bounds in (3.38), (3.39) hold without assuming condition (3.23), that is the inclusion is allowed to touch the boundary of Ω. This will be clear from the proof of Theorem 3.3 given in Section 5.

Remark 3.5. The analytical procedure by which size estimates are found is indeed constructive, but, in practice, is likely to lead to rather pessimistic evaluations of the constants $C_1^\pm$, $C_2^\pm$. For this reason, it is interesting and useful for concrete applications to obtain realistic estimates of such constants. A detailed investigation attempting to estimate these constants by numerical simulations is currently under preparation and will be the object of a forthcoming paper. In the sequel, we shall consider some special cases for which the exact solution to (3.16)–(3.19) is available and one can find theoretical upper and lower bounds to the size of the inclusion D.

More precisely, we consider a rectangular plate $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | \ 0 < x_1 < a, \ 0 < x_2 < b\}$, with constant thickness h, made of isotropic elastic material with constant Lamè moduli λ, μ satisfying (3.4), and positive Poisson coefficient ν. To simplify the notation, let us denote by $\ell_{\{x_1=0\}}$, $\ell_{\{x_1=a\}}$ the two sides of Ω belonging to the straight lines $x_1 = 0$ and $x_1 = a$, respectively. Similarly, $\ell_{\{x_2=0\}}$, $\ell_{\{x_2=b\}}$ are the other two sides of Ω. The transverse force field $\overline{Q}$ at the boundary $\partial\Omega$ is assumed to be absent, whereas the couple field $\overline{M}$ is given as follows:

Case 1) $\overline{M} = -Me_2$ on $\ell_{\{x_1=a\}}$, $\overline{M} = Me_2$ on $\ell_{\{x_1=0\}}$, $\overline{M} = 0$ on the sides $\ell_{\{x_2=0\}}$, $\ell_{\{x_2=b\}}$;

Case 2) $\overline{M} = -Me_2$ on $\ell_{\{x_1=a\}}$, $\overline{M} = Me_2$ on $\ell_{\{x_1=0\}}$, $\overline{M} = Me_1$ on $\ell_{\{x_2=b\}}$, $\overline{M} = -Me_1$ on $\ell_{\{x_2=0\}}$,

where M is a non vanishing constant. These kinds of loads are rather special, but they are easy to realize in experiments and are commonly employed in non-destructive testing for the characterization of plate-like specimens. For

the two cases above, we can compute the exact solution (φ_0, w_0) to (3.16)–(3.19), obtaining:

$$\text{Case 1)} \quad \widehat{\nabla} \varphi_0 = \frac{M}{B(1 - \nu^2)} \begin{bmatrix} -1 & 0 \\ 0 & \nu \end{bmatrix}, \quad \varphi_0 + \nabla w_0 = 0 \quad \text{in } \overline{\Omega}; \quad (3.40)$$

$$\text{Case 2)} \quad \widehat{\nabla} \varphi_0 = \frac{M}{B(1 + \nu)} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \varphi_0 + \nabla w_0 = 0 \quad \text{in } \overline{\Omega}. \quad (3.41)$$

We assume that the inclusion $D \subset \Omega$ is made by isotropic elastic material with plate tensor

$$\widetilde{\mathbb{P}} = f \mathbb{P}, \quad (3.42)$$

where the *stiffness ratio* f is a positive constant. We notice that $0 < f < 1$ and $f > 1$ correspond to the case of *softer* inclusion and *harder* inclusion, respectively. In the case of softer inclusion, the size estimates (3.39) can be written as

$$C_1^- \frac{W - W_0}{W_0} \leq \frac{|D|}{|\Omega|} \leq C_2^- \frac{W - W_0}{W_0}, \quad (3.43)$$

where the constants C_1^-, C_2^- are given by

$$\text{Case 1)} \quad C_1^- = \frac{f}{1 - f} \cdot \frac{1 - \nu}{1 + \nu^2}, \quad C_2^- = \frac{1}{1 - f} \cdot \frac{1 + \nu}{1 + \nu^2}, \quad (3.44)$$

$$\text{Case 2)} \quad C_1^- = \frac{f}{1 - f}, \quad C_2^- = \frac{1}{1 - f} \cdot \frac{1 + \nu}{1 - \nu}, \quad (3.45)$$

and, in both cases,

$$\frac{C_2^-}{C_1^-} = \frac{1}{f} \cdot \frac{1 + \nu}{1 - \nu}. \quad (3.46)$$

When the inclusion is harder, we have

$$C_1^+ \frac{W_0 - W}{W_0} \leq \frac{|D|}{|\Omega|} \leq C_2^+ \frac{W_0 - W}{W_0}, \quad (3.47)$$

with

$$\text{Case 1)} \quad C_1^+ = \frac{1}{f - 1} \cdot \frac{1 - \nu}{1 + \nu^2}, \quad C_2^+ = \frac{f}{f - 1} \cdot \frac{1 + \nu}{1 + \nu^2}, \quad (3.48)$$

$$\text{Case 2)} \quad C_1^+ = \frac{1}{f - 1}, \quad C_2^+ = \frac{f}{f - 1} \cdot \frac{1 + \nu}{1 - \nu}, \quad (3.49)$$

and

$$\frac{C_2^+}{C_1^+} = f \cdot \frac{1+\nu}{1-\nu}. \tag{3.50}$$

As an example, if we assume $\nu = 0.3$ (Poisson coefficient typical of a mild steel) and $f = \frac{1}{10}$, then

$$\frac{C_2^-}{C_1^-} \simeq 18.5714. \tag{3.51}$$

This last calculation shows that the theoretical estimates may be rather pessimistic, since the angular sector determined in the cartesian plane with coordinates $\left(\frac{|W-W_0|}{W_0}, \frac{|D|}{|\Omega|}\right)$ may be very large. Based on previous results on two and three-dimensional linear elasticity [ABFMRT07], it is expected that the size estimates can improve significantly when the constants C_1, C_2 are evaluated numerically. This is the object of ongoing research.

Finally, the above calculations show that the size estimate from below degenerates both as $f \to 0^+$ and $f \to +\infty$. These two limit cases, e.g., cavities ($f = 0$) and rigid inclusions ($f = +\infty$), need a specific treatment and cannot simply inferred as limit of the present theory, see [MRV13] for analogous results in the Kirchhoff-Love model of thin plate.

4 Unique continuation estimates

The key quantitative estimate of unique continuation for the Reissner-Mindlin reference plate is the following three spheres inequality, which was obtained in [MRV17, Theorem 7.1].

Theorem 4.1. *Under the assumptions made in Section 3, let* $(\varphi_0, w_0) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ *be the solution to problem* (3.16)–(3.19) *normalized by conditions* (3.21). *Let* $\bar{x} \in \Omega$ *and* $R_1 > 0$ *be such that* $B_{R_1}(\bar{x}) \subset \Omega$. *Then there exists* $\theta \in (0, 1)$, θ *depending on* $\alpha_0, \alpha_1, \gamma_0, \frac{\rho_0}{h}$ *only, such that if* $0 < R_3 < R_2 < R_1$ *and* $\frac{R_3}{R_1} \leq \frac{R_2}{R_1} \leq \theta$, *then we have*

$$\int_{B_{R_2}(\bar{x})} |V|^2 \leq C \left(\int_{B_{R_3}(\bar{x})} |V|^2\right)^\tau \left(\int_{B_{R_1}(\bar{x})} |V|^2\right)^{1-\tau} \tag{4.1}$$

where

$$|V|^2 = |\varphi_0|^2 + \frac{1}{\rho_0^2}|w_0|^2, \tag{4.2}$$

$\tau \in (0, 1)$ *depends on* $\alpha_0, \alpha_1, \gamma_0, \frac{R_3}{R_1}, \frac{R_2}{R_1}, \frac{\rho_0}{h}$ *only and* C *depends on* $\alpha_0, \alpha_1, \gamma_0, \frac{R_2}{R_1}, \frac{\rho_0}{h}$ *only.*

