Global stability for an inverse problem in soil-structure interaction *

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Abstract

We consider the inverse problem of determining the Winkler subgrade reaction coefficient of a slab foundation modelled as a thin elastic plate clamped at the boundary. The plate is loaded by a concentrated force and its transversal deflection is measured at the interior points. We prove a global Hölder stability estimate under (mild) regularity assumptions on the unknown coefficient.

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1 Introduction

The soil-structure interaction is an important issue in structural building design. The determination of the contact actions exchanged between foundation and soil is commonly approached by using simplified models of interaction. Among these, the model introduced by Winkler in the second half of the nineteenth century is one of the most popular in engineering [W] and geotechnical [Mo-Ma] applications. In Winkler's model, the foundation rests on a bed of linearly elastic springs of stiffness k, $k \geq 0$, acting along the vertical direction only. The

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springs are independent of each other, that is, the deflection of every spring is not influenced by the other adjacent springs. The accuracy of this model of interaction depends strongly on the values assigned to the subgrade reaction coefficient k. Ranges of average values of k are available in literature from extensive series of in-situ experiments performed on various soil types, [C-G], but these values are quite scattered and, in addition, they may vary significantly from point to point in the case of large foundations. Estimate of the coefficient k becomes even more difficult for existing buildings, since the soil on which the foundation is resting is not directly accessible for experiments. For the reasons stated above, the development of a method for the determination of k is an inverse problem of current interest in practice.

In this paper we consider the stability issue in determining the Winkler's subgrade coefficient of a slab foundation from the measurement of the deflection induced at interior points by a given load condition. The mechanical model is as follows. The slab foundation is described as a thin elastic plate with uniform thickness h and middle surface coinciding with a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$. The plate is assumed to be clamped at the boundary $\partial\Omega$, a condition that occurs when the slab foundation is anchored to sufficiently rigid vertical walls. A concentrated force of intensity f is supposed to act at an internal point $P_0 \in \Omega$. This load condition has the merit of being easy to implement in practice. According to the Winkler model of soil and working in the framework of the Kirchhoff-Love theory of plates, the transversal displacement w of the plate satisfies the fourth order Dirichlet boundary value problem

$$\begin{cases} \operatorname{div}\left(\operatorname{div}\left(\frac{h^3}{12}\mathbb{C}\nabla^2 w\right)\right) + kw = f\delta(P_0), & \text{in } \Omega, \quad (1.1) \\ w = 0, & \text{on } \partial\Omega, \quad (1.2) \\ \frac{\partial w}{\partial n} = 0, & \text{on } \partial\Omega, \quad (1.3) \end{cases}$$

where $\mathbb{C}$ is the elasticity tensor of the material and n is the unit outer normal to $\partial\Omega$. Given the concentrated force $f\delta(P_0)$ and the coefficient k, $k \in L^\infty(\Omega)$, for a strongly convex tensor $\mathbb{C} \in L^\infty(\Omega)$ the problem (1.1)–(1.3) admits a unique solution $w \in H^2_0(\Omega)$.

The inverse problem in which we are interested in consists in studying the stability of the determination of the unknown subgrade coefficient k in (1.1)–(1.3) from a single measurement of w inside Ω. It should be noted that the measurement of the transversal deflection at interior points of the plate can be easily carried out by means of no-contact techniques based on radar methodology ([Be]).

Our main result states that, for $\mathbb{C} \in W^{2,\infty}(\Omega) \cap H^{2+s}(\Omega)$, for some $0 < s < 1$, satisfying a suitable structural condition (see (3.3)), if $w_i \in H^2_0(\Omega)$ is the solution to (1.1)–(1.3) for Winkler coefficient $k = k_i \in L^\infty(\Omega) \cap H^s(\Omega)$, $i = 1, 2$, and if, for a given $\epsilon > 0$,

$$\|w_1 - w_2\|_{L^2(\Omega)} \leq \epsilon f, \quad (1.4)$$

then, for every $\sigma > 0$, we have

$$\|k_1 - k_2\|_{L^2(\Omega_\sigma)} \leq C\epsilon^\beta, \quad (1.5)$$

where $\Omega_\sigma = \{x \in \Omega \mid dist(x, \partial\Omega) > \sigma\}$ and the constants $C > 0$, $\beta \in (0, 1)$ only depend on the a priori data and on σ.

It should be noted that one difficulty of the problem comes from the fact that the displacement w may change sign and vanish somewhere inside Ω. See for instance the examples in [Ga], [K-K-M], [S-T]. Therefore it is necessary to keep under control the possible vanishing rate of w. Thus, the key ingredients of the proof are quantitative versions of the unique continuation principle for the solutions to the equation $\mathrm{div}\,(\mathrm{div}\,(\frac{h^3}{12}\mathbb{C}\nabla^2 w)) + kw = 0$, precisely an estimate of continuation from an open subset to all of the domain (Propositions 3.4) and the A_p property (Proposition 3.5). Another useful tool is a pointwise lower bound in a neighborhood of the point P_0 where the force is acting for solutions to (1.1) (Lemma 3.3).

