REALIZATION OF JETS AND VECTOR FIELDS IN SCALAR PARABOLIC EQUATIONS

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let

$$L'u := \sum_{i,j=i}^{N} \partial_i(a_{ij}(x)\partial_j u), \quad x \in \Omega$$

be a second order strongly elliptic differential operator with smooth symmetric coefficients. Let B denote the Dirichlet or the Neumann boundary operator. We prove the existence of a smooth function $a\colon \overline{\Omega} \to \mathbb{R}$ such that all sufficiently small vector fields on $\mathbb{R}^{N+1}$ can be realized on the center manifold of the semilinear parabolic equation

$$\begin{aligned} u_t &= L'u + a(x)u + f(x,u,\nabla u), && t > 0, x \in \Omega \\ Bu &= 0, && t > 0, x \in \partial\Omega, \end{aligned}$$

with an appropriate nonlinearity $f\colon (x,s,w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto f(x,s,w) \in \mathbb{R}$.

For $N = 2$, $n, k \in \mathbb{N}$, we prove the existence of a smooth function $a\colon \overline{\Omega} \to \mathbb{R}$ such that all sufficiently small k-jets of vector fields on $\mathbb{R}^n$ can be realized on the center manifold of the semilinear parabolic equation

$$\begin{aligned} u_t &= L'u + a(x)u + g(x,u) \cdot \nabla u, && t > 0, x \in \Omega \\ Bu &= 0, && t > 0, x \in \partial\Omega, \end{aligned}$$

with an appropriate nonlinearity $g\colon (x,s) \in \overline{\Omega} \times \mathbb{R} \mapsto g(x,s) \in \mathbb{R}^2$ (here "$\cdot$" denotes the scalar product in $\mathbb{R}^2$).

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let

$$L'u := \sum_{i,j=i}^{N} \partial_i(a_{ij}(x)\partial_j u), \quad x \in \Omega$$

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be a second order strongly elliptic differential operator with smooth symmetric coefficients. Let $a\colon\overline{\Omega}\to\mathbb{R}$ be a smooth function and set

$$Lu := L'u + a(x)u$$

Consider the semilinear parabolic equations

$$\begin{aligned} u_t &= Lu + f(x,u,\nabla u), \quad && t>0, x\in\Omega \\ u &= 0, && t>0, x\in\partial\Omega \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} u_t &= Lu + f(x,u,\nabla u), \quad && t>0, x\in\Omega \\ \frac{\partial u}{\partial \nu} &= 0, && t>0, x\in\partial\Omega. \end{aligned} \tag{1.2}$$

Here $f\colon(x,s,w)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\mapsto f(x,s,w)\in\mathbb{R}$ is some "smooth" nonlinearity. For $p\geq 1$, the operator $-L$ with Dirichlet (or Neumann) boundary condition on $\partial\Omega$ defines a sectorial operator on $X:=L^p(\Omega)$ with the corresponding family X^α of fractional power spaces. If $p>N$, then α can be chosen such that $X^\alpha\subset C^1(\overline{\Omega})$ and then the solutions of (1.1) (or (1.2)) define a local semiflow on X^α.

