Estimating the area of extreme inclusions in Reissner-Mindlin plates *

Antonino Morassi[†] and Edi Rosset[‡]

Thanks Sergio, for what we have learned
and continue to learn working with you. Happy birthday!

Abstract

We derive upper and lower estimates of the area of unknown defects in the form of either cavities or rigid inclusions in Mindlin-Reissner elastic plates in terms of the difference δW of the works exerted by boundary loads on the defected and on the reference plate. It turns out that the upper estimates depend linearly on δW, whereas the lower ones depend quadratically on δW. These results continue a line of research concerning size estimates of extreme inclusions in electric conductors, elastic bodies and plates.

Mathematics Subject Classification (2010): Primary 35B60. Secondary 35B30, 35Q74, 35R30.

Keywords: Inverse problems, Reissner-Mindlin elastic plates, size estimates, cavities, rigid inclusions.

How to cite this paper:
This paper has been accepted in Applicable Analysis and the final publication is available at https://doi.org/10.1080/00036811.2020.1742885.

*Antonino Morassi is supported by PRIN 2015TTJN95 "Identificazione e diagnostica di sistemi strutturali complessi". Edi Rosset is supported by PRIN 201758MTR2 "Direct and inverse problems for partial differential equations: theoretical aspects and applications", by Progetto GNAMPA 2019 "Proprietà delle soluzioni di equazioni alle derivate parziali e applicazioni ai problemi inversi" Istituto Nazionale di Alta Matematica (INdAM) and by FRA2016 "Problemi inversi, dalla stabilità alla ricostruzione", Università degli Studi di Trieste.

[†]Dipartimento Politecnico di Ingegneria e Architettura, Università degli Studi di Udine, via Cotonificio 114, 33100 Udine, Italy. E-mail: antonino.morassi@uniud.it

[‡]Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, via Valerio 12/1, 34127 Trieste, Italy. E-mail: rossedi@units.it

1 Introduction

In the present paper we continue a line of research concerning the identification of unknown defects inside Reissner-Mindlin plates. As is well known, the Reissner-Mindlin theory gives a more refined model for elastic plates with respect to the Kirchhoff-Love theory and, in particular, it allows for an accurate description of moderately thick plates, having thickness h of the order of one tenth of the dimension of the middle plane Ω.

Perhaps, the simplest approach in detecting defects consists in estimating their *size*. In [MRV18] we derived constructive upper and lower bounds of the area of elastic inclusions (*size estimates*) in terms of the difference between the works exerted by given boundary loads in deforming the plate without and with inclusion. Here, we obtain constructive size estimates for *extreme* inclusions in the form of either cavities or rigid inclusions. The interested reader can refer, among others, to the papers [ARS00], [BCOZ], [DiCLW13], [DiCLVW13], [I98], [KKM12], [KM13], [KSS97], [MN12] for results and application of the size estimate approach to various physical contexts.

Let $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ be the plate, with Ω a bounded domain in $\mathbb{R}^2$, and let $\mathbb{P}$ the fourth-order bending tensor and S the shearing matrix of the reference plate (i.e., without defects). Let us denote by $D \times \left[-\frac{h}{2}, \frac{h}{2}\right]$, $D \subset\subset \Omega$, the unknown defect to be determined. Our experiment consists in applying the same (self-equilibrated) transverse force field $\overline{Q}$ and couple field $\overline{M}$ at the boundary of the plate, in presence and in absence of the inclusion.

When D represents a cavity, the infinitesimal transverse displacement w_c and the infinitesimal rigid rotation φ_c (of transverse material fiber to the middle plane Ω) satisfy the following Neumann boundary value problem

$$
\begin{cases}
\operatorname{div}(S(\varphi_c + \nabla w_c)) = 0, & \text{in } \Omega \setminus \overline{D}, \quad (1.1)\\
\operatorname{div}(\mathbb{P}\nabla \varphi_c) - S(\varphi_c + \nabla w_c) = 0, & \text{in } \Omega \setminus \overline{D}, \quad (1.2)\\
S(\varphi_c + \nabla w_c) \cdot n = \overline{Q} & \text{on } \partial\Omega, \quad (1.3)\\
(\mathbb{P}\nabla \varphi_c) n = \overline{M}, & \text{on } \partial\Omega, \quad (1.4)\\
S(\varphi_c + \nabla w_c) \cdot n = 0 & \text{on } \partial D, \quad (1.5)\\
(\mathbb{P}\nabla \varphi_c) n = 0, & \text{on } \partial D, \quad (1.6)
\end{cases}
$$

where n is the outer unit normal to Ω and D, respectively.

In case D is a rigid inclusion, the analogous statical equilibrium problem becomes the following mixed boundary value problem

$$
\begin{cases}
\operatorname{div}(S(\varphi_r + \nabla w_r)) = 0, & \text{in } \Omega \setminus \overline{D}, \quad (1.7)\\
\operatorname{div}(\mathbb{P}\nabla\varphi_r) - S(\varphi_r + \nabla w_r) = 0, & \text{in } \Omega \setminus \overline{D}, \quad (1.8)\\
S(\varphi_r + \nabla w_r)\cdot n = \overline{Q} & \text{on } \partial\Omega, \quad (1.9)\\
(\mathbb{P}\nabla\varphi_r)n = \overline{M}, & \text{on } \partial\Omega, \quad (1.10)\\
\varphi_r = b, & \text{in } \overline{D}, \quad (1.11)\\
w_r = -b\cdot x + a, & \text{in } \overline{D}, \quad (1.12)
\end{cases}
$$

where φ_r and w_r are continuous functions through ∂D, a is any constant and b is any 2-dimensional vector.

When D is empty, the equilibrium of the undefective plate is modelled by

$$
\begin{cases}
\operatorname{div}(S(\varphi_0 + \nabla w_0)) = 0, & \text{in } \Omega, \quad (1.13)\\
\operatorname{div}(\mathbb{P}\nabla\varphi_0) - S(\varphi_0 + \nabla w_0) = 0, & \text{in } \Omega, \quad (1.14)\\
S(\varphi_0 + \nabla w_0)\cdot n = \overline{Q} & \text{on } \partial\Omega, \quad (1.15)\\
(\mathbb{P}\nabla\varphi_0)n = \overline{M}, & \text{on } \partial\Omega. \quad (1.16)
\end{cases}
$$

Let us define the following boundary integrals, which express the works produced by the given boundary loads $\overline{Q}$, $\overline{M}$ when D is a cavity, a rigid inclusion, or D is absent:

$$
W_c = \int_{\partial\Omega} \overline{Q} w_c + \overline{M}\cdot\varphi_c, \quad W_r = \int_{\partial\Omega} \overline{Q} w_r + \overline{M}\cdot\varphi_r, \quad W_0 = \int_{\partial\Omega} \overline{Q} w_0 + \overline{M}\cdot\varphi_0. \tag{1.17}
$$

Our size estimates are formulated in terms of the *normalized work gap*

$$
\frac{\delta W}{W_0}, \tag{1.18}
$$

where

$$
\delta W = W_c - W_0, \qquad \delta W = W_0 - W_r, \tag{1.19}
$$

respectively.

Upper and lower estimates require different mathematical tools and, also, present different dependence on the work gap. Precisely, upper estimates have *linear character*, but require additional a priori assumptions on the material (isotropy) and on the defect D, namely the following *Fatness Condition*:

$$
\operatorname{area}\{x \in D \mid \operatorname{dist}(x, \partial D) > h_1\} \geq \frac{1}{2}\operatorname{area}(D), \tag{1.20}
$$

where h_1 is a given parameter.

The isotropy assumption ensures the unique continuation property in the form of a quantitative three spheres inequality for the strain energy density of solutions to the reference problem (1.13)–(1.16), which was obtained in [MRV18]. Let us observe that the above Fatness Condition could be removed provided a doubling inequality were at disposal. In that case, the upper estimate would have Hölder character, see, for example, [AMR02, Theorems 2.6 and 2.8] for an electric conductor, and [MR03, Theorems 2.7 and 2.9] in the context of linear elasticity.

The estimates from below are quite different, both for the a priori assumptions and the techniques of proof. In fact, we need to assume Lipschitz regularity of the boundary of D, precisely

$$\partial D \text{ is of Lipschitz class, with constants } r_D, L_D, \tag{1.21}$$

whereas, on the other hand, we can replace the Fatness Condition (1.20) with the weaker Scale Invariant Fatness Condition

$$\text{diam}(D) \leq Q_D r_D, \tag{1.22}$$

where, in both conditions, r_D is *unknown*. This last hypothesis avoids collapsing of D to an empty set having null 2-dimensional Lebesgue measure. Moreover, the dependence of $area(D)$ on δW has a quadratic character.

Main mathematical tools for cavities are constructive Poincaré-type and Korn-type inequalities and, in particular, a generalized Korn inequality recently obtained in [MRV17], suitable to handle the Reissner-Mindlin system.

The treatment of rigid inclusions requires, in addition, boundary estimates in L^2 for the boundary value problem (1.7)–(1.12). These estimates are based on identities of Rellich type (see [R], [PW58]), and are in the style of the solvability in L^2 of the regularity and Neumann problems formulated in [FKP91], [KP93].

The paper is organized as follows. In Section 2 we introduce some useful notation. Direct problems are described in Section 3, and the size estimates are stated in Section 4. Finally, proofs are given in Section 5 and 6, for cavities and rigid inclusions, respectively.

2 Notation

Let $P = (x_1(P), x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the open disc in $\mathbb{R}^2$ of radius r and center P and by $R_{a,b}(P)$ the rectangle $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a,\ |x_2 - x_2(P)| < b\}$. To simplify the notation, we shall denote $B_r = B_r(O)$, $R_{a,b} = R_{a,b}(O)$.

Definition 2.1. (Lipschitz regularity) Let G be a bounded domain in $\mathbb{R}^2$. We say that a portion Σ of ∂G is of Lipschitz class with constants ρ, L if, for any $P \in \Sigma$, there exists a rigid transformation of coordinates under which we have $P = O$ and

$$G \cap R_{\rho,L\rho} = \{x = (x_1, x_2) \in R_{\rho,L\rho} \quad | \quad x_2 > \psi(x_1)\},$$

where ψ is a Lipschitz continuous function on $(-\rho, \rho)$ satisfying

$$\psi(0) = 0,$$

$$\|\psi\|_{C^{0,1}(-\rho,\rho)} \leq L\rho.$$

Remark 2.2. We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous with the L^∞ norm and coincide with the standard definition when the dimensional parameter equals one. For instance, the norm appearing above is meant as follows

$$\|\psi\|_{C^{0,1}(-\rho,\rho)} = \|\psi\|_{L^\infty(-\rho,\rho)} + \rho\|\psi'\|_{L^\infty(-\rho,\rho)}. \tag{2.1}$$

Given $G \subset \mathbb{R}^2$, for any $t > 0$ we denote

$$G_t = \{x \in G \mid \mathrm{dist}(x, \partial G) > t\}, \tag{2.2}$$

$$G^r = \{x \in \mathbb{R}^2 \quad | \quad 0 < \mathrm{dist}(x, G) < r\}. \tag{2.3}$$

Let

$$\begin{aligned} \mathcal{A} &= \{z = (\varphi, w) \mid \varphi = b, w = -b \cdot x + a, a \in \mathbb{R}, b \in \mathbb{R}^2\} \\ &= \{z = (\varphi, w) \mid \nabla\varphi = 0, \varphi + \nabla w = 0\}. \end{aligned} \tag{2.4}$$

We denote by $\mathbb{M}^2$ the space of 2×2 real valued matrices and by $\mathcal{L}(X, Y)$ the space of bounded linear operators between Banach spaces X and Y.

For every 2×2 matrices A, B and for every $\mathbb{L} \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$, we use the following notation:

$$(\mathbb{L}A)_{ij} = L_{ijkl}A_{kl}, \tag{2.5}$$

$$A \cdot B = A_{ij}B_{ij}, \quad |A| = (A \cdot A)^{\frac{1}{2}}, \quad tr(A) = A_{ii}, \tag{2.6}$$

$$(A^T)_{ij} = A_{ji}, \quad \widehat{A} = \frac{1}{2}(A + A^T). \tag{2.7}$$

Notice that here and in the sequel summation over repeated indexes is implied.