In order to obtain the size estimates we need an estimate analogous to (4.1) for the strain energy density

$$E(\varphi_0, w_0) = \left(|\widehat{\nabla}\varphi_0|^2 + \frac{1}{\rho_0^2}|\varphi_0 + \nabla w_0|^2 \right)^{\frac{1}{2}}. \qquad (4.3)$$

Theorem 4.2. *Under the assumptions made in Section 3, let $(\varphi_0, w_0) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ be the solution to problem (3.16)–(3.19) normalized by conditions (3.21). There exist $\theta \in (0,1)$, $\tau \in (0,1)$, $C > 0$ only depending on α_0, α_1, γ_0, $\frac{\rho_0}{h}$, such that for every $\rho \in (0, \rho_0)$ and for every $\bar{x} \in \Omega$ such that $dist(\bar{x}, \partial\Omega) \geq \frac{7}{2\theta}\rho$, we have*

$$\int_{B_{3\rho}(\bar{x})} E^2(\varphi_0, w_0) \leq C \left(\frac{\rho_0}{\rho}\right)^2 \left(\int_{B_\rho(\bar{x})} E^2(\varphi_0, w_0)\right)^\tau \left(\int_{B_{\frac{7}{2\theta}\rho}(\bar{x})} E^2(\varphi_0, w_0)\right)^{1-\tau}. \qquad (4.4)$$

The main tool used to derive inequality (4.4) from inequality (4.1) is the following Korn's inequality of constructive type, which was established in [MRV17, Theorem 4.3].

Theorem 4.3 (Generalized second Korn inequality)**.** *Let Ω be a bounded domain in $\mathbb{R}^2$, with boundary of Lipschitz class with constants ρ_0, M_0, satisfying (3.1), (3.2). There exists a positive constant C only depending on M_0, M_1 and s_0, such that, for every $\varphi \in H^1(\Omega, \mathbb{R}^2)$ and for every $w \in H^1(\Omega, \mathbb{R})$,*

$$\|\nabla\varphi\|_{L^2(\Omega)} \leq C \left(\|\widehat{\nabla}\varphi\|_{L^2(\Omega)} + \frac{1}{\rho_0}\|\varphi + \nabla w\|_{L^2(\Omega)} \right). \qquad (4.5)$$

It is also convenient to recall the following Poincaré inequalities.

Proposition 4.4 (Poincaré inequalities)**.** *Let Ω be a bounded domain in $\mathbb{R}^2$, with boundary of Lipschitz class with constants ρ_0, M_0, satisfying (3.1). There exists a positive constant C_P only depending on M_0 and M_1, such that for every $u \in H^1(\Omega, \mathbb{R}^n)$, $n = 1, 2$,*

$$\|u - u_\Omega\|_{L^2(\Omega)} \leq C_P \rho_0 \|\nabla u\|_{L^2(\Omega)}, \qquad (4.6)$$

$$\|u - u_G\|_{H^1(\Omega)} \leq \left(1 + \left(\frac{|\Omega|}{|G|}\right)^{\frac{1}{2}}\right) \sqrt{1 + C_P^2} \, \rho_0 \|\nabla u\|_{L^2(\Omega)}, \qquad (4.7)$$

where G, $G \subseteq \Omega$, is any measurable subset of Ω with positive measure and $u_G = \frac{1}{|G|}\int_G u$.

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We refer to [AMR08, Example 3.5] and also [AMR02b] for a quantitative evaluation of the constant C_P.

Proof of Theorem 4.2. Let us apply Theorem 4.1 to the solution (φ^*, w^*) to (3.16)–(3.19), where

$$\varphi^* = \varphi_0 - c_\rho, \quad w^* = w_0 + c_\rho \cdot (x - \bar{x}) - d_\rho, \qquad (4.8)$$

with

$$c_\rho = \frac{1}{|B_\rho|} \int_{B_\rho(\bar{x})} \varphi_0, \quad d_\rho = \frac{1}{|B_\rho|} \int_{B_\rho(\bar{x})} w_0, \qquad (4.9)$$

$$R_3 = \rho, \quad R_2 = \frac{7}{2}\rho, \quad R_1 = \frac{7}{2\theta}\rho. \qquad (4.10)$$

Since $\varphi^* + \nabla w^* = \varphi_0 + \nabla w_0$ and $\nabla \varphi^* = \nabla \varphi_0$, we have

$$\int_{B_{\frac{7}{2}\rho}(\bar{x})} |\varphi^*|^2 + \frac{1}{\rho_0^2}|w^*|^2 \le C \left(\int_{B_\rho(\bar{x})} |\varphi^*|^2 + \frac{1}{\rho_0^2}|w^*|^2 \right)^\tau \left(\int_{B_{\frac{7}{2\theta}\rho}(\bar{x})} |\varphi^*|^2 + \frac{1}{\rho_0^2}|w^*|^2 \right)^{1-\tau},$$
(4.11)

where $\tau \in (0,1)$, $C > 0$ only depend on α_0, α_1, γ_0 and $\frac{\rho_0}{h}$.

By applying Poincaré inequality (4.6) to the functions w^* and φ^* and Korn inequality (4.5) to φ^* in the domain $B_\rho(\bar{x})$ where these functions have zero mean value, we have

$$\int_{B_\rho(\bar{x})} |\varphi^*|^2 + \frac{1}{\rho_0^2}|w^*|^2 \le C \int_{B_\rho(\bar{x})} |\varphi^*|^2 + \frac{\rho^2}{\rho_0^2}|\nabla w_0 + c_\rho|^2 \le$$

$$\le C \int_{B_\rho(\bar{x})} |\varphi^*|^2 + \frac{\rho^2}{\rho_0^2}|\nabla w_0 + \varphi_0|^2 + \frac{\rho^2}{\rho_0^2}|\varphi_0 - c_\rho|^2 \le$$

$$\le C \int_{B_\rho(\bar{x})} \left(1 + \frac{\rho^2}{\rho_0^2}\right) |\varphi^*|^2 + \frac{\rho^2}{\rho_0^2}|\nabla w_0 + \varphi_0|^2 \le$$

$$\le C \int_{B_\rho(\bar{x})} \rho^2 \left(1 + \frac{\rho^2}{\rho_0^2}\right) |\nabla \varphi^*|^2 + \frac{\rho^2}{\rho_0^2}|\nabla w_0 + \varphi_0|^2 \le$$

$$\le C \int_{B_\rho(\bar{x})} \rho^2 \left(1 + \frac{\rho^2}{\rho_0^2}\right) |\widehat{\nabla} \varphi_0|^2 + \left(1 + \frac{\rho^2}{\rho_0^2}\right) |\nabla w_0 + \varphi_0|^2 \le$$

$$\le C\rho_0^2 \left(\int_{B_\rho(\bar{x})} |\widehat{\nabla} \varphi_0|^2 + \frac{1}{\rho_0^2}|\nabla w_0 + \varphi_0|^2 \right), \quad (4.12)$$

with C an absolute constant.

Similarly, we can estimate the integral over $B_{\frac{7}{2\theta}\rho}(\bar{x})$ by using Poincaré inequality (4.7) with $G = B_\rho(\bar{x})$, $\Omega = B_{\frac{7}{2\theta}\rho}(\bar{x})$, obtaining

$$\int_{B_{\frac{7}{2\theta}\rho}(\bar{x})} |\varphi^*|^2 + \frac{1}{\rho_0^2}|w^*|^2 \leq C\rho_0^2 \left(\int_{B_{\frac{7}{2\theta}\rho}(\bar{x})} |\widehat{\nabla}\varphi_0|^2 + \frac{1}{\rho_0^2}|\nabla w_0 + \varphi_0|^2 \right), \quad (4.13)$$

with $C > 0$ only depending on α_0, α_1, γ_0, $\frac{\rho_0}{h}$.