It is known that for $N=1$ the dynamics of (1.1) (or (1.2)) is very simple, as all bounded solutions are convergent. On the other side, if the nonlinearity f is independent of gradient terms, then the local semiflow generated by (1.1) (or (1.2)) is gradient-like and so the dynamics is again rather simple and nonchaotic. The situation is completely different if $N\geq 2$ and if f depends explicitly on gradient terms. It has recently been proved that the dynamics of (1.1) (or (1.2)) can be very complicated, in fact even 'arbitrary'. A first result of this kind was given by Poláčik in [8]. More specifically, he proved that every finite jet of a vector field on $\mathbb{R}^n$ can be realized on the center manifold of (1.1) with an appropriate nonlinearity f provided the kernel of the operator L (with Dirichlet boundary conditions on $\partial\Omega$) has dimension n and the corresponding eigenfunctions satisfy a certain nondegeneracy condition (called Poláčik condition). In this case $n=N$ or $n=N+1$ and Poláčik also gave examples of operators satisfying this condition, both with $n=N$ (and Ω being the unit ball) and $n=N+1$ (with Ω being smooth and smoothly diffeomorphic to the unit ball), and with L of the form $L=\Delta+a(x)$. In [18] Rybakowski showed that under the Poláčik condition actually all sufficiently smooth and sufficiently small vector fields v on $\mathbb{R}^n$ can be realized on the center manifold of (1.1) with an appropriate nonlinearity f. The method of proof used in [18] (the Nash-Moser implicit mapping theorem) leads to a loss of derivatives: g is less smooth that v. In [14] Poláčik and Rybakowski proved that if L has analytic coefficients and Poláčik condition holds then a vector field realization result holds without loss of derivatives. They also showed that there are real analytic functions a on $\mathbb{R}^N$ such that the operator $Lu=\Delta u+a(x)u$ satisfies the Poláčik condition on a ball of $\mathbb{R}^N$ with $n=N+1$. These results lead to a restriction in the space dimension of (1.1): to get realizability of any vector field of $\mathbb{R}^n$ we have to choose $n=N$ or $n=N+1$. Therefore the question arises what is the least possible space dimension that allows arbitrary dynamics in (1.1) and (1.2). In [12] it was shown by P. Poláčik that every finite jet of a vector field on $\mathbb{R}^n$ can be realized on the center manifold of (1.1) with an appropriate polynomial nonlinearity f and an appropriate two-dimensional domain (close to a square). In [12]the form of the nonlinearity f involves high powers of the gradient of the solution u. On the other hand, when modelling scientific phenomena by equations (1.1) and (1.2), one usually tries to make the convection terms (i.e. the terms depending on ∇u) as simple as possible. In [16] it is shown that arbitrary jets can be realized in (1.1) even for functions f depending on the gradient in a linear fashion.

All the above realization results were proved only on very particular domains, diffeomophic to a ball or close to a square, and for operators of the form $L = \Delta + a(x)$. One can ask if it is possible to extend such results to the case of arbitrary (sufficiently regular) domains and general second order elliptic operators in divergence form.

A first affirmative answer to this question was given by K. Rybakowski and the present author in [17]. More specifically, they proved that the vector field realization result from [14] is valid for the Laplacian on an arbitrary bounded domain Ω of class $C^{2,\gamma}$, $0 < \gamma < 1$.

The goal of this paper is to extend all the above realization results to the case of a general second order elliptic operator on an arbitrary spatial domain, both with Dirichlet and Neumann boundary conditions. In order to achieve this result, we exploit some of the techniques used in [17] together with a "localization lemma" (Lemma 5.1), which is the main contribution of this paper.

The paper is organized as follows: in Sections 1 and 2 we recall some basic realization results presenting them in a more abstract form; in Section 4 we slightly refine some perturbation results of [17]; in Section 5 we state and prove the above mentioned "localization lemma"; finally, in Sections 6 and 7 we show that the abstract assumptions in Sections 1 and 2 actually are satisfied with any symmetric strongly elliptic second order differential operator in divergence form on an arbitrary spatial domain.

2. Vector field realizations

Throughout this section let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\gamma}$ with $0 < \gamma < 1$. Let L be a differential operator of the form

$$Lu = \sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j u) + au.$$

We assume throughout that L is uniformly elliptic and its coefficient functions satisfy $a_{ij} \in C^{1,\gamma}(\overline{\Omega})$, i, $j = 1, \ldots, N$, and $a \in C^{\gamma}(\overline{\Omega})$. Consider the semilinear parabolic equations

$$(2.1_g) \qquad \begin{aligned} u_t &= Lu + g(x, u, \nabla u), && t > 0,\ x \in \Omega \\ u(x,t) &= 0, && t > 0,\ x \in \partial\Omega \end{aligned}$$

and

$$(2.2_g) \qquad \begin{aligned} u_t &= Lu + g(x, u, \nabla u), && t > 0,\ x \in \Omega \\ \frac{\partial u}{\partial \nu}(x,t) &= 0, && t > 0,\ x \in \partial\Omega. \end{aligned}$$

Here

$$g\colon (x,s,w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto g(x,s,w) \in \mathbb{R}$$

is some nonlinearity.