3 Formulation of the direct problems

Let us consider a plate, with constant thickness h, represented by a bounded domain Ω in $\mathbb{R}^2$ having boundary of Lipschitz class, with constants ρ_0 and L_0, and satisfying

$$\text{diam}(\Omega) \leq Q_0\rho_0, \tag{3.1}$$

for some $Q_0 > 0$, and

$$O \in \Omega. \tag{3.2}$$

The *reference* plate is assumed to be made by linearly elastic isotropic material with Lamé moduli λ and μ satisfying the strong convexity conditions

$$\mu(x) \geq \alpha_0, \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0, \quad \text{in } \overline{\Omega}, \tag{3.3}$$

for given positive constants α_0, γ_0, and the regularity condition

$$\|\lambda\|_{C^{0,1}(\overline{\Omega})} + \|\mu\|_{C^{0,1}(\overline{\Omega})} \leq \alpha_1, \tag{3.4}$$

where α_1 is a given constant. Therefore, the shearing and bending plate tensors take the form

$$SI_2, \quad S = h\mu, \quad S \in C^{0,1}(\overline{\Omega}), \tag{3.5}$$

$$\mathbb{P}A = B\left[(1-\nu)\widehat{A} + \nu tr(A)I_2\right], \quad \mathbb{P} \in C^{0,1}(\overline{\Omega}), \tag{3.6}$$

where I_2 is the two-dimensional unit matrix, A denotes a 2×2 matrix and

$$B = \frac{Eh^3}{12(1-\nu^2)}, \tag{3.7}$$

with Young's modulus E and Poisson's coefficient ν given by

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}, \quad \nu = \frac{\lambda}{2(\mu + \lambda)}. \tag{3.8}$$

By (3.3) and (3.4), the ellipticity conditions for S and $\mathbb{P}$ become

$$h\alpha_0 \leq S \leq h\alpha_1, \quad \text{in } \overline{\Omega}, \tag{3.9}$$

and

$$\frac{h^3}{12}\xi_0|\widehat{A}|^2 \leq \mathbb{P}A \cdot A \leq \frac{h^3}{12}\xi_1|\widehat{A}|^2, \quad \text{in } \overline{\Omega}, \tag{3.10}$$

for every 2×2 matrix A, with

$$\xi_0 = \min\{2\alpha_0, \gamma_0\}, \quad \xi_1 = 2\alpha_1. \tag{3.11}$$

Moreover,

$$\|S\|_{C^{0,1}(\overline{\Omega})} \leq h\alpha_1, \qquad \|\mathbb{P}\|_{C^{0,1}(\overline{\Omega})} \leq Ch^3, \tag{3.12}$$

with $C>0$ only depending on α_0, α_1, γ_0.

Let the boundary of the plate $\partial\Omega$ be subject to a transverse force field $\overline{Q}$ and a couple field $\overline{M}$ satisfying

$$\overline{Q} \in H^{-\frac{1}{2}}(\partial\Omega), \quad \overline{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2), \tag{3.13}$$

and the compatibility conditions

$$\int_{\partial\Omega} \overline{Q} = 0, \quad \int_{\partial\Omega} (\overline{Q}x - \overline{M}) = 0. \tag{3.14}$$

Throughout the paper, the defect is represented by an open set D satisfying

$$D \subset\subset \Omega, \qquad \Omega\setminus\overline{D} \text{ is connected.} \tag{3.15}$$

When D represents a cavity, the statical equilibrium is governed by the Neumann boundary value problem

$$\begin{cases}
\operatorname{div}(S(\varphi_c+\nabla w_c)) = 0, & \text{in } \Omega\setminus\overline{D}, & (3.16)\\
\operatorname{div}(\mathbb{P}\nabla\varphi_c) - S(\varphi_c+\nabla w_c) = 0, & \text{in } \Omega\setminus\overline{D}, & (3.17)\\
S(\varphi_c+\nabla w_c)\cdot n = \overline{Q} & \text{on } \partial\Omega, & (3.18)\\
(\mathbb{P}\nabla\varphi_c)n = \overline{M}, & \text{on } \partial\Omega, & (3.19)\\
S(\varphi_c+\nabla w_c)\cdot n = 0 & \text{on } \partial D, & (3.20)\\
(\mathbb{P}\nabla\varphi_c)n = 0, & \text{on } \partial D, & (3.21)
\end{cases}$$

where n is the outer unit normal to Ω and to D, respectively. A weak solution to the above system is a pair $(\varphi_c, w_c) \in H^1(\Omega\setminus\overline{D}, \mathbb{R}^2)\times H^1(\Omega\setminus\overline{D})$ satisfying

$$\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla\varphi_c\cdot\nabla\psi + \int_{\Omega\setminus\overline{D}} S(\varphi_c+\nabla w_c)\cdot(\psi+\nabla v) = \int_{\partial\Omega} \overline{Q}v + \overline{M}\cdot\psi, \tag{3.22}$$

for every $\psi \in H^1(\Omega\setminus\overline{D}, \mathbb{R}^2)$ and for every $v\in H^1(\Omega\setminus\overline{D})$.

Problem (3.16)–(3.21) admits a weak solution $(\varphi_c, w_c) \in H^1(\Omega\setminus\overline{D}, \mathbb{R}^2)\times H^1(\Omega\setminus\overline{D})$, which is uniquely determined up to addition of an element $z\in\mathcal{A}$.

Since D has Lipschitz boundary, one can continue a solution pair (φ_c, w_c) to a $H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ function pair, which we continue to call (φ_c, w_c):

$$(\varphi_c, w_c) = \begin{cases} (\varphi_c^+, w_c^+) & \text{in } \Omega \setminus \overline{D}, \\ \\ (\varphi_c^-, w_c^-) & \text{in } D, \end{cases} \tag{3.23}$$

where (φ_c^+, w_c^+) is the given solution (φ_c, w_c) and (φ_c^-, w_c^-) is defined as the weak solution of the Dirichlet problem

$$\begin{cases} \mathrm{div}(S(\varphi_c^- + \nabla w_c^-)) = 0, & \text{in } D, \\ \mathrm{div}(\mathbb{P}\nabla\varphi_c^-) - S(\varphi_c^- + \nabla w_c^-) = 0, & \text{in } D, \\ \varphi_c^- = \varphi_c^+|_{\partial D}, & \text{on } \partial D, \\ w_c^- = w_c^+|_{\partial D}, & \text{on } \partial D. \end{cases} \tag{3.24}$$

When D represents a rigid inclusion, the statical equilibrium is governed by the mixed boundary value problem

$$\begin{cases} \mathrm{div}(S(\varphi_r^+ + \nabla w_r^+)) = 0, & \text{in } \Omega \setminus \overline{D}, & (3.25) \\ \mathrm{div}(\mathbb{P}\nabla\varphi_r^+) - S(\varphi_r^+ + \nabla w_r^+) = 0, & \text{in } \Omega \setminus \overline{D}, & (3.26) \\ S(\varphi_r^+ + \nabla w_r^+) \cdot n = \overline{Q} & \text{on } \partial\Omega, & (3.27) \\ (\mathbb{P}\nabla\varphi_r^+)n = \overline{M}, & \text{on } \partial\Omega, & (3.28) \\ \varphi_r^- + \nabla w_r^- = 0, & \text{in } \overline{D}, & (3.29) \\ \nabla\varphi_r^- = 0, & \text{in } \overline{D}, & (3.30) \\ w_r^- = w_r^+, & \text{on } \partial D, & (3.31) \\ \varphi_r^- = \varphi_r^+, & \text{on } \partial D, & (3.32) \end{cases}$$

where we have denoted by (φ_r^-, w_r^-) and (φ_r^+, w_r^+) the restriction of the solution (φ_r, w_r) in $\overline{D}$ and in $\Omega \setminus \overline{D}$, respectively, and n is the outer unit normal to Ω. For future reference, we notice that the compatibility conditions (3.14) together with the above formulation (3.25)–(3.32) imply

$$\int_{\partial D} S(\varphi_r^+ + \nabla w_r^+) \cdot n = 0, \tag{3.33}$$

$$\int_{\partial D} ((\mathbb{P}\nabla\varphi_r^+)n - (S(\varphi_r^+ + \nabla w_r^+) \cdot n)x) = 0. \tag{3.34}$$

From the mechanical point of view, the last two conditions state the force balance and the couple balance of the rigid inclusion D, respectively.

Let us introduce

$$H^1_D(\Omega,\mathbb{R}^2)\times H^1_D(\Omega)=\{(\varphi,w)\in H^1(\Omega,\mathbb{R}^2)\times H^1(\Omega)|\ (\varphi,w)_{|_D}\in\mathcal{A}\}. \tag{3.35}$$

A pair $(\varphi_r,w_r)\in H^1_D(\Omega,\mathbb{R}^2)\times H^1_D(\Omega)$ is a weak solution to (3.25)–(3.32) if for every $(\psi,v)\in H^1_D(\Omega,\mathbb{R}^2)\times H^1_D(\Omega)$ we have

$$\int_\Omega \mathbb{P}\nabla\varphi_r\cdot\nabla\psi+\int_\Omega S(\varphi_r+\nabla w_r)\cdot(\psi+\nabla v)=\int_{\partial\Omega}\overline{Q}v+\overline{M}\cdot\psi. \tag{3.36}$$

Since $H^1_D(\Omega,\mathbb{R}^2)\times H^1_D(\Omega)$ is a closed linear subspace of $H^1(\Omega,\mathbb{R}^2)\times H^1(\Omega)$, by standard variational methods it can be proven that (3.25)–(3.32) has a weak solution $(\varphi_c,w_c)\in H^1(\Omega,\mathbb{R}^2)\times H^1(\Omega)$, which is uniquely determined up to addition of an element $z\in\mathcal{A}$.

It is convenient to consider also the reference plate, in absence of inclusions, whose statical equilibrium is governed by the following Neumann boundary value problem

$$\begin{cases}\operatorname{div}(S(\varphi_0+\nabla w_0))=0, & \text{in } \Omega, \quad (3.37)\\ \operatorname{div}(\mathbb{P}\nabla\varphi_0)-S(\varphi_0+\nabla w_0)=0, & \text{in } \Omega, \quad (3.38)\\ S(\varphi_0+\nabla w_0)\cdot n=\overline{Q}, & \text{on } \partial\Omega, \quad (3.39)\\ (\mathbb{P}\nabla\varphi_0)n=\overline{M}, & \text{on } \partial\Omega. \quad (3.40)\end{cases}$$

A weak solution to the above Neumann problem is a pair $(\varphi_0,w_0)\in H^1(\Omega,\mathbb{R}^2)\times H^1(\Omega)$ satisfying

$$\int_\Omega \mathbb{P}\nabla\varphi_0\cdot\nabla\psi+\int_\Omega S(\varphi_0+\nabla w_0)\cdot(\psi+\nabla v)=\int_{\partial\Omega}\overline{Q}v+\overline{M}\cdot\psi, \tag{3.41}$$

for every $\psi\in H^1(\Omega,\mathbb{R}^2)$ and for every $v\in H^1(\Omega)$. The equilibrium problem (3.37)–(3.40) has a weak solution $(\varphi_0,w_0)\in H^1(\Omega,\mathbb{R}^2)\times H^1(\Omega)$, which is uniquely determined up to addition of an element $z\in\mathcal{A}$.