Next we need to derive a suitable Caccioppoli type inequality, see [C51] for the classical version for elliptic equations and [Gi83, Proposition 2.1] for a recent reference for elliptic systems. To this aim, let us consider a function $\eta \in C_0^\infty(\mathbb{R}^2)$, having compact support contained in $B_{\frac{7}{2}\rho}(\bar{x})$, satisfying $\eta \equiv 1$ in $B_{3\rho}(\bar{x})$, $\eta \geq 0$, $|\nabla\eta| \leq \frac{C}{\rho}$, $C > 0$ being an absolute constant. Inserting in the weak formulation (3.20) the test functions $\psi = \eta^2\varphi^*$, $v = \eta^2 w^*$, we have

$$\int_{B_{\frac{7}{2}\rho}(\bar{x})} \eta^2 \mathbb{P}\widehat{\nabla}\varphi^* \cdot \widehat{\nabla}\varphi^* + S(\varphi^* + \nabla w^*) \cdot (\eta^2(\varphi^* + \nabla w^*)) \leq$$

$$\leq C \int_{B_{\frac{7}{2}\rho}(\bar{x})} \left(h^{\frac{3}{2}}|\nabla\eta||\varphi^*| \right) \left(h^{\frac{3}{2}}\eta|\widehat{\nabla}\varphi^*| \right) + \left(h^{\frac{1}{2}}\eta|\varphi^* + \nabla w^*| \right) \left(h^{\frac{1}{2}}|\nabla\eta||w^*| \right),$$

$$(4.14)$$

where C is an absolute constant.

By applying the ellipticity assumptions (3.10), (3.11) and by using the standard inequality $2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$, $\epsilon > 0$, we have

$$\int_{B_{\frac{7}{2}\rho}(\bar{x})} \eta^2 h^3|\widehat{\nabla}\varphi^*|^2 + h\eta^2|\varphi^* + \nabla w^*|^2 \leq$$

$$\leq C\epsilon \int_{B_{\frac{7}{2}\rho}(\bar{x})} \eta^2 h^3|\widehat{\nabla}\varphi^*|^2 + h\eta^2|\varphi^* + \nabla w^*|^2 + \frac{C}{\epsilon\rho^2} \int_{B_{\frac{7}{2}\rho}(\bar{x})} h^3|\varphi^*|^2 + h|w^*|^2,$$

$$(4.15)$$

with C only depending on α_0, α_1, γ_0. For a suitable value of ϵ, only depending on α_0, α_1, γ_0, we obtain the following Caccioppoli type inequality

$$\int_{B_{3\rho}(\bar{x})} |\widehat{\nabla}\varphi_0|^2 + \frac{1}{\rho_0^2}|\varphi_0 + \nabla w_0|^2 \leq \frac{C}{\rho^2} \int_{B_{\frac{7}{2}\rho}(\bar{x})} |\varphi^*|^2 + \frac{1}{\rho_0^2}|w^*|^2, \quad (4.16)$$

with C only depending on α_0, α_1, γ_0 and $\frac{\rho_0}{h}$. By (4.11), (4.12), (4.13), (4.16), the thesis follows. $\qquad\square$

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Finally, the last mathematical tool of quantitative unique continuation is the following result, whose proof is deferred in Section 6.

Theorem 4.5 (Lipschitz propagation of smallness)**.** *Under the assumptions made in Section 3, for every $\rho > 0$ and for every $x \in \Omega_{\frac{7}{2\theta}\rho}$, we have*

$$\int_{B_\rho(x)} E^2(\varphi_0, w_0) \geq C_\rho \int_\Omega E^2(\varphi_0, w_0), \tag{4.17}$$

where C_ρ only depends on α_0, α_1, γ_0, $\frac{\rho_0}{h}$, M_0, M_1, s_0, $\mathcal{F}$ and ρ, and $\theta \in (0, 1)$ has been introduced in Theorem 4.1, θ depending on $\alpha_0, \alpha_1, \gamma_0, \frac{\rho_0}{h}$ only.

5 Proof of Theorem 3.3

The basic result connecting the presence of an inclusion to the difference of the works corresponding to problems (3.16)–(3.19) and (3.31)–(3.34) is the following Lemma.

Lemma 5.1 (Energy Lemma)**.** *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of Lipschitz class. Let $S, \widetilde{S} \in L^\infty(\Omega, \mathbb{M}^2)$ satisfy (3.24) and $\mathbb{P}, \widetilde{\mathbb{P}} \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$ satisfy (3.25). Let us assume that the jumps $(\widetilde{S} - S)$ and $(\widetilde{\mathbb{P}} - \mathbb{P})$ satisfy either (3.26)–(3.27) or (3.28)–(3.29). Let (φ_0, w_0), $(\varphi, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ be the weak solutions to problems (3.16)–(3.19), (3.31)–(3.34), respectively.*
    If (3.26)–(3.27) hold, then we have

$$\frac{\eta}{\delta} \int_D \frac{h^3}{12} \xi_0 |\widehat{\nabla}\varphi_0|^2 + h\sigma_0 |\varphi_0 + \nabla w_0|^2 \leq \int_{\partial\Omega} \overline{Q}(w_0 - w) + \overline{M} \cdot (\varphi_0 - \varphi) \leq$$

$$\leq (\delta - 1) \int_D \frac{h^3}{12} \xi_1 |\widehat{\nabla}\varphi_0|^2 + h\sigma_1 |\varphi_0 + \nabla w_0|^2. \tag{5.1}$$

If (3.28)–(3.29) hold, then we have

$$\eta \int_D \frac{h^3}{12} \xi_0 |\widehat{\nabla}\varphi_0|^2 + h\sigma_0 |\varphi_0 + \nabla w_0|^2 \leq \int_{\partial\Omega} \overline{Q}(w_0 - w) + \overline{M} \cdot (\varphi_0 - \varphi) \leq$$

$$\leq \frac{1 - \delta}{\delta} \int_D \frac{h^3}{12} \xi_1 |\widehat{\nabla}\varphi_0|^2 + h\sigma_1 |\varphi_0 + \nabla w_0|^2. \tag{5.2}$$

Proof of Theorem 3.3. Let us notice that by (4.5) and (4.6), and by the trivial estimate $\|\nabla w_0\|_{L^2(\Omega)} \leq \|\varphi_0 + \nabla w_0\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Omega)}$, we have

$$\|\varphi_0\|_{H^1(\Omega)} + \frac{1}{\rho_0}\|w_0\|_{H^1(\Omega)} \leq \frac{C}{\rho_0}\left(\|\widehat{\nabla}\varphi_0\|_{L^2(\Omega)} + \frac{1}{\rho_0}\|\varphi_0 + \nabla w_0\|_{L^2(\Omega)}\right) \leq$$

$$\leq C\left(\int_\Omega E^2(\varphi_0, w_0)\right)^{\frac{1}{2}}, \quad (5.3)$$

with C only depending on M_0, M_1, s_0.