Let us mention that this method, essentially based on quantitative estimates of unique continuation, has some similarities, although with a different underlying equation and with a different kind of data, with the one used in [Al], for another inverse problem with interior measurements arising in hybrid imaging.

The paper is organized as follows. Section 2 contains the notation, the formulation of the direct problem and a regularity result in fractional Sobolev spaces (Proposition 2.3). Section 3 is devoted to the formulation and analysis of the inverse problem.

2 The direct problem

2.1 Notation

We shall denote by $B_r(P)$ the open disc in $\mathbb{R}^2$ of radius r and center P.

For any $U \subset \mathbb{R}^2$ and for any $r > 0$, we denote

$$U_r = \{x \in U \mid \mathrm{dist}(x, \partial U) > r\}. \tag{2.1}$$

Definition 2.1. *($C^{k,\alpha}$ regularity) Let Ω be a bounded domain in $\mathbb{R}^2$. Given k, α, with $k = 0, 1, 2, ...$, $0 < \alpha \le 1$, we say that a portion S of $\partial\Omega$ is of class $C^{k,\alpha}$ with constants ρ_0, $M_0 > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = O$ and*

$$\Omega \cap B_{\rho_0}(O) = \{x = (x_1, x_2) \in B_{\rho_0}(O) \quad | \quad x_2 > \psi(x_1)\},$$

where ψ is a $C^{k,\alpha}$ function defined in $I_{\rho_0} = (-\rho_0, \rho_0)$ satisfying

$$\psi(0) = 0,$$

$$\psi'(0) = 0, \quad \text{when } k \ge 1,$$

$$\|\psi\|_{C^{k,\alpha}(I_{\rho_0})} \le M_0\rho_0.$$

When $k = 0$, $\alpha = 1$, we also say that S is of Lipschitz class with constants ρ_0, M_0.

We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous with the argument of the norm and coincide with the standard definition when the dimensional parameter equals one. For instance, the norm appearing above is meant as follows

$$\|\psi\|_{C^{k,\alpha}(I_{\rho_0})} = \sum_{i=0}^{k} \rho_0^i \|\psi^{(i)}\|_{L^\infty(I_{\rho_0})} + \rho_0^{k+\alpha} |\psi^{(k)}|_{\alpha, I_{\rho_0}},$$

where

$$|\psi^{(k)}|_{\alpha, I_{\rho_0}} = \sup_{\substack{x_1,\ y_1 \in I_{\rho_0} \\ x_1 \neq y_1}} \frac{|\psi^{(k)}(x_1) - \psi^{(k)}(y_1)|}{|x_1 - y_1|^\alpha}$$

and $\psi^{(i)}$ denotes the i-order derivative of ψ.

Similarly, given a function $u : \Omega \mapsto \mathbb{R}$, where $\partial\Omega$ satisfies Definition 2.1, and denoting by $\nabla^i u$ the vector which components are the derivatives of order i of the function u, we denote

$$\|u\|_{L^2(\Omega)} = \rho_0^{-1} \left(\int_\Omega u^2 \right)^{\frac{1}{2}},$$

$$\|u\|_{H^k(\Omega)} = \rho_0^{-1} \left(\sum_{i=0}^{k} \rho_0^{2i} \int_\Omega |\nabla^i u|^2 \right)^{\frac{1}{2}}, \quad k = 0, 1, 2, ...$$

Moreover, for $k = 0, 1, 2, ...$, and $s \in (0, 1)$, we denote

$$\|u\|_{H^{k+s}(\Omega)} = \|u\|_{H^k(\Omega)} + \rho_0^{s-1} [u]_s,$$

where the semi-norm $[\,\cdot\,]_s$ is given by

$$[u]_s = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}}\, dx\, dy \right)^{\frac{1}{2}}. \tag{2.2}$$

For every 2×2 matrices A, B and every $\mathbb{L} \in \mathcal{L}(\mathbb{M}^2 \times \mathbb{M}^2)$, we use the following notation:

$$(\mathbb{L}A)_{ij} = L_{ijkl} A_{kl}, \tag{2.3}$$

$$A \cdot B = A_{ij} B_{ij}, \tag{2.4}$$

$$|A| = (A \cdot A)^{\frac{1}{2}}. \tag{2.5}$$

Finally, we denote by A^T the transpose of the matrix A.

2.2 Formulation of the direct problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary is of Lipschitz class with constants ρ_0, M_0 and assume that

$$|\Omega| \leq M_1 \rho_0^2. \tag{2.6}$$

We consider a thin plate $\Omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ with middle surface represented by Ω and whose thickness h is much smaller than the characteristic dimension of Ω, that is $h << \rho_0$. The plate is made by linearly elastic material with elasticity tensor $\mathbb{C}(\cdot) \in L^\infty(\Omega, \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2))$ with cartesian components $C_{\alpha\beta\gamma\delta}$ satisfying the symmetry conditions

$$\mathbb{C}A = (\mathbb{C}A)^T, \tag{2.7}$$

$$\mathbb{C}A \cdot B = A \cdot \mathbb{C}B, \tag{2.8}$$

for every 2×2 matrices A, B, and the strong convexity condition

$$\xi_0 |A|^2 \leq \mathbb{C}A \cdot A \leq \xi_1 |A|^2, \tag{2.9}$$

for every 2×2 symmetric matrix A, where ξ_0, ξ_1 are positive constants.