To study (2.1_g) and (2.2_g) we shall rewrite this problems in a more abstract way. Set

$$X := L^p(\Omega), \quad \text{for some } p > N.$$

The operator $-L$ with Dirichlet (resp. Neumann) boundary conditions on $\partial\Omega$ defines a sectorial operator A on X with domain $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (resp. $W_N^{2,p}(\Omega)$, where $W_N^{2,p}(\Omega)$ is the space of all functions in $W^{2,p}(\Omega)$ that satisfy the Neumann condition on $\partial\Omega$ in the sense of traces). The operator A generates the corresponding family X^α of fractional power spaces and fixing α with

$$(N+p)/(2p) < \alpha < 1$$

we have that

$$X^\alpha \subset C^1(\overline{\Omega})$$

with continuous inclusion. Define

$$X_0 := \ker A$$

and suppose that

$$n := \dim X_0 \geq 1.$$

Let P be the $L^2(\Omega)$-orthogonal projection of X onto X_0. Fix an arbitrary L^2-orthonormal basis $\phi_1, \ldots, \phi_n$ of X_0 and write

$$\phi(x) := (\phi_1(x), \ldots, \phi_n(x)).$$

Note that the assignment

$$Q\colon \mathbb{R}^n \to X_0, \qquad Q\xi := \xi \cdot \phi = \sum_{i=1}^n \xi_i \phi_i$$

is a linear isomorphism.

For $m \in \mathbb{N}_0$ let $C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ be the set of all maps

$$h\colon \mathbb{R}^n \to \mathbb{R}^n$$

such that for all $0 \leq k \leq m$ the Fréchet derivative $\mathrm{D}^k h$ exists and is continuous and bounded on $\mathbb{R}^n$.

$C_b^m(\mathbb{R}^n, \mathbb{R}^n)$ is a linear space which becomes a Banach space when endowed with the norm

$$|h|_m := \sup_{y \in \mathbb{R}^n} \sup_{0 \leq k \leq m} |\mathrm{D}^k h(y)|_{\mathcal{L}^k((\mathbb{R}^n)^k, \mathbb{R}^n)}.$$

Furthermore, let Y_m be the set of all functions

$$g\colon (x,s,w) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \mapsto g(x,s,w) \in \mathbb{R}$$

such that for all $0 \le k \le m$ the Fréchet derivative $\mathrm{D}^k_{(s,w)} g$ exists and is continuous and bounded on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$.

Y_m is a linear space which becomes a Banach space when endowed with the norm

$$|g|_m := \sup_{(x,s,w)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N} \sup_{0\le k\le m} |\mathrm{D}^k_{(s,w)} g(x,s,w)|_{\mathcal{L}^k((\mathbb{R}\times\mathbb{R}^N)^k,\mathbb{R})}.$$

For $g \in Y_m$ the formula

$$\hat{g}(y)(x) := g(x, y(x), \nabla y(x)), \quad y \in X^\alpha, x \in \overline{\Omega}$$

defines the Nemitski operator

$$\hat{g}\colon X^\alpha \to X$$

of class C^m_b. We can rewrite problem (2.1_g) in the form

$$\dot{y} = -Ay + \hat{g}(y). \tag{2.3_g}$$

Let $u_0 \in X^\alpha$; *a solution of* (2.3_g) *on* $]0,T[$ (where $T \in]0,\infty]$) *through* u_0 is, by definition, a continuous map $u\colon [0,T[\to X$ with $u\colon]0,T[\to X$ differentiable, $u(t) \in \operatorname{dom} A$ for $t \in]0,T[$, $t \to \hat{g}(u(t)) \in X$ is locally Hölder continuous on $]0,T[$, $\int_0^a |g(u(s))|\, ds < \infty$ for some $a \in]0,T]$ and

$$\dot{u}(t) + Au(t) = \hat{g}(u(t)), \qquad \text{for all } t \in]0,T[.$$

For every $g \in Y_1$ and $u_0 \in X^\alpha$ there is a maximal $T(u_0) \in]0,\infty]$ and a unique solution $u(\cdot, u_0)$ of (2.3_g) on $]0,T(u_0)[$ through u_0. Writing

$$\Pi(t,u_0) := u(t,u_0), \qquad t \in]0,T(u_0)[$$

we obtain a local semiflow $\Pi = \Pi_g$ on X^α.