By standard regularity estimates for elliptic systems (see [Cam80, Theorem 6.1]), by (5.3) and by the weak formulation of the Neumann problem (3.16)–(3.19), we have

$$\|\varphi_0\|_{L^\infty(D)} + \rho_0\|\widehat{\nabla}\varphi_0\|_{L^\infty(D)} + \|\nabla w_0\|_{L^\infty(D)} \leq C\left(\|\varphi_0\|_{H^1(\Omega)} + \frac{1}{\rho_0}\|w_0\|_{H^1(\Omega)}\right) \leq$$

$$\leq C\left(\int_\Omega E^2(\varphi_0, w_0)\right)^{\frac{1}{2}} \leq \frac{C}{\rho_0^{\frac{3}{2}}}\left(\int_{\partial\Omega}\overline{Q}w_0 + \overline{M}\cdot\varphi_0\right)^{\frac{1}{2}}, \quad (5.4)$$

where the constant C depends only on M_0, M_1, s_0, α_0, α_1, γ_0, $\frac{\rho_0}{h}$, d_0.

The lower bound for $|D|$ in (3.38), (3.39) follows from the right hand side of (5.1), (5.2) and from (5.4).

Now, let us prove the upper bound for $|D|$ in (3.38), (3.39). Note that

$$\int_D \frac{h^3}{12}\xi_0|\widehat{\nabla}\varphi_0|^2 + h\sigma_0|\varphi_0 + \nabla w_0|^2 \geq C\int_D E^2(\varphi_0, w_0), \quad (5.5)$$

with C only depending on α_0, γ_0, $\frac{\rho_0}{h}$.

Let us cover D_{h_1} with internally non overlapping closed squares Q_j of side l, for $j = 1, ..., J$, with $l = \frac{4\theta h_1}{2\sqrt{2}\theta + 7}$, where $\theta \in (0, 1)$ is as in Theorem 4.5. By the choice of l the squares Q_j are contained in D. Hence

$$\int_D E^2(\varphi_0, w_0) \geq \int_{\bigcup_{j=1}^J Q_j} E^2(\varphi_0, w_0) \geq \frac{|D_{h_1}|}{l^n}\int_{Q_{\bar{j}}} E^2(\varphi_0, w_0), \quad (5.6)$$

where $\bar{j}$ is such that $\int_{Q_{\bar{j}}} E^2(\varphi_0, w_0) = \min_j \int_{Q_j} E^2(\varphi_0, w_0)$. Let $\bar{x}$ be the center of $Q_{\bar{j}}$. From (5.5), (5.6), estimate (4.17) with $x = \bar{x}$ and $\rho = l/2$, (3.10), (3.11) and from the weak formulation of (3.16)–(3.19) we have

$$\int_D \frac{h^3}{12}\xi_0|\widehat{\nabla}\varphi_0|^2 + h\sigma_0|\varphi_0 + \nabla w_0|^2 \geq K|D|W_0, \quad (5.7)$$

where K depends only on α_0, α_1, γ_0, M_0, M_1, s_0, $\frac{\rho_0}{h}$, h_1 and $\mathcal{F}$.

The upper bound for $|D|$ in (3.38), (3.39) follows from the left hand side of (5.1),(5.2) and from (5.7). $\qquad\square$

6   Proof of Theorem 4.5

Let us premise the following Proposition.

Proposition 6.1. *Let Ω be a bounded domain in $\mathbb{R}^2$, with boundary of Lipschitz class with constants ρ_0, M_0, satisfying (3.1). Let $S \in C^{0,1}(\overline{\Omega}, \mathbb{M}^2)$ and $\mathbb{P} \in C^{0,1}(\overline{\Omega}, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$ given by (3.6), (3.7) with the Lamé moduli satisfying (3.4), (3.5). Let $\overline{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$ and $\overline{Q} \in H^{-\frac{1}{2}}(\partial\Omega)$ satisfy the compatibility conditions (3.14). Let $(\varphi_0, w_0) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ be the solution of the problem (3.16)–(3.19), normalized by the conditions (3.21). Then there exists a positive constant C only depending on M_0, M_1, α_0, α_1, γ_0, $\frac{\rho_0}{h}$, such that*

$$\|\overline{M}\|_{H^{-1}(\partial\Omega,\mathbb{R}^2)} + \rho_0\|\overline{Q}\|_{H^{-1}(\partial\Omega)} \leq C\rho_0^2 \left(\|\varphi_0\|_{L^2(\partial\Omega,\mathbb{R}^2)} + \frac{1}{\rho_0}\|w_0\|_{L^2(\partial\Omega)} \right). \tag{6.1}$$

Remark 6.2. Let us highlight that the above Proposition, as well as Lemma 6.3, on which its proof is based, hold true for anisotropic materials.

Proof of Theorem 4.5. By Proposition 5.5 in [ARRV09] and by (3.3), there exists $h_2 > 0$ only depending on M_0 such that $\Omega_{\frac{4}{\theta}\rho}$ is connected and of Lipschitz class with constant ρ_0, M_0, for every $\rho \leq \frac{\theta}{4}h_2\rho_0$. Let $\rho \leq \frac{\theta}{4}h_2\rho_0$.

Given any point $y \in \Omega_{\frac{4}{\theta}\rho}$, let γ be an arc in $\Omega_{\frac{4}{\theta}\rho}$ joining x and y. Let us define the points $\{x_i\}$, $i = 1, ..., L$, as follows: $x_1 = x$, $x_{i+1} = \gamma(t_i)$, where $t_i = \max\{t \text{ s.t. } |\gamma(t) - x_i| = 2\rho\}$ if $|x_i - y| > 2\rho$, otherwise let $i = L$ and stop the process. By construction, the disks $B_\rho(x_i)$ are pairwise disjoint and $|x_{i+1} - x_i| = 2\rho$, $i = 1, ..., L - 1$, $|x_L - y| \leq 2\rho$.

By applying Theorem 4.2 and denoting $E(\varphi_0, w_0) = E$ to simplify the notation, we have

$$\int_{B_\rho(x_{i+1})} E^2 \leq C \left(\frac{\rho_0}{\rho}\right)^2 \left(\int_{B_\rho(x_i)} E^2\right)^\tau \left(\int_{B_{\frac{7}{2\theta}\rho}(x_i)} E^2\right)^{1-\tau}, \tag{6.2}$$

for $i = 1, ..., L - 1$, where $\tau \in (0, 1)$ and $C > 0$ only depend on α_0, α_1, γ_0 and $\frac{\rho_0}{h}$.

Let us apply the Caccioppoli inequality (4.16) to estimate from above the second integral on the right hand side of (6.2), namely

$$\int_{B_{\frac{7}{2\theta}\rho}(x_i)} E^2 \leq \frac{C}{\rho^2}\int_{B_{\frac{4}{\theta}\rho}(x_i)} |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2 \leq \frac{C}{\rho^2}\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2, \tag{6.3}$$

where $\varphi_0^* = \varphi_0 - c$, $w_0^* = w_0 + c \cdot (x - \overline{x}) - d$, with $c \in \mathbb{R}^2$, $d \in \mathbb{R}$, $\overline{x} \in \mathbb{R}^2$ to be chosen later, and where $C > 0$ only depends on α_0, α_1, γ_0 and $\frac{\rho_0}{h}$.

By (6.2) and (6.3), and using an iteration argument, we have

$$\frac{\rho^2 \int_{B_\rho(y)} E^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2} \leq C \left(\frac{\rho_0}{\rho}\right)^2 \left(\frac{\rho^2 \int_{B_\rho(x)} E^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2}\right)^{\tau^L}, \qquad (6.4)$$

where, by (3.1), $L \leq C_1 \left(\frac{\rho_0}{\rho}\right)^2$, with $C_1 > 0$ only depending on M_1, and C is as above.