The plate is resting on a Winkler soil with subgrade reaction coefficient

$$k \in L^\infty(\Omega), \quad k \geq 0 \ \text{ a.e. in } \Omega. \tag{2.10}$$

The boundary $\partial\Omega$ is clamped and we assume that a concentrated force is acting at a point $P_0 \in \Omega$ along a direction orthogonal to the middle surface Ω. According to the Kirchhoff-Love theory of thin plates subject to infinitesimal deformation, the statical equilibrium of the plate is described by the following Dirichlet boundary value problem

$$\begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w)) + kw = f\,\frac{\delta(P_0)}{\rho_0^2}, & \text{in } \Omega, \quad (2.11) \\ w = 0, & \text{on } \partial\Omega, \quad (2.12) \\ \frac{\partial w}{\partial n} = 0, & \text{on } \partial\Omega, \quad (2.13) \end{cases}$$

where the plate tensor $\mathbb{P}$ is given by

$$\mathbb{P} = \frac{h^3}{12}\mathbb{C}; \tag{2.14}$$

the subgrade reaction coefficient k satisfies

$$0 \leq k(x) \leq \frac{\overline{k}}{\rho_0^4}, \quad \text{a.e. in } \ \Omega, \tag{2.15}$$

for some positive constant $\overline{k}$; the concentrated force is positive, i.e.,

$$f \in \mathbb{R}, \ f > 0; \tag{2.16}$$

$w = w(x)$ is the transversal displacement at the point $x \in \Omega$ and n is the unit outer normal to $\partial\Omega$.

We notice that the presence in (2.11) and (2.15) of the parameter ρ_0 (which has the dimension of a length) allows for a scaling-invariant formulation of the plate equation.

Proposition 2.2. *Under the above assumptions, there exists a unique weak solution $w \in H^2_0(\Omega)$ to* (2.11)–(2.13)*, which satisfies*

$$\|w\|_{H^2(\Omega)} \leq Cf, \tag{2.17}$$

where the constant $C > 0$ only depends on h, M_0, M_1, ξ_0.

Proof. The weak formulation of the problem (2.11)–(2.13) consists in finding $w \in H^2_0(\Omega)$ such that

$$\int_\Omega \mathbb{P}\nabla^2 w \cdot \nabla^2 v + \int_\Omega kwv = \frac{f}{\rho_0^2} v(P_0), \quad \text{for every } v \in H^2_0(\Omega). \tag{2.18}$$

Let us notice that

$$H^2_0(\Omega) \subset C^{0,\alpha}(\overline{\Omega}), \quad \text{for every } \alpha < 1 \tag{2.19}$$

and, therefore, the linear functional

$$F : H^2_0(\Omega) \to \mathbb{R}$$

$$F(v) = \frac{f}{\rho_0^2} v(P_0)$$

is bounded and the symmetric bilinear form $B(u, v) = \int_\Omega \mathbb{P}\nabla^2 w \cdot \nabla^2 v + \int_\Omega kwv$ is bounded and coercive on $H^2_0(\Omega) \times H^2_0(\Omega)$. By Riesz representation theorem a solution to (2.18) exists and is unique. By choosing $v = w$ in (2.18), by (2.9) and using Poincaré inequality, we have

$$\frac{fw(P_0)}{\rho_0^2} \geq \int_\Omega \mathbb{P}\nabla^2 w \cdot \nabla^2 w \geq \frac{C}{\rho_0^2} \|w\|^2_{H^2(\Omega)}, \tag{2.20}$$

where the constant $C > 0$ only depends on h, M_0, M_1, ξ_0. By (2.20) and the embedding (2.19), the thesis follows. □

In the analysis of the inverse problem, we shall need the following regularity result when the coefficients of the plate operator belong to a fractional Sobolev space.

Proposition 2.3 (H^s-regularity)**.** *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of Lipschitz class with constants ρ_0, M_0, satisfying* (2.6)*. Given $g \in H^s(\Omega)$, let $w \in H^2(\Omega)$ be a solution to*

$$div(div(\mathbb{P}\nabla^2 w)) = g, \quad in\ \Omega, \tag{2.21}$$

where $\mathbb{P}$ is given by (2.14)*, with $\mathbb{C}$ satisfying* (2.7)–(2.9) *and, for some $s \in (0, 1)$,*

$$\|\mathbb{C}\|_{W^{2,\infty}(\Omega)} \leq M_2, \tag{2.22}$$

$$\|\mathbb{C}\|_{H^{2+s}(\Omega)} \leq M_3. \tag{2.23}$$

Then, for every $\sigma > 0$, we have

$$\|w\|_{H^{4+s}(\Omega_{\sigma\rho_0})} \leq C\left(\|w\|_{H^2(\Omega_{\frac{\sigma}{2}\rho_0})} + \rho_0^4\|g\|_{H^s(\Omega)}\right), \tag{2.24}$$

where the constant $C > 0$ only depends on h, M_0, M_1, M_2, M_3, ξ_0, s, σ.