Let us denote by W_c, W_r, W_0 the works exerted by the surface forces and couples $\overline{Q}$ and $\overline{M}$ when D is a cavity, a rigid inclusion, or it is absent, respectively:

$$W_c=\int_{\partial\Omega}\overline{Q}w_c+\overline{M}\cdot\varphi_c=\int_{\Omega\setminus\overline{D}}\mathbb{P}\nabla\varphi_c\cdot\nabla\varphi_c+\int_{\Omega\setminus\overline{D}}S(\varphi_c+\nabla w_c)\cdot(\varphi_c+\nabla w_c), \tag{3.42}$$

$$W_r=\int_{\partial\Omega}\overline{Q}w_r^++\overline{M}\cdot\varphi_r^+=\int_{\Omega\setminus\overline{D}}\mathbb{P}\nabla\varphi_r^+\cdot\nabla\varphi_r^++\int_{\Omega\setminus\overline{D}}S(\varphi_r^++\nabla w_r^+)\cdot(\varphi_r^++\nabla w_r^+), \tag{3.43}$$

$$W_0=\int_{\partial\Omega}\overline{Q}w_0+\overline{M}\cdot\varphi_0=\int_\Omega\mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_0+\int_\Omega S(\varphi_0+\nabla w_0)\cdot(\varphi_0+\nabla w_0). \tag{3.44}$$

Remark 3.1. Let us notice that, in view of the compatibility conditions (3.14), the works W_c, W_r, W_0 are well defined, that is they are invariant with respect to the addition of any element z in $\mathcal{A}$ to the solution pair (φ_c, w_c), (φ_r, w_r), (φ_0, w_0), respectively.

Throughout the paper, we shall choose the following normalization conditions for (φ_0, w_0):

$$\int_\Omega \varphi_0 = 0, \quad \int_\Omega w_0 = 0. \tag{3.45}$$

4 The inverse problems: main results

Let us start considering the size estimates for cavities. We analyze separately the upper and lower estimates, since they require different a priori assumptions and techniques of proof.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^2$, such that $\partial\Omega$ is of Lipschitz class with constants ρ_0, L_0 and satisfying* (3.1)*. Let D be an open set satisfying* (3.15) *and*

$$|D_{h_1\rho_0}| \geq \frac{1}{2}|D|, \tag{4.1}$$

for a given positive constant h_1. Let the reference plate be made by linearly elastic isotropic material with Lamé moduli λ, μ satisfying (3.3), (3.4)*. Let the transverse force field $\overline{Q} \in H^{-1/2}(\partial\Omega)$ and the couple field $\overline{M} \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$ satisfy*

$$\mathcal{F} = \frac{\|\overline{M}\|_{H^{-1/2}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-1/2}(\partial\Omega)}}{\|\overline{M}\|_{H^{-1}(\partial\Omega)} + \rho_0\|\overline{Q}\|_{H^{-1}(\partial\Omega)}}, \tag{4.2}$$

for some positive constant $\mathcal{F}$. The following estimate holds

$$|D| \leq K\rho_0^2 \frac{W_c - W_0}{W_0}, \tag{4.3}$$

where K only depends on α_0, α_1, γ_0, L_0, Q_0, $\frac{\rho_0}{h}$, h_1 and $\mathcal{F}$.

In order to obtain the estimate from below, we assume that D is a domain satisfying the following a priori assumptions concerning its regularity and shape:

$$\partial D \text{ is of Lipschitz class with constants } r_D, L_D, \tag{4.4}$$

$$\mathrm{diam}(D) \leq Q_D r_D, \tag{4.5}$$

where L_D, Q_D are given a priori parameters, whereas r_D is unknown.

Let us stress the fact that r_D is an *unknown* parameter (otherwise, the size estimates should follow trivially), whereas the parameters L_D and Q_D, which are invariant under scaling, will be considered as a priori information.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^2$, such that $\partial\Omega$ is of Lipschitz class with constants ρ_0, L_0 and satisfying* (3.1)*. Let D be a subdomain of Ω satisfying* (3.15)*,* (4.4)*,* (4.5)*, and such that*

$$dist(D, \partial\Omega) \geq d_0\rho_0, \tag{4.6}$$

with $d_0 > 0$, $r_D < \frac{d_0}{2}\rho_0$. Let the reference plate be made by linearly elastic isotropic material with Lamé moduli λ, μ satisfying (3.3)*,* (3.4)*. Let the transverse force field $\overline{Q} \in H^{-1/2}(\partial\Omega)$ and the couple field $\overline{M} \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$. The following estimate holds*

$$|D| \geq k\rho_0^2 \Psi\left(\frac{W_c - W_0}{W_0}\right), \tag{4.7}$$

where the function Ψ is given by

$$[0, +\infty) \ni t \mapsto \Psi(t) = \frac{t^2}{1+t}, \tag{4.8}$$

and $k > 0$ only depends on α_0, α_1, γ_0, L_0, Q_0, $\frac{\rho_0}{h}$, d_0, L_D, Q_D.

Concerning rigid inclusions, the size estimates are stated in the next two theorems.

Theorem 4.3. *Let Ω be a bounded domain in $\mathbb{R}^2$, such that $\partial\Omega$ is of Lipschitz class with constants ρ_0, L_0 and satisfying* (3.1)*. Let D be an open set satisfying* (3.15) *and the fatness condition* (4.1)*. Let the reference plate be made by linearly elastic isotropic material with Lamé moduli λ, μ satisfying* (3.3)*,* (3.4)*. Let the transverse force field $\overline{Q} \in H^{-1/2}(\partial\Omega)$ and the couple field $\overline{M} \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$ satisfy* (4.2)*.*

The following estimate holds

$$|D| \leq K\rho_0^2 \frac{W_0 - W_r}{W_0}, \tag{4.9}$$

where K only depends on α_0, α_1, γ_0, L_0, Q_0, $\frac{\rho_0}{h}$, h_1 and $\mathcal{F}$.

Theorem 4.4. *Let Ω be a bounded domain in $\mathbb{R}^2$, such that $\partial\Omega$ is of Lipschitz class with constants ρ_0, L_0 and satisfying* (3.1)*. Let D be a subdomain of Ω satisfying* (3.15), (4.4), (4.5)*, and such that*

$$dist(D, \partial\Omega) \geq d_0\rho_0, \tag{4.10}$$

with $d_0 > 0$, $r_D < \frac{d_0}{2}\rho_0$. Let the reference plate be made by linearly elastic isotropic material with Lamé moduli λ, μ satisfying (3.3), (3.4)*. Let the transverse force field $\overline{Q} \in H^{-1/2}(\partial\Omega)$ and the couple field $\overline{M} \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$. The following estimate holds*

$$|D| \geq C\rho_0^2 \Phi\left(\frac{W_0 - W_r}{W_0}\right), \tag{4.11}$$

where the function Φ is given by

$$[0,1) \ni t \mapsto \Phi(t) = \frac{t^2}{1-t}, \tag{4.12}$$

and $C > 0$ only depends on α_0, α_1, γ_0, L_0, Q_0, $\frac{\rho_0}{h}$, d_0, L_D, Q_D.

Remark 4.5. Let us notice that the estimates from below stated in Theorems 4.2 and 4.4 hold for the general context of anisotropic plates with bounded coefficients satisfying the strong convexity assumption, since unique continuation estimates are not needed for the proofs of these theorems.

Moreover, Theorems 4.2 and 4.4 can be extended to the case when D is made of a finite *unknown* number of connected components. Precisely, it suffices to assume that $D = \cup_{j=1}^J D_j$, $j = 1, ..., J$, with $\Omega \setminus \overline{D}$ connected, ∂D_j of Lipschitz class with constant r_j, L_D, such that $\mathrm{diam}(D_j) \leq Q_D r_j$, $\mathrm{dist}(D_i, D_j) \geq \frac{3}{2}(r_i + r_j)$, $r_j \leq \frac{d_0}{2}\rho_0$. The proofs can be extended to this general case by applying the same arguments to each connected component D_j taking care to replace the integrals over $\Omega \setminus \overline{D}$ with integrals over a neighborhood of ∂D_j in $\Omega \setminus \overline{D}$, by summing up the estimates obtained for each j, and by applying the Cauchy-Schwarz inequality.

5 Proof of the size estimates for cavities

The proofs of both Theorems 4.1, 4.2 are based on the following energy lemma.

Lemma 5.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $D \subset\subset \Omega$ a measurable set. Let $S, \mathbb{P}$ given in* (3.5), (3.6) *satisfy the strong convexity conditions* (3.3)*. Let $(\varphi_c, w_c) \in H^1(\Omega \setminus \overline{D}, \mathbb{R}^2) \times H^1(\Omega \setminus \overline{D})$, $(\varphi_0, w_0) \in H^1(\Omega, \mathbb{R}^2) \times H^1(\Omega)$ be* *the weak solutions to problems* (3.16)–(3.21) *and* (3.37)–(3.40)*, respectively. We have*

$$
(5.1)\quad \int_D \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_0+\int_D S(\varphi_0+\nabla w_0)\cdot(\varphi_0+\nabla w_0)\le W_c-W_0=
$$
$$
=\int_D \mathbb{P}\nabla\varphi_c\cdot\nabla\varphi_0+\int_D S(\varphi_c+\nabla w_c)\cdot(\varphi_0+\nabla w_0).
$$

Proof. For every weak solution $(\varphi,w)\in H^1(\Omega\setminus\overline{D},\mathbb{R}^2)\times H^1(\Omega\setminus\overline{D})$ to the system (3.16)–(3.17), we have that

$$
(5.2)\quad \int_{\Omega\setminus\overline{D}}\mathbb{P}\nabla\varphi\cdot\nabla\psi+\int_{\Omega\setminus\overline{D}}S(\varphi+\nabla w)\cdot(\psi+\nabla v)=
$$
$$
=\int_{\partial\Omega}(S(\varphi+\nabla w)\cdot n)v+(\mathbb{P}\nabla\varphi)n\cdot\psi-\int_{\partial D}(S(\varphi+\nabla w)\cdot n)v+(\mathbb{P}\nabla\varphi)n\cdot\psi,
$$

for every $\psi\in H^1(\Omega\setminus\overline{D},\mathbb{R}^2)$ and for every $v\in H^1(\Omega\setminus\overline{D})$, where n denotes the exterior unit normal to Ω and D, respectively.

Choosing in the above identity $(\varphi,w)=(\varphi_0,w_0)$, $(\psi,v)=(\varphi_c,w_c)$, we have

$$
(5.3)\quad \int_{\Omega\setminus\overline{D}}\mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_c+\int_{\Omega\setminus\overline{D}}S(\varphi_0+\nabla w_0)\cdot(\varphi_c+\nabla w_c)=
$$
$$
=W_c-\int_{\partial D}(S(\varphi_0+\nabla w_0)\cdot n)w_c+(\mathbb{P}\nabla\varphi_0)n\cdot\varphi_c.
$$

Similarly, choosing in (5.2) $(\varphi,w)=(\varphi_c,w_c)$, $(\psi,v)=(\varphi_0,w_0)$ and recalling the boundary conditions (3.20)–(3.21), we have

$$
(5.4)\qquad \int_{\Omega\setminus\overline{D}}\mathbb{P}\nabla\varphi_c\cdot\nabla\varphi_0+\int_{\Omega\setminus\overline{D}}S(\varphi_c+\nabla w_c)\cdot(\varphi_0+\nabla w_0)=W_0.
$$

By subtracting (5.4) from (5.3),

$$
(5.5)\qquad W_c-W_0=\int_{\partial D}(S(\varphi_0+\nabla w_0)\cdot n)w_c+(\mathbb{P}\nabla\varphi_0)n\cdot\varphi_c.
$$

On the other hand, by the weak formulation of the system (3.37)–(3.38) in D, recalling the transmission conditions in (3.24) for (φ_c,w_c) and by (5.5), it follows that

$$
(5.6)\qquad \int_D\mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_c+\int_D S(\varphi_0+\nabla w_0)\cdot(\varphi_c+\nabla w_c)=W_c-W_0,
$$

that is the equality in (5.1) is established.