Let us tessellate $\Omega_{\frac{5}{\theta}\rho}$ with internally non overlapping closed squares of side $l = \frac{2\rho}{\sqrt{2}}$. By (3.1), their number is dominated by $N = \frac{|\Omega|}{2\rho^2} \leq C \left(\frac{\rho_0}{\rho}\right)^2$, with $C > 0$ only depending on M_1. Then, by (6.4) we have

$$\frac{\rho^2 \int_{\Omega_{\frac{5}{\theta}\rho}} E^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2} \leq C \left(\frac{\rho_0}{\rho}\right)^4 \left(\frac{\rho^2 \int_{B_\rho(x)} E^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2}\right)^{\tau^L}, \qquad (6.5)$$

where $C > 0$ only depends on α_0, α_1, γ_0, $\frac{\rho_0}{h}$ and M_1.

In the next step, we shall estimate from below $\int_{\Omega_{\frac{5}{\theta}\rho}} E^2$. Let us choose

$$c = \frac{1}{|\Omega_{\frac{5}{\theta}\rho}|} \int_{\Omega_{\frac{5}{\theta}\rho}} \varphi_0, \quad d = \frac{1}{|\Omega_{\frac{5}{\theta}\rho}|} \int_{\Omega_{\frac{5}{\theta}\rho}} w_0, \quad \overline{x} = \frac{1}{|\Omega_{\frac{5}{\theta}\rho}|} \int_{\Omega_{\frac{5}{\theta}\rho}} x \qquad (6.6)$$

and let $\rho \leq \frac{\theta}{5} h_2 \rho_0$, so that $\Omega_{\frac{5}{\theta}\rho}$ is connected and of Lipschitz class with constants ρ_0, M_0. By using Korn inequality (4.5) and Poincaré inequality (4.6) in (6.5), and recalling that $E(\varphi_0^*, w_0^*) = E(\varphi_0, w_0)$, we have

$$C \left(\frac{\rho}{\rho_0}\right)^6 \frac{\int_{\Omega_{\frac{5}{\theta}\rho}} |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2} \leq \left(\frac{\rho^2 \int_{B_\rho(x)} E^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2}\right)^{\tau^L}, \qquad (6.7)$$

where $C > 0$ only depends on M_0, M_1, s_0, α_0, α_1, γ_0 and $\frac{\rho_0}{h}$.

Recalling that $\int_\Omega \varphi_0 = 0$, $\int_\Omega w_0 = 0$, and since

$$|\Omega \setminus \Omega_{\frac{5}{\theta}\rho}| \leq C\rho\rho_0, \quad |\Omega_{\frac{5}{\theta}\rho}| \geq C\rho_0^2, \qquad (6.8)$$

with $C > 0$ only depending on M_0 and M_1 (see [AR98, Appendix] for details), by Hölder inequality we have

$$|c| \leq \frac{C}{\rho_0} \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} \left(\int_{\Omega \setminus \Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2\right)^{\frac{1}{2}}, \qquad (6.9)$$

$$|d| \le \frac{C}{\rho_0} \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} \left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |w_0|^2\right)^{\frac{1}{2}} \tag{6.10}$$

and, therefore,

$$\left|\left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0^*|^2\right)^{\frac{1}{2}} - \left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2\right)^{\frac{1}{2}}\right| \le C_1 \frac{\rho}{\rho_0} \left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2\right)^{\frac{1}{2}}, \tag{6.11}$$

$$\left|\left(\int_{\Omega} |\varphi_0^*|^2\right)^{\frac{1}{2}} - \left(\int_{\Omega} |\varphi_0|^2\right)^{\frac{1}{2}}\right| \le C_1 \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} \left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2\right)^{\frac{1}{2}}, \tag{6.12}$$

where $C_1 > 0$ depends only on M_0 and M_1. Assuming, in addition, $\rho \le \min\{\frac{1}{2C_1}, \frac{1}{4C_1^2}\}\rho_0$, from (6.11), (6.12) we have

$$\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0^*|^2 \le \frac{9}{4} \int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2, \tag{6.13}$$

$$\int_{\Omega} |\varphi_0^*|^2 \ge \frac{1}{4} \int_{\Omega} |\varphi_0|^2. \tag{6.14}$$

By (6.9), (6.10) we can estimate

$$\left|\left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |w_0^*|^2\right)^{\frac{1}{2}} - \left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |w_0|^2\right)^{\frac{1}{2}}\right| \le \left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |c\cdot(x-\overline{x})+d|^2\right)^{\frac{1}{2}} \le$$

$$\le C(\rho_0|c|+|d|)|\Omega\setminus\Omega_{\frac{5}{\theta}\rho}|^{\frac{1}{2}} \le C_2\rho \left(\left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2\right)^{\frac{1}{2}} + \frac{1}{\rho_0}\left(\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |w_0|^2\right)^{\frac{1}{2}}\right) \tag{6.15}$$

and, taking the squares, we obtain

$$\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |w_0^*|^2 \le \left(2 + 4C_2^2\left(\frac{\rho}{\rho_0}\right)^2\right) \int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |w_0|^2 + 4C_2^2\rho^2 \int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2, \tag{6.16}$$

where $C_2 > 0$ only depends on M_0 and M_1. From (6.13) and (6.16), and assuming also $\rho \le \frac{3}{4C_2}\rho_0$, we have

$$\int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0^*|^2 + \frac{1}{\rho_0^2}|w_0^*|^2 \le \frac{9}{2} \int_{\Omega\setminus\Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2 + \frac{1}{\rho_0^2}|w_0|^2. \tag{6.17}$$

By repeating calculations similar to those performed in obtaining (6.15), we have

$$\left| \left(\int_\Omega |w_0^*|^2 \right)^{\frac{1}{2}} - \left(\int_\Omega |w_0|^2 \right)^{\frac{1}{2}} \right| \leq C_3 (\rho \rho_0)^{\frac{1}{2}} \left(\left(\int_\Omega |\varphi_0|^2 \right)^{\frac{1}{2}} + \frac{1}{\rho_0} \left(\int_\Omega |w_0|^2 \right)^{\frac{1}{2}} \right),$$
(6.18)

where $C_3 > 0$ only depends on M_0 and M_1. Taking the squares, we deduce

$$\frac{1}{\rho_0^2} \int_\Omega |w_0^*|^2 \geq \frac{1}{\rho_0^2} \int_\Omega |w_0|^2 + \left(2C_3^2 \frac{\rho}{\rho_0} - 2\sqrt{2} C_3 \left(\frac{\rho}{\rho_0} \right)^{\frac{1}{2}} \right) \left(\int_\Omega |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2 \right),$$
(6.19)

where $C_3 > 0$ only depends on M_0 and M_1. By (6.14) and (6.19), and taking $\rho \leq \frac{1}{2 \cdot 16^2 C_3^2} \rho_0$, we have

$$\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2} |w_0^*|^2 \geq \frac{1}{8} \int_\Omega |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2.$$
(6.20)

Let us rewrite the quotient appearing on the left hand side of (6.7) as

$$\frac{\int_{\Omega_{\frac{5}{\theta}\rho}} |\varphi_0^*|^2 + \frac{1}{\rho_0^2} |w_0^*|^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2} |w_0^*|^2} = 1 - \frac{\int_{\Omega \setminus \Omega_{\frac{5}{\theta}\rho}} |\varphi_0^*|^2 + \frac{1}{\rho_0^2} |w_0^*|^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2} |w_0^*|^2}.$$
(6.21)

By (6.17) and (6.20) we have

$$\frac{\int_{\Omega \setminus \Omega_{\frac{5}{\theta}\rho}} |\varphi_0^*|^2 + \frac{1}{\rho_0^2} |w_0^*|^2}{\int_\Omega |\varphi_0^*|^2 + \frac{1}{\rho_0^2} |w_0^*|^2} \leq 36 \frac{\int_{\Omega \setminus \Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2}{\int_\Omega |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2}.$$
(6.22)