Proof. When $\mathbb{C} \in C^\infty(\Omega)$ the estimate (2.24) is a form of the well-known classical Garding's inequality. Under the less restrictive condition (2.22), (2.23), the proof of (2.24) can be carried out following the same path traced in the classical case ([Ag], [F]) taking care to control the lower order terms by means of M_2 and M_3. We omit the details. □

3 The inverse problem

In order to derive our stability result for the inverse problem we need further a priori information.

Concerning the point P_0 of the plate in which the concentrated force is acting, we assume that

$$dist(P_0, \partial\Omega) \geq d\rho_0, \tag{3.1}$$

for some positive constant d. On the elasticity tensor $\mathbb{C} = \{C_{\alpha\beta\gamma\delta}\}$ we further assume the stronger regularity (2.22), (2.23) and, moreover, we introduce a structural condition. Precisely, denoting by $a_0 = C_{1111}$, $a_1 = 4C_{1112}$, $a_2 = 2C_{1122} + 4C_{1212}$, $a_3 = 4C_{2212}$, $a_4 = C_{2222}$ and by $S(x)$ the following matrix

$$S(x) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\ 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\ 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\ 0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \end{pmatrix}, \tag{3.2}$$

we assume that

$$\mathcal{D}(x) = 0, \quad \text{for every } x \in \Omega, \tag{3.3}$$

where

$$\mathcal{D}(x) = \frac{1}{a_0}|\det S(x)|. \tag{3.4}$$

Let us recall that condition (3.3) includes the class of orthotropic materials and, in particular, the isotropic Lamé case, see [M-R-V]. Concerning the subgrade reaction coefficient k, we require the additional regularity

$$\rho_0^{s-1}[k]_{H^s(\Omega)} \leq \frac{\overline{k}}{\rho_0^4}. \tag{3.5}$$

Remark 3.1. Let us emphasize that the assumption $k \in H^s(\Omega)$ is not merely a mathematical technicality, but it can be grounded on realistic mechanical considerations. If, for instance, k is piecewise constant and is represented as

$$k(x) = \sum_{j=1}^{J} k_j \chi_{E_j}(x), \quad \text{for every } x \in \mathbb{R}^2, \tag{3.6}$$

where $k_j \in \mathbb{R}$ and E_1,...,E_J is a partition of Ω into disjoint subsets of finite perimeter (in the sense of Caccioppoli, that is $\chi_{E_j} \in BV(\mathbb{R}^2)$ for every j), then k belongs to $H^s(\mathbb{R}^2)$ for every s, $0 < s < \frac{1}{2}$. In fact one has

$$[k]_s^2 \leq C_s \|k\|_{L^\infty}^{2s} \left(\int_{\mathbb{R}^2} |k|^2 \right)^{1-2s} \left(\int_{\mathbb{R}^2} |Dk| \right)^{2s},$$

for every $k \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$. Here C_s only depends on $s \in (0, \frac{1}{2})$ and $\int_{\mathbb{R}^2} |Dk|$ denotes the total variation of k. For a proof see [M-P, formula (2.15)] and also [Gi, Remark 1.16] for the convergence properties of the mollifications of BV functions.

In particular, if k is of the form (3.6) and we assume

$$P(E_j) = \int_{\mathbb{R}^2} |D\chi_{E_j}| \leq \mathcal{P}\rho_0, \quad \text{for every } j = 1, ..., J,$$

for a given $\mathcal{P} > 0$, then we obtain

$$[k]_s^2 \leq C_s \overline{k}^2 M_1^{1-2s} (J\mathcal{P})^{2s} \rho_0^{-6-2s}.$$

Hereinafter, we shall refer to h, d, M_0, M_1, M_2, M_3, ξ_0, $\overline{k}$, s as the *a priori data.*

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary of Lipschitz class with constants ρ_0, M_0, satisfying* (2.6)*. Let $\mathbb{P}$ given by* (2.14)*, with $\mathbb{C} \in W^{2,\infty}(\Omega) \cap H^{2+s}(\Omega)$ satisfying* (2.7)*–*(2.9)*,* (2.22)*,* (2.23) *for some $s \in (0, 1)$, and* (3.3)*. Let $P_0 \in \Omega$ satisfying* (3.1)*.*