Choosing in (5.2) $(\varphi, w) = (\psi, v) = (\varphi_c - \varphi_0, w_c - w_0)$, recalling that (φ_c, w_c) and (φ_0, w_0) satisfy the same Neumann conditions on $\partial\Omega$ and that (φ_c, w_c) satisfies homogeneous Neumann conditions on ∂D, we have

$$\begin{aligned}(5.7)\quad &\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla(\varphi_c - \varphi_0)\cdot\nabla(\varphi_c - \varphi_0) + \\ &+ \int_{\Omega\setminus\overline{D}} S((\varphi_c - \varphi_0) + \nabla(w_c - w_0))\cdot((\varphi_c - \varphi_0) + \nabla(w_c - w_0)) = \\ &= \int_{\partial D} (S(\varphi_0 + \nabla w_0)\cdot n)(w_c - w_0) + (\mathbb{P}\nabla\varphi_0)n\cdot(\varphi_c - \varphi_0).\end{aligned}$$

Summing (5.4) and (5.6), we obtain

$$(5.8)\qquad \int_\Omega \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_c + \int_\Omega S(\varphi_0 + \nabla w_0)\cdot(\varphi_c + \nabla w_c) = W_c.$$

Subtracting (3.44) from (5.8) and recalling (5.6), we have

(5.9)

$$\begin{aligned}&\int_\Omega \mathbb{P}\nabla\varphi_0\cdot\nabla(\varphi_c - \varphi_0) + \int_\Omega S(\varphi_0 + \nabla w_0)\cdot((\varphi_c - \varphi_0) + \nabla(w_c - w_0)) = \\ &= W_c - W_0 = \int_D \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_c + \int_D S(\varphi_0 + \nabla w_0)\cdot(\varphi_c + \nabla w_c).\end{aligned}$$

By splitting the domain of integration on the left hand side of (5.9) into the union of $\Omega\setminus\overline{D}$ and D, the following identity easily follows

(5.10)

$$\begin{aligned}&\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla(\varphi_c - \varphi_0)\cdot\nabla\varphi_0 + \int_{\Omega\setminus\overline{D}} S((\varphi_c - \varphi_0) + \nabla(w_c - w_0))\cdot(\varphi_0 + \nabla w_0) = \\ &= \int_D \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_0 + \int_D S(\varphi_0 + \nabla w_0)\cdot(\varphi_0 + \nabla w_0).\end{aligned}$$

By adding and subtracting to the left hand side of (5.10) the term $\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla\varphi_c\cdot \nabla(\varphi_c - \varphi_0) + \int_{\Omega\setminus\overline{D}} S(\varphi_c + \nabla w_c)\cdot((\varphi_c - \varphi_0) + \nabla(w_c - w_0))$ and recalling (3.42) and (5.4), we derive

$$\begin{aligned}(5.11)\quad &\int_D \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_0 + \int_D S(\varphi_0 + \nabla w_0)\cdot(\varphi_0 + \nabla w_0) = \\ &= W_c - W_0 - \int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla(\varphi_c - \varphi_0)\cdot\nabla(\varphi_c - \varphi_0) - \\ &- \int_{\Omega\setminus\overline{D}} S((\varphi_c - \varphi_0) + \nabla(w_c - w_0))\cdot((\varphi_c - \varphi_0) + \nabla(w_c - w_0)) \le W_c - W_0,\end{aligned}$$

that is the inequality in (5.1). $\square$

It is convenient to introduce the strain energy density

$$E(\varphi_0, w_0) = \left(|\widehat{\nabla}\varphi_0|^2 + \frac{1}{\rho_0^2}|\varphi_0 + \nabla w_0|^2 \right)^{\frac{1}{2}}. \tag{5.12}$$

Let us notice that, by (3.9)–(3.11), the following double inequality holds

$$m\rho_0^3 E^2(\varphi_0, w_0) \leq \mathbb{P}\nabla\varphi_0 \cdot \nabla\varphi_0 + S(\varphi_0 + \nabla w_0)\cdot(\varphi_0 + \nabla w_0) \leq M\rho_0^3 E^2(\varphi_0, w_0), \tag{5.13}$$

where $m = \min\left\{ \frac{\xi_0}{12}\left(\frac{h}{\rho_0}\right)^3, \alpha_0 \frac{h}{\rho_0} \right\}$, $M = \max\left\{ \frac{\xi_1}{12}\left(\frac{h}{\rho_0}\right)^3, \alpha_1 \frac{h}{\rho_0} \right\}$ only depend on α_0, α_1, γ_0 and $\frac{\rho_0}{h}$.

The second key tool for proving Theorem 4.1 is the following unique continuation property for solutions to (3.37)–(3.40).

Proposition 5.2 (Lipschitz propagation of smallness)**.** *Under the assumptions of Theorem 4.1, for every $\rho > 0$ and for every $x \in \Omega_{\frac{7}{2\theta}\rho}$, we have*

$$\int_{B_\rho(x)} E^2(\varphi_0, w_0) \geq C_\rho \int_\Omega E^2(\varphi_0, w_0), \tag{5.14}$$

where C_ρ only depends on α_0, α_1, γ_0, $\frac{\rho_0}{h}$, L_0, Q_0, $\mathcal{F}$, $\frac{\rho}{\rho_0}$, and where $\theta \in (0,1)$ only depends on $\alpha_0, \alpha_1, \gamma_0, \frac{\rho_0}{h}$.

The above proposition was established in [MRV18, Theorem 4.5].

Proof of Theorem 4.1. Let us cover $D_{h_1\rho_0}$ with internally non overlapping closed squares Q_j of side l, for $j = 1, ..., J$, with $l = \frac{4\theta h_1}{2\sqrt{2}\theta+7}\rho_0$, where $\theta \in (0,1)$ is as in Proposition 5.2. By the choice of l the squares Q_j are contained in D. Hence

$$\int_D E^2(\varphi_0, w_0) \geq \int_{\bigcup_{j=1}^J Q_j} E^2(\varphi_0, w_0) \geq \frac{|D_{h_1\rho_0}|}{l^2} \int_{Q_{\bar{j}}} E^2(\varphi_0, w_0), \tag{5.15}$$

where $\bar{j}$ is such that $\int_{Q_{\bar{j}}} E^2(\varphi_0, w_0) = \min_j \int_{Q_j} E^2(\varphi_0, w_0)$. Let $\bar{x}$ be the center of $Q_{\bar{j}}$. By applying estimate (5.14) with $x = \bar{x}$ and $\rho = l/2$ and using (5.15) and (4.1) we have

$$\int_D E^2(\varphi_0, w_0) \geq C\frac{|D|}{\rho_0^2} \int_\Omega E^2(\varphi_0, w_0), \tag{5.16}$$

where C only depends on α_0, α_1, γ_0, L_0, Q_0, $\frac{\rho_0}{h}$, h_1 and $\mathcal{F}$.

From the left hand side of (5.1), (5.13), (5.16) and (3.44), we have

$$W_c - W_0 \geq m\rho_0^3 \int_D E^2(\varphi_0, w_0) \geq C\rho_0|D| \int_\Omega E^2(\varphi_0, w_0) \geq C\frac{|D|}{\rho_0^2} W_0, \tag{5.17}$$

with C only depending on α_0, α_1, γ_0, L_0, Q_0, $\frac{\rho_0}{h}$, h_1 and $\mathcal{F}$, so that (4.3) follows. □

Let us premise some auxiliary propositions concerning Poincaré and Korn inequalities, which will be used for the proof of Theorem 4.2. In the following three propositions G is meant to be a bounded measurable domain in $\mathbb{R}^2$ having boundary of Lipschitz class with constants ρ and L and satisfying

$$\operatorname{diam}(G) \leq Q\rho. \tag{5.18}$$

Given $u \in H^1(G)$ and given $\Gamma \subset \partial G$, we shall denote

$$u_G = \frac{1}{|G|}\int_G u, \qquad u_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma u. \tag{5.19}$$

Proposition 5.3 (Poincaré-type inequalities). *For every $u \in H^1(G)$ we have*

$$\int_G |u - u_G|^2 \leq C_1\rho^2 \int_G |\nabla u|^2, \tag{5.20}$$

$$\int_G |u - u_\Gamma|^2 \leq C_2\left(1 + \frac{|G|}{\rho|\Gamma|}\right)\rho^2 \int_G |\nabla u|^2, \tag{5.21}$$

where C_1 and C_2 only depend on L, Q.

Moreover, if $u \in H^1(G^\rho)$ then

$$\int_{\partial G} |u - u_{\partial G}|^2 \leq C_3\rho \int_{G^\rho} |\nabla u|^2, \tag{5.22}$$

where $C_3 > 0$ only depends on L, Q.

The above Poincaré-type inequalities are well-known. A precise evaluation of the constants C_1, C_2, C_3 in terms of the scale invariant parameters L, Q regarding the regularity and the shape of G, can be found in the proof of [AMR02, Proposition 3.2].

Proposition 5.4 (Second Korn's inequality). *For every $\varphi \in H^1(G,\mathbb{R}^2)$ satisfying*

$$\int_G (\nabla\varphi - (\nabla\varphi)^T) = 0, \tag{5.23}$$

we have

$$\int_G |\nabla\varphi|^2 \leq C \int_G |\widehat{\nabla}\varphi|^2, \tag{5.24}$$

where $C > 0$ only depends on L and Q.

For a proof of this classical inequality see, for instance, [F], [N].

Proposition 5.5 (Generalized second Korn inequality). *For every $\varphi \in H^1(G,\mathbb{R}^2)$ and for every $w \in H^1(G,\mathbb{R})$,*

$$\|\nabla\varphi\|_{L^2(G)} \leq C \left(\|\widehat{\nabla}\varphi\|_{L^2(G)} + \frac{1}{\rho}\|\varphi + \nabla w\|_{L^2(G)} \right), \tag{5.25}$$

where C only depends on L and Q.

The above Generalized Korn inequality, established in [MRV17, Theorem 4.3], turned out to be a key result in dealing with the direct Neumann problem for Reissner-Mindlin plates (see Proposition 5.2 in [MRV17]) and in deriving unique continuation estimates for system (3.37)–(3.38) (see Theorem 4.2 in [MRV18]) and the Lipschitz propagation of smallness stated in Proposition 5.2. Let us notice that in the statement of Theorem 4.3 in [MRV17] it was made the explicit assumption that the domain contains a disc of radius $s_0\rho$, since this condition plays a fundamental role in the proof and, consequently, the constant C appearing in the inequality (5.25) depends on s_0. This hypothesis, which was emphasized in [MRV17] because of its relevance for the derivation of the estimate, can be deduced from the boundary regularity, with $s_0 = \frac{L}{L^2+1+\sqrt{L^2+1}}$ and therefore it is omitted here.