From Hölder's inequality, Sobolev embedding theorem and (6.8) we have

$$\int_{\Omega \setminus \Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2 \leq C \rho^{1 - \frac{2}{p}} \rho_0^{1 + \frac{2}{p}} \int_\Omega |\nabla \varphi_0|^2,$$
(6.23)

$$\int_{\Omega \setminus \Omega_{\frac{5}{\theta}\rho}} |w_0|^2 \leq C \rho^{1 - \frac{2}{p}} \rho_0^{1 + \frac{2}{p}} \int_\Omega |\nabla w_0|^2,$$
(6.24)

with $C > 0$ only depending on M_0 and M_1, and p a given number, $p > 2$, for instance $p = 3$. By (6.23) and (6.24), we have

$$\frac{\int_{\Omega \setminus \Omega_{\frac{5}{\theta}\rho}} |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2}{\int_\Omega |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2} \leq C \rho^{1 - \frac{2}{p}} \rho_0^{1 + \frac{2}{p}} \frac{\int_\Omega |\nabla \varphi_0|^2 + \frac{1}{\rho_0^2} |\nabla w_0|^2}{\int_\Omega |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2},$$
(6.25)

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with C and p as above.

Now, let us recall the following trace inequality (see [Gr85, Theorem 1.5.1.10])

$$\int_{\partial\Omega} |w_0|^2 \leq C \left(\left(\int_{\Omega} |\nabla w_0|^2 \right)^{\frac{1}{2}} \cdot \left(\int_{\Omega} |w_0|^2 \right)^{\frac{1}{2}} + \frac{1}{\rho_0} \int_{\Omega} |w_0|^2 \right), \qquad (6.26)$$

with C only depending on M_0 and M_1. Therefore, by (6.26) and Poincaré inequality (4.6),

$$\int_{\partial\Omega} |w_0|^2 \leq C\rho_0 \left(\int_{\Omega} |w_0|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla \varphi_0|^2 + \frac{1}{\rho_0^2} |\nabla w_0|^2 \right)^{\frac{1}{2}}, \qquad (6.27)$$

where $C > 0$ only depends on M_0 and M_1. Similarly, by a trace inequality analogous to (6.26) and by Poincaré inequality (4.6), we have

$$\int_{\partial\Omega} |\varphi_0|^2 \leq C \left(\int_{\Omega} |\varphi_0|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla \varphi_0|^2 + \frac{1}{\rho_0^2} |\nabla w_0|^2 \right)^{\frac{1}{2}}, \qquad (6.28)$$

with $C > 0$ only depending on M_0 and M_1. Therefore, by (6.27) and (6.28) we have

$$\int_{\Omega} |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2 \geq C \frac{\left(\int_{\partial\Omega} |\varphi_0|^2 \right)^2 + \frac{1}{\rho_0^4} \left(\int_{\partial\Omega} |w_0|^2 \right)^2}{\int_{\Omega} |\nabla \varphi_0|^2 + \frac{1}{\rho_0^2} |\nabla w_0|^2}. \qquad (6.29)$$

with $C > 0$ only depending on M_0 and M_1. From (3.22), (6.1) and (6.29), we deduce

$$\frac{\int_{\Omega} |\nabla \varphi_0|^2 + \frac{1}{\rho_0^2} |\nabla w_0|^2}{\int_{\Omega} |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2} \leq \frac{C}{\rho_0^2} \mathcal{F}^4, \qquad (6.30)$$

with $C > 0$ only depending on M_0, M_1, s_0, α_0, α_1, γ_0 and $\frac{\rho_0}{h}$. From (6.25) and (6.30), there exists $C > 0$ only depending on M_0, M_1, s_0, α_0, α_1, γ_0 and $\frac{\rho_0}{h}$, such that if we further assume $\rho \leq \left(\frac{1}{72 C \mathcal{F}^4} \right)^{\frac{p}{p-2}} \rho_0$, where $p > 2$ is as in (6.25), then

$$\frac{\int_{\Omega \setminus \Omega_{\frac{5}{\theta} \rho}} |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2}{\int_{\Omega} |\varphi_0|^2 + \frac{1}{\rho_0^2} |w_0|^2} \leq \frac{1}{72}. \qquad (6.31)$$

Therefore, from (6.7), (6.21), (6.22) and (6.31), we have

$$\left(\frac{\rho^2 \int_{B_\rho(x)} E^2}{\int_{\Omega} |\varphi_0^*|^2 + \frac{1}{\rho_0^2} |w_0^*|^2} \right)^{\tau^L} \geq C \left(\frac{\rho}{\rho_0} \right)^6 \qquad (6.32)$$

and, by (6.20),

$$\int_{B_\rho(x)} E^2 \geq C \left(\frac{\rho}{\rho_0}\right)^{\frac{6}{\tau L}} \frac{1}{\rho^2} \int_\Omega |\varphi_0|^2 + \frac{1}{\rho_0^2}|w_0|^2, \qquad (6.33)$$

with $C > 0$ only depending on M_0, M_1, s_0, α_0, α_1, γ_0 and $\frac{\rho_0}{h}$.

The integral on the right hand side of (6.33) can be estimated from below first by using (6.30), namely

$$\int_{B_\rho(x)} E^2 \geq \frac{C}{\mathcal{F}^4} \left(\frac{\rho}{\rho_0}\right)^{\frac{6}{\tau L} - 2} \int_\Omega |\nabla\varphi_0|^2 + \frac{1}{\rho_0^2}|\nabla w_0|^2, \qquad (6.34)$$

and then by Poincaré inequality, obtaining

$$\int_{B_\rho(x)} E^2 \geq \frac{C}{\mathcal{F}^4} \left(\frac{\rho}{\rho_0}\right)^{\frac{6}{\tau L} - 2} \int_\Omega E^2, \qquad (6.35)$$

with $C > 0$ only depending on M_0, M_1, s_0, α_0, α_1, γ_0 and $\frac{\rho_0}{h}$. Hence (4.17) holds for $\rho \leq \overline{\gamma}\rho_0$, with $\overline{\gamma}$ depending on M_0, M_1, s_0, α_0, α_1, γ_0 and $\frac{\rho_0}{h}$. If $\rho \geq \overline{\gamma}\rho_0$, then the thesis follows *a fortiori*. $\qquad \square$

In order to prove Proposition 6.1, let us introduce the following Lemma.

Lemma 6.3. *Under the hypotheses of Proposition 6.1, let us assume that $\varphi_{|\partial\Omega} \in H^1(\partial\Omega, \mathbb{R}^2)$ and $w_{|\partial\Omega} \in H^1(\partial\Omega)$. Then there exists a positive constant C only depending on M_0, M_1, α_0, α_1, γ_0, $\frac{\rho_0}{h}$, such that*

$$\|\overline{M}\|_{L^2(\partial\Omega, \mathbb{R}^2)} + \rho_0\|\overline{Q}\|_{L^2(\partial\Omega)} \leq C\rho_0^2 \left(\|\varphi_0\|_{H^1(\partial\Omega, \mathbb{R}^2)} + \frac{1}{\rho_0}\|w_0\|_{H^1(\partial\Omega)} \right). \quad (6.36)$$