Given $f > 0$, let $w_i \in H_0^2(\Omega)$, $i = 1, 2$, be the solution to

$$\begin{cases} div\,(div\,(\mathbb{P}\nabla^2 w_i)) + k_i w_i = f\,\frac{\delta(P_0)}{\rho_0^2}, & \text{in } \Omega, \quad (3.7)\\ w_i = 0, & \text{on } \partial\Omega, \quad (3.8)\\ \frac{\partial w_i}{\partial n} = 0, & \text{on } \partial\Omega, \quad (3.9) \end{cases}$$

for $k_i \in L^\infty(\Omega) \cap H^s(\Omega)$ satisfying (2.15) *and* (3.5)*.*

If, for some $\epsilon > 0$,

$$\|w_1 - w_2\|_{L^2(\Omega)} \leq \epsilon f, \tag{3.10}$$

then for every $\sigma > 0$ we have

$$\|k_1 - k_2\|_{L^2(\Omega_{\sigma\rho_0})} \leq \frac{C}{\rho_0^4} \epsilon^\beta, \tag{3.11}$$

where the constants $C > 0$ and $\beta \in (0, 1)$ only depend on the a priori data and on σ.

As is obvious, the above stability result also implies uniqueness. Indeed, by the following arguments it is easily seen that, under the above stated structural conditions on $\mathbb{C}$ (3.2)–(3.4), uniqueness continues to hold by merely assuming $k \in L^\infty$ and $\mathbb{C} \in W^{2,\infty}$.

Let us premise to the proof of Theorem 3.2 some auxiliary propositions concerning quantitative versions of the unique continuation principle (Lemma 3.3 and Propositions 3.4 and 3.5 below).

Lemma 3.3. *Let Ω be a bounded domain with boundary $\partial\Omega$ of Lipschitz class with constants ρ_0, M_0, satisfying* (2.6). *Let $P_0 \in \Omega$ satisfying* (3.1). *Let $\mathbb{P}$ given by* (2.14), *with $\mathbb{C}$ satisfying* (2.7)–(2.9), *and let k and f satisfy* (2.15), (2.16), *respectively. Let $w \in H^2_0(\Omega)$ be the solution to* (2.11)–(2.13). *There exists $\overline{\sigma} > 0$, only depending on h, d, M_0, M_1, ξ_0, ξ_1, such that*

$$w(x) \geq C d^2 f, \quad \forall x \in B_{2\overline{\sigma}\rho_0}(P_0), \tag{3.12}$$

where $C > 0$ only depends on h, M_0, M_1, ξ_0 and ξ_1,

$$\int_{B_{2\sigma\rho_0}(P_0)\setminus B_{\sigma\rho_0}(P_0)} w^2 \geq C\sigma^2 d^2 \rho_0^2 \|w\|^2_{H^2(\Omega)}, \quad \textit{for every } \sigma,\ 0 < \sigma \leq \overline{\sigma} \tag{3.13}$$

where $C > 0$ only depends on h, M_0, M_1, ξ_0, ξ_1, $\overline{k}$.

Proof. By (2.9) and (2.18), we have that for every $v \in H^2_0(\Omega)$

$$f|v(P_0)| \leq C\|v\|_{H^2(\Omega)}\|w\|_{H^2(\Omega)}, \tag{3.14}$$

so that

$$\|\delta(P_0)\|_{H^{-2}(\Omega)} = \sup_{\substack{v\in H^2_0(\Omega)\\ \|v\|_{H^2(\Omega)}=1}} |v(P_0)| \leq \frac{C}{f}\|w\|_{H^2(\Omega)}, \tag{3.15}$$

where $C > 0$ only depends on h, ξ_1, $\overline{k}$. Since $B_{d\rho_0}(P_0) \subset \Omega$ by (3.1), we have

$$\|\delta(P_0)\|_{H^{-2}(\Omega)} \geq \|\delta(P_0)\|_{H^{-2}(B_{d\rho_0}(P_0))} \geq Cd, \tag{3.16}$$

where $C > 0$ is an absolute constant. From (3.15), (3.16) it follows that

$$\|w\|_{H^2(\Omega)} \geq Cdf, \tag{3.17}$$

where $C > 0$ only depends on h, ξ_1, $\overline{k}$. By (2.9), (2.18) and Poincaré inequality, we have

$$w(P_0) \geq \frac{C}{f}\|w\|^2_{H^2(\Omega)}, \tag{3.18}$$

where $C > 0$ only depends on h, M_0, M_1, ξ_0. By (3.17), (3.18) and by the embedding inequality (2.19), we have

$$w(P_0) \geq Cd\|w\|_{H^2(\Omega)}, \tag{3.19}$$

$$w(P_0) \geq c_0 d \|w\|_{C^{0,\alpha}(\overline{\Omega})}, \tag{3.20}$$

where $C > 0$ and $c_0 > 0$ only depend on h, M_0, M_1, ξ_0, ξ_1, $\overline{k}$. Let

$$\overline{\sigma} = \min\left(\frac{d}{4}, \frac{1}{2}\left(\frac{c_0 d}{2}\right)^{\frac{1}{\alpha}}\right). \tag{3.21}$$