Proof of Theorem 4.2. Let us consider $D^{r_D} \subset \Omega$ and its boundary $\partial D^{r_D} = \partial D \cup \Gamma^{r_D}$, where $\Gamma^{r_D} = \{x \in \Omega \setminus D \mid dist(x, \partial D) = r_D\}$. Let us tessellate $\mathbb{R}^2$ with internally nonoverlapping closed squares having side $l = \frac{r_D}{2\sqrt{2}}$ and let $Q_1, \ldots, Q_N$ be those squares having nonempty intersection with D^{r_D}. Let us define $\widetilde{D}^{r_D}$ the interior of $\cup_{i=1}^N Q_i \setminus D$. We have that $\partial\widetilde{D}^{r_D} = \partial D \cup \Sigma^{r_D}$, where $\Sigma^{r_D} \subset \cup_{j\in J}\partial Q_j$, with $J = \{j \mid Q_j \cap \Gamma^{r_D} \neq \emptyset\}$. As a portion of the boundary of $\widetilde{D}^{r_D}$, ∂D is of Lipschitz class with constants $\frac{r_D}{\sqrt{L_D^2+1}}$ and L_D. By construction, Σ^{r_D} is of Lipschitz class with constants $\frac{r_D}{8}$ and 1. Therefore $\partial\widetilde{D}^{r_D}$ is of

Lipschitz class with constants γr_D, L', where $\gamma = \left(\max\left\{8, \sqrt{L_D^2+1}\right\}\right)^{-1}$ and $L' = \max\{1, L_D\}$. Moreover, $\operatorname{diam}(\widetilde{D}^{r_D}) \leq (Q_D+3)r_D$. Let

$$x_{\partial D} = \frac{1}{|\partial D|}\int_{\partial D} x$$

be the center of mass of ∂D. Let

$$a = \frac{1}{|\partial D|}\int_{\partial D} w_c,$$

$$b = \frac{1}{|\partial D|}\int_{\partial D} \varphi_c,$$

$$W = \frac{1}{2|\widetilde{D}^{r_D}|}\int_{\widetilde{D}^{r_D}} \nabla\varphi_c - \nabla^T\varphi_c,$$

$$r = b + W(x - x_{\partial D}),$$

$$\varphi_c^* = \varphi_c - r,$$

$$w_c^* = w_c + b\cdot(x - x_{\partial D}) + a.$$

By these definitions, φ_c^* and w_c^* have zero mean on ∂D, and

$$\varphi_c^* + \nabla w_c^* = \varphi_c + \nabla w_c - W(x - x_{\partial D}).$$

By the weak formulation of the Reissner-Mindlin system satisfied by (φ_0, w_0) in D choosing the test pair (φ_c^*, w_c^*), and recalling the right hand side of (5.1), we have

(5.26)
$$\begin{aligned}W_c - W_0 = \int_D \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_c^* + \int_D S(\varphi_0+\nabla w_0)\cdot(\varphi_c^*+\nabla w_c^*+W(x-x_{\partial D})) = \\ = \int_{\partial D}(\mathbb{P}\nabla\varphi_0 n)\cdot\varphi_c^* + \int_{\partial D}(S(\varphi_0+\nabla w_0)\cdot n)w_c^* + \int_D S(\varphi_0+\nabla w_0)\cdot W(x-x_{\partial D}) = \\ = I_1+I_2+I_3.\end{aligned}$$

By applying Hölder inequality and by (3.12),

$$|I_1| \leq C\left(h^3\int_{\partial D}|\nabla\varphi_0|^2\right)^{\frac{1}{2}}\left(h^3\int_{\partial D}|\varphi_c^*|^2\right)^{\frac{1}{2}}, \tag{5.27}$$

with C only depending on α_0, γ_0, α_1.

Recalling (4.6), we can apply interior regularity estimates (see [C, Theorem 6.1]) and then, by taking into account the normalization condition (3.45), by applying Proposition 5.5 to (φ_0, w_0) in Ω, and recalling (5.13) and (3.44), we have

(5.28)
$$h^3 \int_{\partial D} |\nabla \varphi_0|^2 \leq h^3 |\partial D| \|\nabla \varphi_0\|^2_{L^\infty(D)} \leq Ch^3|\partial D| \left(\|\varphi_0\|^2_{H^1(\Omega)} + \frac{1}{\rho_0^2}\|w_0\|^2_{H^1(\Omega)} \right) \leq$$
$$\leq Ch^3|\partial D| \left(\|\widehat{\nabla}\varphi_0\|^2_{L^2(\Omega)} + \frac{1}{\rho_0^2}\|\varphi_0 + \nabla w_0\|^2_{L^2(\Omega)} \right) \leq \frac{C}{\rho_o^2}|\partial D| W_0,$$

with C only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0.

By (5.22), (5.23) and (5.13) we have

$$h^3 \int_{\partial D} |\varphi_c^*|^2 \leq Ch^3 r_D \int_{D^{r_D}} |\nabla \varphi_c^*|^2 \leq Ch^3 r_D \int_{\widetilde{D}^{r_D}} |\nabla \varphi_c^*|^2 \leq$$
$$\leq Ch^3 r_D \int_{\widetilde{D}^{r_D}} |\widehat{\nabla} \varphi_c|^2 \leq C r_D W_c, \tag{5.29}$$

with C only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_D, Q_D.

By using arguments similar to those in [AR98, Lemma 2.8], we have that

$$|\partial D| \leq C\frac{|D|}{r_D}, \tag{5.30}$$

with C only depending on L_D.

By (5.27)–(5.30),

$$|I_1| \leq \frac{C}{\rho_0}|D|^{\frac{1}{2}} W_0^{\frac{1}{2}} W_c^{\frac{1}{2}}, \tag{5.31}$$

with C only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, L_D, Q_D, d_0.

Let us estimate I_3. By Hölder inequality and by (3.12),

$$|I_3| \leq Ch \left(\int_D |\varphi_0 + \nabla w_0|^2 \right)^{\frac{1}{2}} \left(\int_D |W(x - x_{\partial D})|^2 \right)^{\frac{1}{2}} \leq$$
$$\leq Ch \|\varphi_0 + \nabla w_0\|_{L^\infty(D)} |D|^{\frac{1}{2}} |W| \left(\int_D |x - x_{\partial D}|^2 \right)^{\frac{1}{2}}, \tag{5.32}$$

with C only depending on α_1.

By interior regularity estimates, by using the normalization conditions (3.45), the Poincaré inequality (5.20), the Generalized Korn inequality (5.25), and recalling (5.13) and (3.44), we have

$$(5.33)\quad h\|\varphi_0+\nabla w_0\|_{L^\infty(D)}\leq Ch\left(\|\varphi_0\|_{H^1(\Omega)}+\frac{1}{\rho_0}\|w_0\|_{H^1(\Omega)}\right)\leq$$

$$\leq Ch\left(\int_\Omega|\widehat{\nabla}\varphi_0|^2+\frac{1}{\rho_0^2}|\varphi_0+\nabla w_0|^2\right)^{\frac{1}{2}}\leq\frac{C}{\sqrt{\rho_0}}W_0^{\frac{1}{2}},$$

with with C only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, d_0, L_0, Q_0.

By Hölder inequality, by the Generalized Korn inequality (5.25), noticing that $|\widetilde{D}^{r_D}|\geq C{r_D}^2$, with C only depending on L_D, using $r_D<\frac{d_0}{2}\rho_0$, and recalling (5.13) and (3.42), we have

(5.34)

$$|W|\leq\frac{C}{|\widetilde{D}^{r_D}|^{\frac{1}{2}}}\left(\int_{\widetilde{D}^{r_D}}|\nabla\varphi_c|^2\right)^{\frac{1}{2}}\leq\frac{C}{r_D}\left(\int_{\widetilde{D}^{r_D}}|\widehat{\nabla}\varphi_c|^2+\frac{1}{{r_D}^2}|\varphi_c+\nabla w_c|^2\right)^{\frac{1}{2}}\leq$$

$$\leq\frac{C}{{r_D}^2}\left(\int_{\widetilde{D}^{r_D}}\rho_0^2|\widehat{\nabla}\varphi_c|^2+|\varphi_c+\nabla w_c|^2\right)^{\frac{1}{2}}\leq\frac{C}{{r_D}^2\sqrt{\rho_0}}W_c^{\frac{1}{2}},$$

with C only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_D, Q_D, d_0. Moreover,

$$(5.35)\qquad\left(\int_D|x-x_{\partial D}|^2\right)^{\frac{1}{2}}\leq|D|^{\frac{1}{2}}\mathrm{diam}(D)\leq C{r_D}^2$$

with C only depending on Q_D.

By (5.32)–(5.35), it follows that

$$(5.36)\qquad|I_3|\leq\frac{C}{\rho_0}W_0^{\frac{1}{2}}W_c^{\frac{1}{2}}|D|^{\frac{1}{2}},$$

with with C only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, d_0, L_0, Q_0, L_D, Q_D.

By applying Hölder inequality, by (3.12), (5.22), (5.30) and (5.33), we get

$$(5.37)\qquad|I_2|\leq\frac{C}{\sqrt{\rho_0}}W_0^{\frac{1}{2}}|D|^{\frac{1}{2}}\left(\int_{\widetilde{D}^{r_D}}|\nabla w_c^*|^2\right)^{\frac{1}{2}},$$

with with C only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, d_0, L_0, Q_0, L_D, Q_D.

On the other hand,

(5.38)

$$\int_{\widetilde{D}^{r_D}}|\nabla w_c^*|^2=\int_{\widetilde{D}^{r_D}}|\nabla w_c+b|^2\leq2\int_{\widetilde{D}^{r_D}}|\nabla w_c+\varphi_c|^2+2\int_{\widetilde{D}^{r_D}}|\varphi_c-b|^2\leq$$

$$\leq2\int_{\widetilde{D}^{r_D}}|\nabla w_c+\varphi_c|^2+4\int_{\widetilde{D}^{r_D}}|\varphi_c-b-W(x-x_{\partial D})|^2+4\int_{\widetilde{D}^{r_D}}|W(x-x_{\partial D})|^2.$$

By applying the Poincaré inequality (5.21) and the Korn inequality (5.24), using (5.13) and (5.34) and recalling that $\mathrm{diam}(\widetilde{D}^{r_D}) \leq (Q_D+3)r_D$ and $|\widetilde{D}^{r_D}| \leq C{r_D}^2$, with C only depending on Q_D, we have

(5.39)
$$\int_{\widetilde{D}^{r_D}} |\nabla w_c^*|^2 \leq C\int_{\widetilde{D}^{r_D}} |\nabla w_c+\varphi_c|^2 + C{r_D}^2\int_{\widetilde{D}^{r_D}} |\widehat{\nabla}\varphi_c|^2 + C|W|^2\int_{\widetilde{D}^{r_D}} |x-x_{\partial D}|^2 \leq$$
$$\leq C\int_{\widetilde{D}^{r_D}} (|\varphi_c+\nabla w_c|^2 + \rho_0^2|\widehat{\nabla}\varphi_c|^2) + \frac{C}{\rho_0}W_c \leq \frac{C}{\rho_0}W_c,$$

where C only depends on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, d_0, L_D, Q_D.

By (5.37) and (5.39), it follows that

$$|I_2| \leq \frac{C}{\rho_0}W_0^{\frac{1}{2}}W_c^{\frac{1}{2}}|D|^{\frac{1}{2}}, \tag{5.40}$$

where C only depends on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, d_0, L_0, Q_0, L_D, Q_D.

Finally, by (5.26), (5.31), (5.37) and (5.40),

$$W_c - W_0 \leq \frac{C}{\rho_0}W_0^{\frac{1}{2}}W_c^{\frac{1}{2}}|D|^{\frac{1}{2}}, \tag{5.41}$$

where C only depends on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, d_0, L_0, Q_0, L_D, Q_D, and the thesis follows by straightforward calculations. □

6 Proof of the size estimates for rigid inclusions

The comparison between the works W_0 and W_r is stated in the following lemma.

Lemma 6.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ and $D \subset\subset \Omega$ a measurable set. Let S, $\mathbb{P}$ given in* (3.5), (3.6) *satisfy the strong convexity conditions* (3.3)*. Let $(\varphi_r, w_r) \in H^1_D(\Omega,\mathbb{R}^2)\times H^1_D(\Omega)$, $(\varphi_0, w_0) \in H^1(\Omega,\mathbb{R}^2)\times H^1(\Omega)$ be the weak solutions to problems* (3.25)*–*(3.32) *and* (3.37)*–*(3.40)*, respectively. We have*

(6.1)
$$\int_D \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_0 + S(\varphi_0+\nabla w_0)\cdot(\varphi_0+\nabla w_0) \leq W_0 - W_r =$$
$$= \int_{\partial D} (\mathbb{P}\nabla\varphi_r^+)n\cdot\varphi_0 + (S(\varphi_r^+ + \nabla w_r^+)\cdot n)w_0.$$

Proof. The proof of this *energy lemma* can be obtained by adapting the proof of the corresponding result in linear elasticity, see [MR03]. Therefore, we skip the details and we report the main steps of the proof.