Now let Y and $\tilde{Y}$ be arbitrary Banach spaces and Π (resp. $\tilde{\Pi}$) be a local semiflow on Y (resp. $\tilde{Y}$). We say that $\tilde{\Pi}$ imbeds in Π if there is an imbedding $\Lambda\colon \tilde{Y} \to Y$ such that whenever I is an interval in $\mathbb{R}$ and $z\colon I \to \tilde{Y}$ is a solution of $\tilde{\Pi}$ then $\Lambda \circ z\colon I \to Y$ is a solution of Π. (Here by imbedding we mean that Λ is injective, of class C^1, $\Lambda^{-1}\colon \Lambda(\tilde{Y}) \to \tilde{Y}$ is continuous, and for every $\tilde{y} \in \tilde{Y}$, $\mathrm{D}\Lambda(\tilde{y})$ is injective and its image splits, i.e. admits a topological complement.) In this case, $\mathcal{M} := \Lambda(\tilde{Y})$ is a C^1-submanifold of Y which is invariant relative to the local semiflow Π and Π restricted to $\mathcal{M}$ is 'isomorphic' to the local semiflow $\tilde{\Pi}$. We are particularly interested in the case where $Y = X^\alpha$, $\Pi = \Pi_g$ for some nonlinearity g, $\tilde{Y} := \mathbb{R}^n$ and $\tilde{\Pi}$ is generated by an ordinary differential equation

$$\dot{\xi} = h(\xi), \qquad \xi \in \mathbb{R}^n \tag{2.4_h}$$

where $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitzian. If $\tilde{\Pi}$ imbeds in Π (via the imbedding Λ) then we say the PDE (2.3_g) realizes the vector field $h\colon \mathbb{R}^n \to \mathbb{R}^n$ (via the imbedding Λ on the invariant manifold $\mathcal{M} = \Lambda(\mathbb{R}^n)$). In this case the qualitative behavior of the ODE is completely simulated by the corresponding PDE (restricted to the invariant manifold $\mathcal{M}$). An important candidate for the manifold $\mathcal{M}$ is given by the (global) center manifold:

The *global center manifold* $\mathcal{M}_c$ *of* (2.3_g) is, by definition, the set of all $u_0 \in X^\alpha$ for which there exists a solution $u: \mathbb{R} \to X^\alpha$ of (2.3_g) satisfying $u(0) = u_0$ and such that $\sup_{t\in\mathbb{R}} |(I-P)u(t)|_{X^\alpha} < \infty$. Obviously, the global center manifold of (2.3_g) is an invariant set for the local semiflow defined by that equation. We call an imbedding $\Lambda: \mathbb{R}^n \to X^\alpha$ *canonical* if $P\Lambda(\xi) = Q\xi$ for all $\xi \in \mathbb{R}^n$. It follows from the center manifold theory that for $g \in Y_1$ with $|g|_1$ small there is a canonical imbedding $\Lambda: \mathbb{R}^n \to X^\alpha$ such that $\Lambda(\mathbb{R}^n)$ is the global center manifold of (2.3_g).

Let us recall the following fundamental concept:

Definition 2.1. We say that the operator L satisfies the *Poláčik condition on* Ω if $\dim\ker A = N+1$ and for some (hence every) basis $\phi_1, \dots, \phi_{N+1}$ of $\ker A$, $R(x) \neq 0$ for some $x \in \Omega$, where

$$R(\phi_1, \dots, \phi_{N+1})(x) := \det \begin{pmatrix} \phi_1(x) & \nabla\phi_1(x) \\ \vdots & \vdots \\ \phi_{N+1}(x) & \nabla\phi_{N+1}(x) \end{pmatrix}, \quad x \in \Omega.$$

Remark. We have $n = N+1$ in case the Poláčik condition holds. One can also define a (weaker and less interesting) version of the Poláčik condition with $n = N$ (cf. [18]).