Let us notice that, by this choice of $\overline{\sigma}$, $dist(P_0, \partial\Omega) \geq 4\overline{\sigma}\rho_0$ and, recalling (3.20), we have

$$w(x) \geq w(P_0) - |w(x) - w(P_0)| \geq w(P_0) - (2\overline{\sigma})^\alpha \|w\|_{C^{0,\alpha}(\overline{\Omega})} \geq \frac{w(P_0)}{2},$$
$$\text{for every } x \in B_{2\overline{\sigma}\rho_0}(P_0). \tag{3.22}$$

Choosing $\alpha = \frac{1}{2}$, (3.12) follows from (3.17), (3.18) and (3.22) whereas (3.13) follows, restricting to the annulus $B_{2\sigma\rho_0}(P_0) \setminus B_{\sigma\rho_0}(P_0)$, which is contained in $B_{2\overline{\sigma}\rho_0}(P_0)$ for $\sigma \leq \overline{\sigma}$, from (3.19) and (3.22). □

Proposition 3.4 (Lipschitz propagation of smallness)**.** *Let U be a bounded Lipschitz domain of $\mathbb{R}^2$ with constants ρ_0, M_0 and satisfying $|U| \leq M_1\rho_0^2$. Let $w \in H^2(U)$ be a solution to*

$$div(div(\mathbb{P}\nabla^2 w)) + kw = 0, \quad in\ U, \tag{3.23}$$

where $\mathbb{P}$, defined in (2.14)*, satisfies* (2.7)*–*(2.9) *and* (2.22) *in U, and k satisfies* (2.15) *in U. Assume*

$$\frac{\|w\|_{H^{\frac{1}{2}}(U)}}{\|w\|_{L^2(U)}} \leq N.$$

There exists a constant $c_1 > 1$, only depending on h, M_2, ξ_0 and $\overline{k}$, such that, for every $\tau > 0$ and for every $x \in U_{c_1\tau\rho_0}$, we have

$$\int_{B_{\tau\rho_0}(x)} w^2 \geq c_\tau \int_U w^2, \tag{3.24}$$

where $c_\tau > 0$ only depends on h, M_0, M_1, M_2, ξ_0, $\overline{k}$, τ and on N.

The proof of the above proposition is based on the three spheres inequality obtained in [L-N-W].

Proposition 3.5 (A_p property)**.** *In the same hypotheses of Proposition 3.4, there exists a constant $c_2 > 1$, only depending on h, M_0, M_1, M_2, ξ_0, $\overline{k}$, such that, for every $\tau > 0$ and for every $x \in U_{c_2\tau\rho_0}$, we have*

$$\left(\frac{1}{|B_{\tau\rho_0}(x)|}\int_{B_{\tau\rho_0}(x)} |w|^2\right)\left(\frac{1}{|B_{\tau\rho_0}(x)|}\int_{B_{\tau\rho_0}(x)} |w|^{-\frac{2}{p-1}}\right)^{p-1} \leq B, \tag{3.25}$$

where $B > 0$ and $p > 1$ only depend on h, M_0, M_1, M_2, ξ_0, $\overline{k}$, τ and on N.

The proof of the above proposition follows from the doubling inequality obtained in [dC-L-M-R-V-W], by applying the arguments in [G-L].

Proof of Theorem 3.2. If $\epsilon \geq 1$, then the proof of (3.11) is trivial in view of (2.15). Therefore we restrict the analysis to the case $0 < \epsilon < 1$.

The difference

$$w = w_1 - w_2 \tag{3.26}$$

of the solutions to (3.7)–(3.9) for $i = 1, 2$ satisfies the boundary value problem

$$\begin{cases} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w)) + k_2 w = (k_2 - k_1)w_1, & \text{in } \Omega, \quad (3.27)\\ w = 0, & \text{on } \partial\Omega, \quad (3.28)\\ \frac{\partial w}{\partial n} = 0, & \text{on } \partial\Omega. \quad (3.29) \end{cases}$$

Obviously, it is not restrictive to assume that $\sigma \leq \overline{\sigma}$, where $\overline{\sigma}$ has been defined in (3.21) with $\alpha = \frac{1}{2}$. We have

$$\int_{\Omega_{\sigma\rho_0}} (k_2 - k_1)^2 w_1^2 \leq 2(I_1 + I_2), \tag{3.30}$$

where

$$I_1 = \int_{\Omega_{\sigma\rho_0}} k_2^2 w^2, \tag{3.31}$$

$$I_2 = \int_{\Omega_{\sigma\rho_0}} \left(\operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w))\right)^2. \tag{3.32}$$