Proof of Proposition 6.1. For brevity, we shall write φ, w instead of φ_0, w_0 respectively. Let us consider the standard Dirichlet-to-Neumann map

$$\Lambda = \Lambda_{\mathbb{P},S} : H^{1/2}(\partial\Omega, \mathbb{R}^2) \times H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega, \mathbb{R}^2) \times H^{-1/2}(\partial\Omega),$$

$$\Lambda(g_1, g_2) = ((\mathbb{P}\nabla\varphi)n, S(\varphi + \nabla w) \cdot n),$$

where $(\varphi, w) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ is the unique solution to the Dirichlet problem

$$\begin{cases} \text{div}\,(S(\varphi + \nabla w)) = 0 & \text{in } \Omega, & (6.37) \\ \text{div}\,(\mathbb{P}\nabla\varphi) - S(\varphi + \nabla w) = 0, & \text{in } \Omega, & (6.38) \\ \varphi = g_1, & \text{on } \partial\Omega, & (6.39) \\ w = g_2, & \text{on } \partial\Omega. & (6.40) \end{cases}$$

Here the norm in the domain of Λ is normalized by

$$\|(g_1, g_2)\|_{H^{1/2}(\partial\Omega,\mathbb{R}^2)\times H^{1/2}(\partial\Omega)} = \|g_1\|_{H^{1/2}(\partial\Omega,\mathbb{R}^2)} + \rho_0^{-1}\|g_2\|_{H^{1/2}(\partial\Omega)}$$

and similar normalizations will be implied in the sequel for other norms in the domain of Λ and in the codomain of its adjoint Λ^*, whereas the norm in the codomain of Λ is normalized by

$$\|(h_1, h_2)\|_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)\times H^{-1/2}(\partial\Omega)} = \|h_1\|_{H^{-1/2}(\partial\Omega,\mathbb{R}^2)} + \rho_0\|h_2\|_{H^{-1/2}(\partial\Omega)}$$

and similar normalizations will be implied in the sequel for other norms in the codomain of Λ and in the domain of its adjoint Λ^*.

Let us set

$$E = H^1(\partial\Omega,\mathbb{R}^2) \times H^1(\partial\Omega), \qquad F = L^2(\partial\Omega,\mathbb{R}^2) \times L^2(\partial\Omega).$$

By Lemma 6.3 we know that the map Λ can be defined as a bounded linear operator with domain E and codomain F, precisely

$$\Lambda : E \to F, \tag{6.41}$$

$$\|\Lambda(g_1, g_2)\|_F \le C\rho_0^2\|(g_1, g_2)\|_E, \tag{6.42}$$

where we recall that the norms in E and F, according to the above convention, are defined as follows

$$\|(g_1, g_2)\|_E = \|g_1\|_{H^1(\partial\Omega,\mathbb{R}^2)} + \rho_0^{-1}\|g_2\|_{H^1(\partial\Omega)},$$

$$\|(h_1, h_2)\|_F = \|h_1\|_{L^2(\partial\Omega,\mathbb{R}^2)} + \rho_0\|h_2\|_{L^2(\partial\Omega)}.$$

The idea is to use a duality argument in order to deduce the continuity of Λ as an operator acting between larger spaces. Let us consider the adjoint Λ^* of the Dirichlet-to-Neumann map (6.41)–(6.42). Since F is a reflexive space, the domain of the adjoint operator $D(\Lambda^*)$ can be estended by density to all of F',

$$\Lambda^* : F' \to E'$$

$$< \Lambda^*(h_1, h_2), (g_1, g_2) >_{E',E} = < (h_1, h_2), \Lambda(g_1, g_2) >_{F',F} \quad \forall(g_1, g_2) \in E, \forall(h_1, h_2) \in F'. \tag{6.43}$$

By (6.42)–(6.43), we have

$$\|\Lambda^*(h_1, h_2)\|_{E'} \le C\rho_0^2\|(h_1, h_2)\|_{F'} \quad \forall(h_1, h_2) \in F', \tag{6.44}$$

Given any $(h_1, h_2) \in E \subset F \cong F'$, let us consider the unique weak solution to the Dirichlet problem

$$\begin{cases} \text{div}\,(S(\psi + \nabla v)) = 0 & \text{in } \Omega, & (6.45) \\ \text{div}\,(\mathbb{P}\nabla\psi) - S(\psi + \nabla v) = 0, & \text{in } \Omega, & (6.46) \\ \psi = h_1, & \text{on } \partial\Omega, & (6.47) \\ v = h_2, & \text{on } \partial\Omega. & (6.48) \end{cases}$$

By using the weak formulation of problems (6.37)–(6.40) and (6.45)–(6.48), by the symmetry properties of S and $\mathbb{P}$, see (3.24)-(3.25), and by identifying the reflexive space F with its dual space F', we have

$$< \Lambda^*(h_1, h_2), (g_1, g_2) >_{E',E} = < (h_1, h_2), \Lambda(g_1, g_2) >_{F',F} =$$
$$= \int_{\partial\Omega} h_1 \cdot (\mathbb{P}(\nabla\varphi))n + h_2 S(\varphi + \nabla w) \cdot n = \int_{\partial\Omega} \psi \cdot (\mathbb{P}(\nabla\varphi))n + v S(\varphi + \nabla w) \cdot n =$$
$$= \int_{\Omega} \mathbb{P}\nabla\varphi \cdot \nabla\psi + S(\varphi + \nabla w) \cdot (\psi + \nabla v) = \int_{\Omega} \mathbb{P}\nabla\psi \cdot \nabla\varphi + S(\psi + \nabla v) \cdot (\varphi + \nabla w) =$$
$$= \int_{\partial\Omega} \varphi \cdot (\mathbb{P}(\nabla\psi))n + w S(\psi + \nabla v) \cdot n = \int_{\partial\Omega} g_1 \cdot (\mathbb{P}(\nabla\psi))n + g_2 S(\psi + \nabla v) \cdot n,$$
$$(6.49)$$

that is

$$< \Lambda^*(h_1, h_2), (g_1, g_2) >_{E',E} = < \Lambda(h_1, h_2), (g_1, g_2) >_{F',F} \quad \forall (h_1, h_2), (g_1, g_2) \in E. \tag{6.50}$$

Therefore

$$\Lambda^*(h_1, h_2) = \Lambda(h_1, h_2), \qquad \forall (h_1, h_2) \in E \subset F \cong F'. \tag{6.51}$$

By (6.44), we have

$$\|\Lambda(h_1, h_2)\|_{H^{-1/2}(\partial\Omega, \mathbb{R}^2) \times H^{-1/2}(\partial\Omega)} \leq C\rho_0^2 \|(h_1, h_2)\|_{L^2(\partial\Omega, \mathbb{R}^2) \times L^2(\partial\Omega)} \quad \forall (h_1, h_2) \in E. \tag{6.52}$$

Since E is dense in $L^2(\partial\Omega, \mathbb{R}^2) \times L^2(\partial\Omega)$, the above inequality extends to

$$\|\Lambda(h_1, h_2)\|_{H^{-1}(\partial\Omega, \mathbb{R}^2) \times H^{-1}(\partial\Omega)} \leq C\rho_0^2 \|(h_1, h_2)\|_{L^2(\partial\Omega, \mathbb{R}^2) \times L^2(\partial\Omega)}, \tag{6.53}$$

for every $(h_1, h_2) \in L^2(\partial\Omega, \mathbb{R}^2) \times L^2(\partial\Omega)$.

$\square$

In order to derive Lemma 6.3, we need to premise some notation and two auxiliary lemmas which were proved in [AMR02b] and in [MR03] respectively.

Given the notation for the local representation of the boundary of Ω introduced in Definition 2.1, let us set, for $t < \rho_0$,

$$R_t^+ = \Omega \cap R_{t,M_0 t} = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < t, \psi(x_1) < x_2 < M_0 t\},$$

$$\Delta_t = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < t, x_2 = \psi(x_1)\}.$$

The following Lemma is a straightforward consequence of Lemma 5.2 in [AMR02b] and of Lemma 4.3 in [MR03], which were established in general anisotropic setting.