Let us multiply equations (3.25), (3.26) by w_0, φ_0, respectively. Integrating by parts and summing up, we obtain

$$\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla\varphi_r^+ \cdot \nabla\varphi_0 + S(\varphi_r^+ + \nabla w_r^+)\cdot(\varphi_0+\nabla w_0) = \\ = W_0 - \int_{\partial D} (\mathbb{P}\nabla\varphi_r^+)n\cdot\varphi_0 + (S(\varphi_r^+ + \nabla w_r^+)\cdot n)w_0. \tag{6.2}$$

Next, we multiply equations (3.37), (3.38) by w_r^+, φ_r^+, respectively, and we integrate by parts in $\Omega\setminus\overline{D}$. Summing up, we obtain

$$W_r = \int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla\varphi_r^+ \cdot \nabla\varphi_0 + S(\varphi_r^+ + \nabla w_r^+)\cdot(\varphi_0+\nabla w_0) - \\ - \int_{\partial D} (\mathbb{P}\nabla\varphi_0)n\cdot\varphi_r^+ + (S(\varphi_0 + \nabla w_0)\cdot n)w_r^+ = \\ = \int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla\varphi_r^+ \cdot \nabla\varphi_0 + S(\varphi_r^+ + \nabla w_r^+)\cdot(\varphi_0+\nabla w_0), \tag{6.3}$$

where, in the last step, we have used the fact that the second integral on the right hand side vanishes because of the definition of (φ_r, w_r) in $\overline{D}$. By (6.2) and (6.3), the equality on the right hand side of (6.1) follows.

To obtain the inequality in (6.1), we consider the quadratic form of the strain energy associated to the pair $(\varphi_0-\varphi_r, w_0-w_r)$ in Ω. Recalling the definition of (φ_r^-, w_r^-) in $\overline{D}$, by (3.43), (3.44) and by (6.3), we have

(6.4)
$$\int_\Omega \mathbb{P}\nabla(\varphi_0-\varphi_r)\cdot\nabla(\varphi_0-\varphi_r) + S((\varphi_0-\varphi_r)+\nabla(w_0-w_r))\cdot((\varphi_0-\varphi_r)+\nabla(w_0-w_r)) = \\ = \int_\Omega \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_0 + S(\varphi_0+\nabla w_0)\cdot(\varphi_0+\nabla w_0) + \\ + \int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla\varphi_r^+\cdot\nabla\varphi_r^+ + S(\varphi_r^+ + \nabla w_r^+)\cdot(\varphi_r^+ + \nabla w_r^+) - \\ - 2\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_r^+ + S(\varphi_0+\nabla w_0)\cdot(\varphi_r^+ + \nabla w_r^+) = W_0 - W_r.$$

Noticing that

$$(6.5)\quad \int_D \mathbb{P}\nabla\varphi_0\cdot\nabla\varphi_0+S(\varphi_0+\nabla w_0)\cdot(\varphi_0+\nabla w_0)=$$

$$=\int_D \mathbb{P}\nabla(\varphi_0-\varphi_r^-)\cdot\nabla(\varphi_0-\varphi_r^-)+S((\varphi_0-\varphi_r^-)+\nabla(w_0-w_r^-))\cdot((\varphi_0-\varphi_r^-)+\nabla(w_0-w_r^-))\leq$$

$$\leq\int_\Omega \mathbb{P}\nabla(\varphi_0-\varphi_r)\cdot\nabla(\varphi_0-\varphi_r)+S((\varphi_0-\varphi_r)+\nabla(w_0-w_r))\cdot((\varphi_0-\varphi_r)+\nabla(w_0-w_r)),$$

by (6.4) the thesis follows. $\square$

Let us notice that the estimate from above stated in Theorem 4.3 can be derived as in the proof of Theorem 4.1.

In order to prove Theorem 4.4 we shall use the following proposition.

Proposition 6.2. *Let the hypotheses of Theorem 4.4 be satisfied. The contact actions exerted by the material in $\Omega\setminus\overline{D}$ on D throughout the boundary ∂D are square summable on ∂D, e.g., $(\mathbb{P}\nabla\varphi_r^+)n\in L^2(\partial D,\mathbb{R}^2)$ and $S(\varphi_r^++\nabla w_r^+)\cdot n\in L^2(\partial D)$, and the following estimate holds*

$$(6.6)\quad \int_{\partial D}|(\mathbb{P}\nabla\varphi_r^+)n|^2+\rho_0^2|S(\varphi_r^++\nabla w_r^+)\cdot n|^2\leq C\frac{\rho_0}{r_D}\int_{\Omega\setminus\overline{D}}\rho_0^5|\widehat{\nabla}\varphi_r^+|^2+\rho_0^3|\varphi_r^++\nabla w_r^+|^2,$$

where n denotes the outer unit normal to D and the constant $C>0$ only depends on Q_0, d_0, L_D, Q_D, α_0, α_1, γ_0.

Proof of Theorem 4.4. By using (3.33) and (3.34), the right hand side of (6.1) can be written as

$$(6.7)\quad W_0-W_r=\int_{\partial D}((\mathbb{P}\nabla\varphi_r^+)n-(S(\varphi_r^++\nabla w_r^+)\cdot n)x)\cdot(\varphi_0-\varphi_{0,\partial D})+$$

$$+\int_{\partial D}(S(\varphi_r^++\nabla w_r^+)\cdot n)x\cdot\varphi_0+\int_{\partial D}(S(\varphi_r^++\nabla w_r^+)\cdot n)(w_0-w_{0,\partial D})\equiv I_1+I_2+I_3.$$

By applying Hölder's inequality and Poincaré's inequality (5.22) we have

$$(6.8)\quad |I_1|\leq Cr_D^{1/2}\left(\int_D|\nabla\varphi_0|^2\right)^{1/2}\left(\int_{\partial D}|(\mathbb{P}\nabla\varphi_r^+)n|^2+\rho_0^2|S(\varphi_r^++\nabla w_r^+)\cdot n|^2\right)^{1/2},$$

where $C>0$ only depends on Q_0, L_D, Q_D.

By interior regularity estimates, by the generalized Korn inequality (5.25) (applied to (φ_0,w_0) in Ω), and by recalling (5.13) and (3.44), we have

$$(6.9)\quad \left(\int_D|\nabla\varphi_0|^2\right)^{1/2}\leq\frac{C}{\rho_0^{5/2}}|D|^{1/2}W_0^{1/2},$$

where $C>0$ only depends on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0. Therefore, by (6.8) and (6.9), we have
(6.10)
$$|I_1|\leq \frac{C}{\rho_0^{5/2}}r_D^{1/2}|D|^{1/2}W_0^{1/2}\left(\int_{\partial D}|(\mathbb{P}\nabla\varphi_r^+)n|^2+\rho_0^2|S(\varphi_r^++\nabla w_r^+)\cdot n|^2\right)^{1/2},$$
where the constant $C>0$ only depends on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0, L_D, Q_D.

By using similar estimates, we get

$$|I_3|\leq \frac{C}{\rho_0^{5/2}}r_D^{1/2}|D|^{1/2}W_0^{1/2}\left(\int_{\partial D}\rho_0^2|S(\varphi_r^++\nabla w_r^+)\cdot n|^2\right)^{1/2}, \tag{6.11}$$

where the constant $C>0$ only depends on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0, L_D, Q_D.

By (3.33) and by using Hölder's inequality, the integral I_2 can be dominated as follows

$$\begin{aligned} I_2 &= \int_{\partial D}(S(\varphi_r^++\nabla w_r^+)\cdot n)(x\cdot\varphi_0-(x\cdot\varphi_0)_{\partial D})\leq \\ &\leq\left(\int_{\partial D}|x\cdot\varphi_0-(x\cdot\varphi_0)_{\partial D}|^2\right)^{1/2}\left(\int_{\partial D}|S(\varphi_r^++\nabla w_r^+)\cdot n|^2\right)^{1/2}. \end{aligned}\tag{6.12}$$

Noticing that $\nabla(x\cdot\varphi_0)=\varphi_0+(\nabla\varphi_0)^Tx$, the first integral on the right hand side of (6.12) can be estimated by using Proposition 5.3, interior regularity estimates for $\nabla\varphi_0$, the generalized Korn's inequality (5.25) (applied to (φ_0,w_0) in Ω), inequality (5.13) and the definition of W_0 in (3.44), obtaining

(6.13)
$$\begin{aligned}\int_{\partial D}|x\cdot\varphi_0-(x\cdot\varphi_0)_{\partial D}|^2 &\leq Cr_D\int_D|\nabla(x\cdot\varphi_0)|^2\leq Cr_D\int_D|\varphi_0|^2+|x|^2|\nabla\varphi_0|^2\leq \\ &\leq Cr_D|D|(\|\varphi_0\|^2_{L^\infty(D)}+\rho_0^2\|\nabla\varphi_0\|^2_{L^\infty(D)})\leq\frac{C}{\rho_0^3}r_D|D|W_0,\end{aligned}$$

where $C>0$ only depends on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0, L_D, Q_D. Inserting the above estimate in (6.12) we have

$$I_2\leq \frac{C}{\rho_0^{3/2}}r_D^{1/2}|D|^{1/2}W_0^{1/2}\left(\int_{\partial D}|S(\varphi_r^++\nabla w_r^+)\cdot n|^2\right)^{1/2}, \tag{6.14}$$

where the constant $C>0$ only depends on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0, L_D, Q_D.

By (6.7), (6.10), (6.11), (6.14) and by Proposition 6.2, we have

$$W_0 - W_r \leq \frac{C}{\rho_0^{3/2}}|D|^{1/2}W_0^{1/2}\left(\int_{\Omega\setminus\overline{D}}\rho_0^4|\widehat{\nabla}\varphi_r^+|^2+\rho_0^2|\varphi_r^+ + \nabla w_r^+|^2\right)^{1/2}, \tag{6.15}$$

with $C>0$ only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0, L_D, Q_D.

To conclude, by the strong convexity of $\mathbb{P}$ and S, recalling (5.13) and (3.43), we have

$$\begin{aligned}\int_{\Omega\setminus\overline{D}}\rho_0^4|\widehat{\nabla}\varphi_r^+|^2+\rho_0^2|\varphi_r^+ + \nabla w_r^+|^2 \leq \\ \leq C\rho_0\int_{\Omega\setminus\overline{D}}\mathbb{P}\nabla\varphi_r^+\cdot\nabla\varphi_r^+ + S(\varphi_r^+ + \nabla w_r^+)\cdot(\varphi_r^+ + \nabla w_r^+) = C\rho_0 W_r,\end{aligned} \tag{6.16}$$

with $C>0$ only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0, L_D, Q_D. Therefore, by (6.15) and (6.16), we have

$$W_0 - W_r \leq \frac{C}{\rho_0}|D|^{1/2}W_0^{1/2}W_r^{1/2}, \tag{6.17}$$

with $C>0$ only depending on α_0, γ_0, α_1, $\frac{\rho_0}{h}$, L_0, Q_0, d_0, L_D, Q_D. By some algebra, estimate (4.11) follows. □

The remaining part of the section is devoted to the proof of Proposition 6.2. The main idea consists in estimating the $L^2(\partial D)$-norm of the conormal derivatives $(\mathbb{P}\nabla\varphi_r^+)n$, $S\nabla w_r^+\cdot n$ in terms of the strain energy stored in $\Omega\setminus\overline{D}$ and the $L^2(\partial D)$-norm of the tangential component of the gradient of φ_r^+ and w_r^+.

We start by introducing some notation.

Given $\rho>0$, $L>0$ and a Lipschitz continuous function $\psi:(-2\rho,2\rho)\to\mathbb{R}$ satisfying $\psi(0)=0$, $\|\psi\|_{C^{0,1}((-2\rho,2\rho))}\leq 2\rho L$, let us define for every t, $0<t\leq 2\rho$,

$$C_t^+ = \{(x_1,x_2)\in\mathbb{R}^2 \mid |x_1|<t,\ \psi(x_1)<x_2<Lt\}, \tag{6.18}$$

$$\Delta_t = \{(x_1,x_2)\in\mathbb{R}^2 \mid |x_1|<t,\ x_2=\psi(x_1)\}. \tag{6.19}$$

We shall use the following two-dimensional version of the constructive Korn-type inequality on cylindrical domains due to Kondrat'ev and Oleinik [KO89].