By (2.15) and (3.10), we have

$$I_1 \leq \frac{\overline{k}^2}{\rho_0^6}\epsilon^2. \tag{3.33}$$

By (2.22), we have

$$I_2 \leq C\frac{h^3 M_2^2}{12\rho_0^6}\|w\|^2_{H^4(\Omega_{\sigma\rho_0})}. \tag{3.34}$$

with $C > 0$ an absolute constant. Let $g = (k_2 - k_1)w_1 - k_2 w$. Note that, by (2.15), (2.17), (2.19), and (3.5),

$$\|g\|_{H^s(\Omega)} \leq C\frac{\overline{k}f}{\rho_0^4}, \tag{3.35}$$

where $C > 0$ only depends on h, M_0, M_1, ξ_0. By applying Proposition 2.3, we have

$$\|w\|_{H^{4+s}(\Omega_{\sigma\rho_0})} \leq Cf\overline{k}, \tag{3.36}$$

with $C > 0$ only depending on h, M_0, M_1, M_2, M_3, ξ_0, s, σ. From the well-known interpolation inequality

$$\|w\|_{H^4(\Omega_{\sigma\rho_0})} \leq C\|w\|^{\frac{4}{4+s}}_{H^{4+s}(\Omega_{\sigma\rho_0})}\|w\|^{\frac{s}{4+s}}_{L^2(\Omega_{\sigma\rho_0})}, \tag{3.37}$$

and recalling (3.36) and (3.10), we obtain

$$\|w\|_{H^4(\Omega_{\sigma\rho_0})} \leq C f \epsilon^{\frac{s}{4+s}}, \tag{3.38}$$

where $C > 0$ only depends on h, M_0, M_1, M_2, M_3, ξ_0, $\overline{k}$, s, σ. From (3.30), (3.33), (3.34), (3.38), it follows that

$$\int_{\Omega_{\sigma\rho_0}} (k_2 - k_1)^2 w_1^2 \leq \frac{C}{\rho_0^6} f^2 \epsilon^{\frac{2s}{4+s}}, \tag{3.39}$$

where $C > 0$ only depends on h, M_0, M_1, M_2, M_3, ξ_0, $\overline{k}$, s, σ.

Let us first estimate $|k_2 - k_1|$ in a disc centered at P_0. Notice that, by the choice of $\overline{\sigma}$, $\Omega_{\sigma\rho_0} \supset B_{2\overline{\sigma}\rho_0}(P_0)$, for every $\sigma \leq \overline{\sigma}$. By applying (3.12) for $w = w_1$, and by (3.39) with $\sigma = \overline{\sigma}$, we obtain

$$\int_{B_{2\overline{\sigma}\rho_0}(P_0)} (k_2 - k_1)^2 \leq \frac{C}{\rho_0^6 d^4} \epsilon^{\frac{2s}{4+s}}, \tag{3.40}$$

where $C > 0$ only depends on h, M_0, M_1, M_2, M_3, ξ_0, $\overline{k}$, s and d.

Now, let us control $|k_2 - k_1|$ in

$$\widetilde{\Omega}_{\sigma\rho_0} = \Omega_{\sigma\rho_0} \setminus B_{2\overline{\sigma}\rho_0}. \tag{3.41}$$

This estimate is more involved and requires arguments of unique continuation, precisely the A_p-property and the Lipschitz propagation of smallness.

By applying Hölder inequality and (3.39), we can write, for every $p > 1$,

$$\begin{aligned}
\int_{\widetilde{\Omega}_{\sigma\rho_0}} (k_2 - k_1)^2 &= \int_{\widetilde{\Omega}_{\sigma\rho_0}} |w_1|^{\frac{2}{p}} (k_2 - k_1)^2 |w_1|^{-\frac{2}{p}} \leq \\
&\leq \left(\int_{\widetilde{\Omega}_{\sigma\rho_0}} (k_2 - k_1)^2 w_1^2 \right)^{\frac{1}{p}} \left(\int_{\widetilde{\Omega}_{\sigma\rho_0}} (k_2 - k_1)^2 |w_1|^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p}} \leq \\
&\leq \frac{C}{\rho_0^{\frac{6}{p}}} f^{\frac{2}{p}} \epsilon^{\frac{2s}{p(4+s)}} \left(\int_{\widetilde{\Omega}_{\sigma\rho_0}} (k_2 - k_1)^2 |w_1|^{-\frac{2}{p-1}} \right)^{\frac{p-1}{p}},
\end{aligned} \tag{3.42}$$

where $C > 0$ only depends on h, M_0, M_1, M_2, M_3, ξ_0, $\overline{k}$, s, σ.

Let us cover $\widetilde{\Omega}_{\sigma\rho_0}$ with internally non overlapping closed squares $Q_l(x_j)$ with center x_j and side $l = \frac{\sqrt{2}}{2\max\{2, c_1, c_2\}} \sigma\rho_0$, $j = 1, ..., J$, where c_1 and c_2 have been introduced in Proposition 3.4 and in Proposition 3.5, respectively. By the choice of l, denoting $r = \frac{\sqrt{2}}{2} l$,

$$\widetilde{\Omega}_{\sigma\rho_0} \subset \bigcup_{j=1}^{J} Q_l(x_j) \subset \bigcup_{j=1}^{J} B_r(x_j) \subset \Omega_{\frac{\sigma}{2}\rho_0} \setminus B_{\overline{\sigma}\rho_0}(P_0), \tag{3.43}$$

so that

$$\int_{\widetilde{\Omega}_{\sigma\rho_0}} (k_2 - k_1)^2 |w_1|^{-\frac{2}{p-1}} \leq \frac{4\overline{k}^2}{\rho_0^8} \int_{\widetilde{\Omega}_{\sigma\rho_0}} |w_1|^{-\frac{2}{p-1}} \leq \frac{4\overline{k}^2}{\rho_0^8} \sum_{j=1}^{J} \int_{B_r(x_j)} |w_1|^{-\frac{2}{p-1}}. \tag{3.44}$$