Lemma 6.4. *Let* $S \in C^{0,1}(\overline{\Omega}, \mathbb{M}^2)$ *and* $\mathbb{P} \in C^{0,1}(\overline{\Omega}, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$ *given by* (3.6), (3.7) *respectively, with Lamé moduli satisfying* (3.4), (3.5).

For every $\widetilde{w} \in H^{3/2}(R_{\rho_0}^+)$ *such that* $\operatorname{div}(S\nabla\widetilde{w}) \in L^2(R_{\rho_0}^+)$ *and* $\widetilde{w} = |\nabla\widetilde{w}| = 0$ *on* $\partial R_{\rho_0}^+ \setminus \Delta_{\rho_0}$, *we have*

$$\int_{\Delta_{\rho_0/2}} |S\nabla\widetilde{w}\cdot n|^2 \le C \left(h^2 \int_{\Delta_{\rho_0}} |\nabla_T\widetilde{w}|^2 + \frac{1}{\rho_0} \int_{R_{\rho_0}^+} h^2|\nabla\widetilde{w}|^2 + h\rho_0|\nabla\widetilde{w}||\operatorname{div}(S\nabla\widetilde{w})| \right), \tag{6.54}$$

where $C > 0$ *only depends on* M_0, α_0 *and* α_1.

For every $\widetilde{\varphi} \in H^{3/2}(R_{\rho_0}^+, \mathbb{R}^2)$ *such that* $\operatorname{div}(\mathbb{P}\nabla\widetilde{\varphi}) \in L^2(R_{\rho_0}^+, \mathbb{R}^2)$ *and* $|\widetilde{\varphi}| = |\nabla\widetilde{\varphi}| = 0$ *on* $\partial R_{\rho_0}^+ \setminus \Delta_{\rho_0}$, *we have*

$$\int_{\Delta_{\rho_0/2}} |(\mathbb{P}\nabla\widetilde{\varphi})n|^2 \le C \left(h^6 \int_{\Delta_{\rho_0}} |\nabla_T\widetilde{\varphi}|^2 + \frac{1}{\rho_0} \int_{R_{\rho_0}^+} h^6|\nabla\widetilde{\varphi}|^2 + \rho_0 h^3|\nabla\widetilde{\varphi}||\operatorname{div}(\mathbb{P}\nabla\widetilde{\varphi})| \right), \tag{6.55}$$

where $C > 0$ *only depends on* M_0, α_0, α_1 *and* γ_0.

Proof of Lemma 6.3. We follow the lines of the proof of Proposition 5.1 in [AMR02b]. As a first step, we assume that $\varphi \in H^{3/2}(R_{\rho_0}^+, \mathbb{R}^2)$ and $w \in H^{3/2}(R_{\rho_0}^+)$. Let us consider a cut-off function in $\mathbb{R}^2$

$$\eta(x_1, x_2) = \chi(x_1)\tau(x_2), \tag{6.56}$$

where

$$\chi \in C_0^\infty(\mathbb{R}), \quad \chi(x_1) \equiv 1 \text{ if } |x_1| \le \frac{\rho_0}{2}, \quad \chi(x_1) \equiv 0 \text{ if } |x_1| \ge \frac{3}{4}\rho_0, \tag{6.57}$$

$$\|\chi'\|_\infty \le C_1\rho_0^{-1}, \quad \|\chi''\|_\infty \le C_1\rho_0^{-2}, \tag{6.58}$$

$$\tau \in C_0^\infty(\mathbb{R}), \quad \tau(x_2) \equiv 1 \text{ if } |x_2| \le \frac{M_0\rho_0}{2}, \quad \tau(x_2) \equiv 0 \text{ if } |x_2| \ge \frac{3}{4}M_0\rho_0, \tag{6.59}$$

$$\|\tau'\|_\infty \le C_2\rho_0^{-1}, \quad \|\tau''\|_\infty \le C_2\rho_0^{-2}, \tag{6.60}$$

where C_1 is an absolute constant and C_2 is a constant only depending on M_0.

Let

$$\widetilde{w} = \eta w,$$

$$\widetilde{\varphi} = \eta\varphi.$$

In view of equations (3.16)–(3.17), it will be useful in the sequel to rewrite $\operatorname{div}(S\nabla w)$ in terms of first derivatives of φ and $\operatorname{div}(\mathbb{P}\nabla\varphi)$ in terms of first derivatives of w and in terms of φ

$$\operatorname{div}(S\nabla w) = -\operatorname{div}(S\varphi), \tag{6.61}$$

$$\operatorname{div}(\mathbb{P}\nabla\varphi) = S(\varphi + \nabla w). \tag{6.62}$$

By (6.61), it follows that $\operatorname{div}(S\nabla\widetilde{w}) \in L^2(R_{\rho_0}^+)$ and by (6.62), it follows that $\operatorname{div}(\mathbb{P}\nabla\widetilde{\varphi}) \in L^2(R_{\rho_0}^+, \mathbb{R}^2)$. Therefore we can apply estimates (6.54) and (6.55) of Lemma 6.4 to $\widetilde{w}$ and $\widetilde{\varphi}$, respectively. Taking into account (6.56)–(6.62) we easily obtain

$$\int_{\Delta_{\rho_0/2}} |S\nabla w \cdot n|^2 \leq$$
$$Ch^2 \left[\int_{\Delta_{\rho_0}} \left(|\nabla_T w|^2 + \frac{w^2}{\rho_0^2} \right) + \frac{1}{\rho_0} \int_{R_{\rho_0}^+} \left(|\nabla w|^2 + \frac{w^2}{\rho_0^2} + \rho_0^2 |\nabla\varphi|^2 + |\varphi|^2 \right) \right], \tag{6.63}$$

$$\int_{\Delta_{\rho_0/2}} |\mathbb{P}\nabla\varphi \cdot n|^2 \leq$$
$$Ch^6 \left[\int_{\Delta_{\rho_0}} \left(|\nabla_T \varphi|^2 + \frac{|\varphi|^2}{\rho_0^2} \right) + \frac{1}{\rho_0} \int_{R_{\rho_0}^+} \left(|\nabla\varphi|^2 + \frac{|\varphi|^2}{\rho_0^2} + \frac{|\nabla w|^2}{\rho_0^2} \right) \right], \tag{6.64}$$

where $C > 0$ only depends on M_0, α_0, α_1 and γ_0.

By (6.63) and (6.64) we have

$$\int_{\Delta_{\rho_0/2}} |\mathbb{P}\nabla\varphi \cdot n|^2 + \rho_0^2 |S(\varphi + \nabla w) \cdot n|^2 \leq$$
$$Ch^6 \left[\int_{\Delta_{\rho_0}} \left(|\nabla_T \varphi|^2 + \frac{|\varphi|^2}{\rho_0^2} + \frac{|\nabla_T w|^2}{\rho_0^2} + \frac{w^2}{\rho_0^4} \right) + \frac{1}{\rho_0} \int_{R_{\rho_0}^+} \left(|\nabla\varphi|^2 + \frac{|\varphi|^2}{\rho_0^2} + \frac{w^2}{\rho_0^4} + \frac{|\nabla w|^2}{\rho_0^2} \right) \right], \tag{6.65}$$

where $C > 0$ only depends on M_0, α_0, α_1 and γ_0.

The hypotheses $\varphi \in H^{3/2}(R_{\rho_0}^+, \mathbb{R}^2)$, $w \in H^{3/2}(R_{\rho_0}^+)$ can be removed by following the lines of the approximation argument used in Step 3 of [AMR02b, Lemma 5.2] and [MR03, Lemma 4.3] respectively, obtaining again (6.65). Finally, by (6.65) and the well-posedness of the Dirichlet problem (6.37)–(6.40), inequality (6.36) follows.

$\square$

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