Proposition 6.3. ([KO89], Theorem 2) *Let*

$$C_{l',l} = \{(x_1,x_2)\in\mathbb{R}^2 \mid |x_1|<l',\ -l<x_2<l\}, \tag{6.20}$$

where $l > l'$. For every $u \in H^1(C_{l',l}, \mathbb{R}^2)$ such that $u = 0$ on $\{x_2 = -l\}$, we have

$$\int_{C_{l',l}} |\nabla u|^2 \leq C\left(1 + \frac{4l^2}{l'^2}\right) \int_{C_{l',l}} |\widehat{\nabla} u|^2, \tag{6.21}$$

where $C > 0$ is an absolute constant.

The next proposition states local boundary estimates in L^2 of the conormal derivatives of solutions to the Mindlin-Reissner plate problem. A proof shall be presented at the end of this section.

Proposition 6.4. *Let S, $\mathbb{P}$ given in (3.5), (3.6) satisfy the strong convexity conditions (3.3).*

Let $w \in H^1(C^+_{2\rho})$ be a solution to

$$\mathrm{div}(S\nabla w) = -\mathrm{div}(S\varphi) \quad \text{in } C^+_{2\rho}, \tag{6.22}$$

with $\varphi \in H^1(C^+_{2\rho}, \mathbb{R}^2)$.

If $w|_{\Delta_{2\rho}} \in H^1(\Delta_{2\rho})$, then $S\nabla w \cdot n \in L^2(\Delta_\rho)$ and we have

$$\int_{\Delta_\rho} |S\nabla w \cdot n|^2 \leq C\left(\int_{\Delta_{2\rho}} \rho_0^2 |\nabla_T w|^2 + \left(1 + \frac{\rho_0}{\rho}\right) \int_{C^+_{2\rho}} \rho_0 |\nabla w|^2 + \int_{C^+_{2\rho}} \rho_0 |\varphi|^2 + \rho_0^3 |\nabla \varphi|^2\right) \tag{6.23}$$

where $\nabla_T w$ is the tangential component of ∇w, and the constant $C > 0$ only depends on L, α_0, γ_0, α_1.

Let $\varphi \in H^1(C^+_{2\rho}, \mathbb{R}^2)$ be a solution to

$$\mathrm{div}(\mathbb{P}\nabla \varphi) = S(\varphi + \nabla w) \quad \text{in } C^+_{2\rho}, \tag{6.24}$$

with $w \in H^1(C^+_{2\rho})$.

If $\varphi|_{\Delta_{2\rho}} \in H^1(\Delta_{2\rho}, \mathbb{R}^2)$, then $(\mathbb{P}\nabla\varphi)n \in L^2(\Delta_\rho, \mathbb{R}^2)$ and we have

$$\int_{\Delta_\rho} |(\mathbb{P}\nabla\varphi)n|^2 \leq C\left(\int_{\Delta_{2\rho}} \rho_0^6 |\nabla_T \varphi|^2 + \left(1 + \frac{\rho_0}{\rho}\right) \int_{C^+_{2\rho}} \rho_0^3 |\varphi|^2 + \rho_0^5 |\nabla\varphi|^2 + \int_{C^+_{2\rho}} \rho_0^3 |\nabla w|^2\right) \tag{6.25}$$

where $\nabla_T \varphi$ is the tangential component of $\nabla\varphi$, and the constant $C > 0$ only depends on L, α_0, γ_0, α_1.

Proof of Proposition 6.2. We follow the lines of the proof derived in [AMR02] (Proposition 3.4) for the analogous estimate in an electric conductor, see also [MR03] (Proposition 3.3).

We cover ∂D with internally non-overlapping closed cubes Q_j, $j=1,...,J$, having side $\widetilde{r}_D=\gamma(L_D)r_D$, where $\gamma(L_D)=\frac{\min\{1,L_D\}}{2\sqrt{2}\sqrt{1+L_D^2}}$. The number of these cubes can be evaluated by a slight modification of the arguments in Lemma 2.8 of [AR98], that is

$$J \le C\frac{|D|}{r_D^2} \le CQ_D^2, \tag{6.26}$$

where $C>0$ only depends on L_D.

For every $j=1,...,J$ there exists $x_0\in\partial D\cap Q_j$ such that $Q_j\cap(\Omega\setminus\overline{D})\subset C^+_{\overline{r}}$, where $\overline{r}=\frac{r_D}{2\sqrt{1+L_D^2}}$ and $C_t^+=\{y=(y_1,y_2)\in\Omega\setminus\overline{D}\mid |y_1|<t,\psi(y_1)<y_2<tL_D\}$ for every t, $0<t\le 2\overline{r}$. Here, ψ is a Lipschitz function in $(-2\overline{r},2\overline{r})$ satisfying $\psi(0)=0$ and $\|\psi\|_{C^{0,1}((-2\overline{r},2\overline{r}))}\le 2\overline{r}L_D$, representing locally the boundary of D in a suitable coordinate system $y=(y_1,y_2)$, $y=Rx$, where R is an orthogonal transformation and $x=(x_1,x_2)$ is the referential cartesian coordinate system.

Recalling (3.29)–(3.32), it is not restrictive to choose (φ_r,w_r) such that

$$\varphi_r\equiv 0,\quad w_r\equiv 0\quad \text{in } \overline{D}. \tag{6.27}$$

By the change of variables $y=Rx$, the pair $(\varphi_r^+=\varphi_r^+(R^Ty),w_r^+=w_r^+(R^Ty))$ satisfies

$$\operatorname{div}_y(S\nabla_y w_r^+)=-\operatorname{div}_y(SR\varphi_r^+)\quad \text{in } C^+_{2\overline{r}} \tag{6.28}$$

and

$$\operatorname{div}_y(\widetilde{\mathbb{P}}(y)\nabla_y\varphi_r^+)=S(\varphi_r^++R^T\nabla_y w_r^+)\quad \text{in } C^+_{2\overline{r}}, \tag{6.29}$$

where $S=S(R^Ty)$ and $\widetilde{\mathbb{P}}(y)[A]=R\mathbb{P}(R^Ty)[R^TAR]R^T$ for every 2×2 matrix A. The tensor $\widetilde{\mathbb{P}}$ belongs to $C^{0,1}(C^+_{2\overline{r}})$, with $\|\widetilde{\mathbb{P}}\|_{C^{0,1}(C^+_{2\overline{r}})}\le Ch^3$, where $C>0$ only depends on α_0, α_1 and γ_0. Moreover, $\widetilde{\mathbb{P}}$ satisfies the strong convexity condition (3.10).

Recalling that $w_r^+=0$ on ∂D and by applying (6.23) with $\rho=\overline{r}$, we have

$$\int_{Q_j\cap\partial D}|S\nabla w_r^+\cdot n|^2\le C\left(1+\frac{\rho_0}{r_D}\right)\int_{C^+_{2\overline{r}}}\rho_0|\nabla w_r^+|^2+C\rho_0\int_{C^+_{2\overline{r}}}|\varphi_r^+|^2+\rho_0^2|\nabla\varphi_r^+|^2, \tag{6.30}$$

where $C>0$ only depends on L_D, α_0, γ_0, α_1. Similarly, since $\varphi_r^+=0$ on ∂D, by applying estimate (6.25) with $\rho=\overline{r}$ we obtain

$$\int_{Q_j\cap\partial D}|(\mathbb{P}\nabla\varphi_r^+)n|^2\le C\left(1+\frac{\rho_0}{r_D}\right)\int_{C^+_{2\overline{r}}}\rho_0^3|\varphi_r^+|^2+\rho_0^5|\nabla\varphi_r^+|^2+C\int_{C^+_{2\overline{r}}}\rho_0^3|\nabla w_r^+|^2, \tag{6.31}$$

where $C>0$ only depends on L_D, α_0, γ_0, α_1.

Let us consider the cylinder

$$C^* = \{(y_1,y_2)\in\mathbb{R}^2 \mid |y_1|<\overline{r},\ |y_2|<L'\overline{r}\}, \tag{6.32}$$

where $L'=\max\{L,2-L\}$, and let $\varphi_r^*\in H^1(C^*,\mathbb{R}^2)$ be defined as follows:

$$\varphi_r^* = \begin{cases} \varphi_r^+ & \text{in } C_{\overline{r}}^+, \\ \\ 0 & \text{in } C^*\setminus C_{\overline{r}}^+. \end{cases} \tag{6.33}$$

By applying the Poincaré inequality $\int_{C_{2\overline{r}}^+}|\varphi_r^+|^2 \leq Cr_D^2\int_{C_{2\overline{r}}^+}|\nabla\varphi_r^+|^2$, with $C>0$ only depending on L_D, the Korn-type inequality (6.21) to φ_r^*, and by (6.31), we have

$$\int_{Q_j\cap\partial D}|(\mathbb{P}\nabla\varphi_r^+)n|^2 \leq C\left(1+\frac{\rho_0}{r_D}\right)\int_{C_{2\overline{r}}^+}\rho_0^5|\widehat{\nabla}\varphi_r^+|^2 + C\int_{C_{2\overline{r}}^+}\rho_0^3|\nabla w_r^+|^2, \tag{6.34}$$

where $C>0$ only depends on L_D, α_0, γ_0, α_1.

Finally, in order to estimate locally the L^2 norm of the contact forces $S(\varphi_r^+ + \nabla w_r^+)\cdot n$ on the boundary of D, we rewrite inequality (6.30) as follows

$$\begin{aligned} &\int_{Q_j\cap\partial D}|S(\varphi_r^+ + \nabla w_r^+)\cdot n|^2 \leq \\ &\leq C\left(1+\frac{\rho_0}{r_D}\right)\int_{C_{2\overline{r}}^+}\rho_0|\nabla w_r^+|^2 + C\rho_0\int_{C_{2\overline{r}}^+}|\varphi_r^+|^2+\rho_0^2|\nabla\varphi_r^+|^2 + C\int_{Q_j\cap\partial D}\rho_0^2|\varphi_r^+|^2, \end{aligned} \tag{6.35}$$

where $C>0$ only depends on L_D, α_0, γ_0, α_1. Recalling that $\varphi_r^+=0$ on ∂D, by using Poincaré inequalities and the Korn-type inequality (6.21), we have

$$\int_{Q_j\cap\partial D}|S(\varphi_r^+ + \nabla w_r^+)\cdot n|^2 \leq C\left(1+\frac{\rho_0}{r_D}\right)\int_{C_{2\overline{r}}^+}\rho_0|\nabla w_r^+|^2 + C\int_{C_{2\overline{r}}^+}\rho_0^3|\widehat{\nabla}\varphi_r^+|^2, \tag{6.36}$$

where $C>0$ only depends on L_D, α_0, γ_0, α_1.

By summing (6.34) and (6.36), using the normalization (6.27), by applying Poincaré's inequality (5.21) and the Korn-type inequality (6.21) we have

$$
\begin{aligned}
(6.37)\quad \int_{Q_j\cap\partial D} |(\mathbb{P}\nabla\varphi_r^+)n|^2 + \rho_0^2|S(\varphi_r^+ + \nabla w_r^+)\cdot n|^2 \leq \\
\leq C\left(1+\frac{\rho_0}{r_D}\right)\int_{C^+_{2\overline{r}}} \rho_0^5|\widehat{\nabla}\varphi_r^+|^2 + \rho_0^3|\nabla w_r^+|^2 \leq \\
\leq C\left(1+\frac{\rho_0}{r_D}\right)\left(\int_{C^+_{2\overline{r}}} \rho_0^5|\widehat{\nabla}\varphi_r^+|^2 + \rho_0^3|\varphi_r^+ + \nabla w_r^+|^2 + C\int_{C^+_{2\overline{r}}}\rho_0^3|\varphi_r^+|^2\right) \leq \\
\leq C\left(1+\frac{\rho_0}{r_D}\right)\int_{C^+_{2\overline{r}}} \rho_0^5|\widehat{\nabla}\varphi_r^+|^2 + \rho_0^3|\varphi_r^+ + \nabla w_r^+|^2,
\end{aligned}
$$

where $C>0$ only depends on L_D, α_0, γ_0, α_1.