By applying the A_p-property (3.25) and the Lipschitz propagation of smallness property (3.24) to $w = w_1$ in $U = \Omega \setminus B_{\overline{\sigma}\rho_0}(P_0)$, with $\tau = \frac{r}{\rho_0} = \frac{\sigma}{2\max\{2, c_1, c_2\}}$, and noticing that, for every j, $j = 1, ..., J$, $dist(x_j, \partial U) \geq c_i r$, $i = 1, 2$, we have

$$\int_{B_r(x_j)} |w_1|^{-\frac{2}{p-1}} \leq \frac{B^{\frac{1}{p-1}} |B_r(x_j)|}{\left(\frac{1}{|B_r(x_j)|} \int_{B_r(x_j)} |w_1|^2\right)^{\frac{1}{p-1}}} \leq \frac{B^{\frac{1}{p-1}} |B_r(x_j)|}{\left(\frac{c_\tau}{|B_r(x_j)|} \int_{\Omega \setminus B_{\overline{\sigma}\rho_0}(P_0)} |w_1|^2\right)^{\frac{1}{p-1}}}, \tag{3.45}$$

where $B > 0$, $p > 1$ and $c_\tau > 0$ only depend on h, M_0, M_1, M_2, ξ_0, $\overline{k}$, σ and on the frequency ratio $\mathcal{F} = \frac{\|w_1\|_{H^{\frac{1}{2}}(\Omega \setminus B_{\overline{\sigma}\rho_0}(P_0))}}{\|w_1\|_{L^2(\Omega \setminus B_{\overline{\sigma}\rho_0}(P_0))}}$. Such a bound can be achieved as follows. Notice that, since $\Omega \setminus B_{\overline{\sigma}\rho_0}(P_0) \supset B_{2\overline{\sigma}\rho_0}(P_0) \setminus B_{\overline{\sigma}\rho_0}(P_0)$, by applying (3.13), we have

$$\mathcal{F} \leq \frac{C}{\overline{\sigma} d}, \tag{3.46}$$

where $C > 0$ only depends on h, M_0, M_1, ξ_0, ξ_1, $\overline{k}$. By applying (3.13) and (3.17) to estimate from below the denominator in the right hand side of (3.45), by (2.6) and (3.44), we obtain

$$\int_{\widetilde{\Omega}_{\sigma\rho_0}} (k_2 - k_1)^2 |w_1|^{-\frac{2}{p-1}} \leq \frac{C|\Omega|}{\rho_0^8 (d^4 f^2)^{\frac{1}{p-1}}} \leq \frac{C}{\rho_0^6 (d^4 f^2)^{\frac{1}{p-1}}}, \tag{3.47}$$

where $C > 0$ only depends on h, M_0, M_1, M_2, ξ_0, $\overline{k}$, d and σ. By (3.42) and (3.47) we have

$$\int_{\widetilde{\Omega}_{\sigma\rho_0}} (k_2 - k_1)^2 \leq \frac{C}{\rho_0^6 d^{\frac{4}{p}}} \epsilon^{\frac{2s}{p(4+s)}}, \tag{3.48}$$

where $C > 0$ only depends on h, M_0, M_1, M_2, M_3, ξ_0, $\overline{k}$, s, d and σ. Finally, by (3.40) and (3.48), and recalling that $p > 1$ and $\epsilon < 1$, estimate (3.11) follows with $\beta = \frac{s}{p(4+s)}$. □

4 Concluding Remarks

In this paper we have shown, by means of a rigorous mathematical analysis, that the nonlinear inverse problem of determining the Winkler coefficient k from the measurement of the transversal displacement w induced by a load concentrated at one point, is only mildly ill-posed.

As is well-known ([Ki], [dH-Q-S]), the Hölder stability estimates we achieved imply convergence of regularized inversion procedures. It would also be noted that, for this specific problem, although it is nonlinear, the inversion can be performed by a cascade of linear inversion procedures. Namely

i) from w obtain kw;

ii) from kw obtain k.

This approach shall be the object of a subsequent study. We emphasize that, from such regularized inversion procedures, it will be possible to test the efficiency of our proposed method with the aid of in-door and field experiments already available in the civil engineering literature.

On the other hand, we are aware that also multiple concentrated loads and distributed loads are of great relevance in civil engineering, and also that different models of foundations, other than the Kirchhoff-Love one, are of interest. However, it is reasonable to expect that under such different modeling assumptions, a different mathematical analysis shall be needed.

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