Since $1+\rho_0/r_D \leq (1+d_0/2)\rho_0/r_D$, and recalling (6.26), we obtain the wished estimate (6.6). □

We conclude the section with a proof of Proposition 6.4, which is based on the following result.

Lemma 6.5. *Let S, $\mathbb{P}$ given in* (3.5), (3.6) *satisfy the strong convexity conditions* (3.9), (3.10) *and the regularity conditions in* (3.12).

For every $w \in H^{3/2}(C^+_{2\rho})$ such that $\mathrm{div}(S\nabla w) \in L^2(C^+_{2\rho})$ *and $w = |\nabla w| = 0$ on $\partial C^+_{2\rho}\setminus\Delta_{2\rho}$, we have*

(6.38)

$$
\int_{\Delta_\rho}|S\nabla w\cdot n|^2 \leq C\left(\rho_0^2\int_{\Delta_{2\rho}}\rho_0|\nabla_T w|^2 + \rho_0\int_{C^+_{2\rho}}|\nabla w|^2 + |\nabla w||\mathrm{div}(S\nabla w)|\right),
$$

where $C>0$ only depends on L, α_0, γ_0, α_1.

For every $\varphi \in H^{3/2}(C^+_{2\rho},\mathbb{R}^2)$ such that $\mathrm{div}(\mathbb{P}\nabla\varphi) \in L^2(C^+_{2\rho},\mathbb{R}^2)$ *and $|\varphi| = |\nabla\varphi| = 0$ on $\partial C^+_{2\rho}\setminus\Delta_{2\rho}$, we have*

(6.39)

$$
\int_{\Delta_\rho}|(\mathbb{P}\nabla\varphi)n|^2 \leq C\left(\int_{\Delta_{2\rho}}\rho_0^6|\nabla_T\varphi|^2 + \int_{C^+_{2\rho}}\rho_0^5|\nabla\varphi|^2 + \rho_0^3|\nabla\varphi||\mathrm{div}(\mathbb{P}\nabla\varphi)|\right),
$$

where $C>0$ only depends on L, α_0, γ_0, α_1.

Proof. The proof follows the lines of the proof of the analogous result obtained in conductivity and elasticity context, see [AMR02] (Lemma 5.2) and [MR03] (Lemma 4.3), respectively. The key mathematical tool is a generalization of the well-known Rellich's identity [R]. □

Proof of Proposition 6.4. The proof can be obtained by adapting the arguments used, for example, in the proof of the analogous result in three-dimensional elasticity [MR03] (Proposition 4.2), see also [AMR02] (Proposition 5.1). Moreover, the proof of the estimates (6.23) and (6.25) follows the same path. Therefore, we sketch the proof of the inequality (6.23) only.

We first prove the thesis under the additional assumption that $w \in H^{3/2}(C_{2t}^+)$ for every $t < \rho$.

Let us introduce the cut-off function in $\mathbb{R}^2$

$$\eta(x_1, x_2) = \chi(x_1)\Psi(x_2), \tag{6.40}$$

where

$$\chi \in C_0^\infty(\mathbb{R}), \quad \chi(x_1) \equiv 1 \text{ if } |x_1| \leq \rho, \tag{6.41}$$

$$\chi(x_1) \equiv 0 \text{ if } |x_1| \geq \frac{3}{2}\rho, \tag{6.42}$$

$$\|\chi'\|_\infty \leq C_1\rho^{-1}, \quad \|\chi''\|_\infty \leq C_1\rho^{-2}, \tag{6.43}$$

$$\Psi \in C_0^\infty(\mathbb{R}), \quad \psi(x_2) \equiv 1 \text{ if } |x_2| \leq \rho L, \tag{6.44}$$

$$\psi(x_2) \equiv 0 \text{ if } |x_2| \geq \frac{3}{2}\rho L, \tag{6.45}$$

$$\|\psi'\|_\infty \leq C_2\rho^{-1}, \quad \|\psi''\|_\infty \leq C_2\rho^{-2}, \tag{6.46}$$

where C_1 is an absolute constant and C_2 is a constant only depending on L.

For every $c \in \mathbb{R}$ the function

$$u = \eta(w - c) \tag{6.47}$$

satisfies the hypotheses of Lemma 6.5 with $\rho = t$, for every $t \in \left(\frac{3}{4}\rho, \rho\right)$.

By substituting (6.47) in (6.38), and recalling (6.22), we have

$$\begin{aligned}
&\int_{\Delta_t} S^2 \left((w-c)^2|\nabla\eta\cdot n|^2 + \eta^2|\nabla w\cdot n|^2 + 2\eta(w-c)(\nabla\eta\cdot n)(\nabla w\cdot n)\right) \leq \\
&\qquad \leq C\rho_0^2 \int_{\Delta_{2t}} (w-c)^2|\nabla_T\eta|^2 + \eta^2|\nabla_T w|^2 + 2\eta(w-c)\nabla_T\eta\cdot\nabla_T w + \\
&+ C\rho_0 \int_{C_{2t}^+} (w-c)^2|\nabla\eta|^2 + \eta^2|\nabla w|^2 + 2\eta(w-c)\nabla\eta\cdot\nabla w + |\varphi|^2 + \rho_0^2|\nabla\varphi|^2,
\end{aligned} \tag{6.48}$$

where $C>0$ only depends on L, α_0, γ_0, α_1.

By recalling (6.41)–(6.46), by using Schwarz inequality and $2ab \leq a^2/\epsilon + \epsilon b^2$, for every $\epsilon > 0$, we obtain

$$\int_{\Delta_t} |S\nabla w\cdot n|^2 \leq \tag{6.49}$$

$$\leq C\left(\rho_0^2\int_{\Delta_{2t}} \frac{(w-c)^2}{t^2} + |\nabla_T w|^2 + \rho_0\int_{C_{2t}^+} \frac{(w-c)^2}{t^2} + |\nabla w|^2 + |\varphi|^2 + \rho_0^2|\nabla\varphi|^2\right),$$

$$\text{for every } t\in\left(\frac{3}{4}\rho,\rho\right),$$

where $C>0$ only depends on L, α_0, γ_0, α_1.

Choosing $c = \frac{1}{|C_{2t}^+|}\int_{C_{2t}^+} w$, by applying trace inequalities and Poincaré's inequality (5.20), we have

$$\int_{\Delta_t} |S\nabla w\cdot n|^2 \leq \tag{6.50}$$

$$\leq C\left(\rho_0^2\int_{\Delta_{2t}} |\nabla_T w|^2 + \left(1+\frac{\rho_0}{t}\right)\int_{C_{2t}^+}\rho_0|\nabla w|^2 + \int_{C_{2t}^+}\rho_0|\varphi|^2 + \rho_0^3|\nabla\varphi|^2\right),$$

$$\text{for every } t\in\left(\frac{3}{4}\rho,\rho\right),$$

where $C>0$ only depends on L, α_0, γ_0, α_1. Passing to the limit for $t\to\rho$, we obtain (6.23).

We notice that if the function Ψ representing the boundary Δ_{2t} is smooth, then the additional assumption made at the beginning of this proof (e.g., $w\in H^{3/2}(C_{2t}^+)$ for every $t<\rho$) is satisfied by regularity estimates up to the boundary for solutions to (6.22). When Δ_{2t} is represented by a Lipschitz function, the thesis can be obtained by following the approximation argument presented in [MR03] (Step 2 of Proposition 4.2). $\square$

References

[AR98] G. Alessandrini and E. Rosset, The inverse conductivity problem with one measurement: bounds on the size of the unknown object, SIAM Journal on Applied Mathematics 58(4) (1998) 1060–1071.

[ARS00] G. Alessandrini, E. Rosset and J.K. Seo, Optimal size estimates for the inverse conductivity problem with one measurement, Proceedings of the Americal Mathematical Society 128 (2000) 53–64.

[AMR02] G. Alessandrini, A. Morassi and E. Rosset, Detecting cavities by electrostatic boundary measurements, Inverse Problems 18 (2002) 1333–1353.

[BCOZ] E. Beretta, C. Cavaterra, J.H. Ortega and S. Zamorano, Size estimates of an obstacle in a stationary Stokes fluid, Inverse Problems 33 (2017) 025008.

[C] S. Campanato, *Sistemi ellittici in forma divergenza. Regolarità all'interno.* Quaderni della Scuola Normale Superiore di Pisa, 1980.

[DiCLW13] M. Di Cristo, C.L. Lin and J.N. Wang, Quantitative uniqueness estimates for the shallow shell system and their application to an inverse problem, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 12 (2013) 43–92.

[DiCLVW13] M. Di Cristo, C.L. Lin, S. Vessella and J.N. Wang, Size estimates of the inverse inclusion problem for the shallow shell equation, SIAM Journal on Mathematical Analysis 45 (2013) 88–100.

[FKP91] R.A. Fefferman, C.E. Kenig and J. Pipher, The theory of weights and the Dirichlet problem for elliptic equations, Annals of Mathematics 134 (1991) 65–124.

[F] K.O. Friedrichs, On the boundary value problems of the theory of elasticity and Korn's inequality, Annals of Mathematics 48 (1947) 441–471.

[I98] M. Ikehata, Size estimation of inclusion, Journal of Inverse and Ill-Posed Problems 6 (1998) 127–140.

[KKM12] H. Kang, E. Kim and G.W. Milton, Sharp bounds on the volume fractions of two materials in a two-dimensional body from electrical boundary measurements: the translation method, Calculus of Variations and Partial Differential Equations 45 (2012) 367–401.

[KM13] H. Kang and G.W. Milton, Bounds on the volume fractions of two materials in a three-dimensional body from boundary measurements by the translation method, SIAM Journal on Applied Mathematics 73 (2013) 475–492.

[KSS97] H. Kang, J.K. Seo and D. Sheen, The inverse conductivity problem with one measurement: stability and estimation of size, SIAM Journal on Mathematical Analysis 128 (1997) 1389–1405.

[KP93] C.E. Kenig and J. Pipher, The Neumann problem for elliptic equations with non-smooth coefficients, Inventiones Mathematicae 113 (1993) 447–509.

[KO89] V.A. Kondrat'ev and O.A. Oleinik, On the dependence of the constant in Korn's inequality on parameters characterizing the geometry of the region, Russian Mathematical Surveys 44 (1989) 187–195.

[MN12] G.W. Milton and L.H. Nguyen, Bounds on the volume fraction of 2-phase, 2-dimensional elastic bodies and on (stress, strain) pairs in composite, Comptes Rendus Mécanique 340 (2012) 193–204.

[MR03] A. Morassi and E. Rosset, Detecting rigid inclusions, or cavities, in an elastic body, Journal of Elasticity 73 (2003) 101–126.

[MRV13] A. Morassi, E. Rosset and S. Vessella, Recent results about the detection of unknown boundaries and inclusions in elastic plates, Journal of Inverse and Ill-Posed Problems 21 (2013) 311–352.

[MRV17] A. Morassi, E. Rosset and S. Vessella, A generalized Korn inequality and strong unique continuation for the Reissner-Mindlin plate system, Journal of Differential Equations 263 (2017) 811–840.

[MRV18] A. Morassi, E. Rosset and S. Vessella, Size estimates for fat inclusions in an isotropic Reissner-Mindlin plate, Inverse Problems, 34 (2018) 025001, 1–26.

[N] J. Necas, *Les méthods directs en théorie des équations élliptiques*, Academia, Prague, 1967.

[PW58] L.E. Payne and H.F. Weinberger, New bounds for solutions of second order elliptic partial differential equations, Pacific Journal of Mathematics 8 (1958) 551–573.

[R] F. Rellich, Darstellung der eigenwerte $\Delta u + \lambda u = 0$ durch ein Randintegral, Mathematische Zeitschrift 46 (1940) 635–646.