The following result was essentially proved in [14]by Poláčik and Rybakowski:

Theorem 2.2. *Let L be as above and let $\kappa > 1$; assume:*

1. *L satisfies the Poláčik codition on Ω;*
2. *$G \subset \Omega$ is an open set;*
3. *$R(x) \neq 0$ for all $x \in G$;*
4. *there is a function $b \in C^\infty(\overline{\Omega})$ with $\operatorname{supp} b \subset G$ such that*

$$\lambda < -\kappa$$

for every eigenvalue λ of the operator $L + b$ on Ω with Dirichlet (or Neumann) boundary condition on $\partial\Omega$.

Then there is a $\delta_1 > 0$ such that for every $h \in C^1_b(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ with $|h|_{C^1_b} < \delta_1$ there is a nonlinearity $f = f_h \in Y_1$ with the property that equation 2.1 (or 2.2) realizes the vector field h on an invariant manifold $\mathcal{M} = \mathcal{M}_h$ via an imbedding $\Lambda = \Lambda_h: \mathbb{R}^{N+1} \to X^\alpha$ of class C^1. Moreover, for each $m \geq 1$ there exists a $\delta_m > 0$ such that if $h \in C^m_b(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ and $|h|_{C^m_b} < \delta_m$ then f_h can be chosen such that $f_h \in Y_m$ and the imbedding $\Lambda_h: \mathbb{R}^{N+1} \to X^\alpha$ is of class C^m.

It was not realized in [14] that, for δ_1 small enough, the manifold $\mathcal{M}$ in the above theorem is actually the global center manifold of (2.3_g), although the imbedding Λ, in general, is not the canonical imbedding; this fact was observed and proved in [17]:

Theorem 2.3. *Let $a_{ij} \in C^{1,\gamma}\overline{\Omega}$ and $a \in C^{\gamma}(\overline{\Omega})$ and assume that the operator L with Dirichlet (or Neumann) boundary condition on $\partial\Omega$ satisfies the Poláčik condition on Ω.*

Then there is a $\delta_1 > 0$ such that for every $h \in C_b^1(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ with $|h|_{C_b^1} < \delta_1$ there is a nonlinearity $g = g_h \in Y_1$ with the property that equation (2.3$_g$) realizes the vector field h on the global center manifold $\mathcal{M}_{\mathrm{c}} = \mathcal{M}_{\mathrm{c}h}$ of (2.3$_g$) via a (not necessarily canonical) imbedding $\Lambda = \Lambda_h \colon \mathbb{R}^{N+1} \to X^\alpha$ of class C^1. If in addition $h \in C_b^m(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ then g_h can be chosen such that $g_h \in Y_m$ and the imbedding $\Lambda_h \colon \mathbb{R}^{N+1} \to X^\alpha$ is of class C^m.

In [14] Polàčik and Rybakowski proved that, if Ω is a ball in $\mathbb{R}^N$, then, for suitable potential functions a and b, the operators $\Delta + a$ and $\Delta + a + b$ with Dirichlet boundary condition on $\partial\Omega$ satisfy all the assumptions in Theorem 2.2. In [17] Rybakowski and the author of the present paper extended this result to the case of an arbitrary smooth bounded domain. In Section 6 we will prove that, given arbitrary $a_{ij} \in C^{1,\gamma}(\overline{\Omega})$, with $a_{ij} \equiv a_{ji}$ for all $i, j = 1, \ldots, N$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \quad x \in \overline{\Omega},\, \xi \in \mathbb{R}^N,$$

for some $c > 0$, then both for Dirichlet and Neumann boundary conditions on $\partial\Omega$ it is possible to construct functions $a, b \colon \overline{\Omega} \to \mathbb{R}$ such that all assumptions in Theorem 2.2 are satisfied.

3. Jet Realizations

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain. Let

$$L := \sum_{i,j=1}^{2} \partial_i(a_{ij}(x)\partial_j) + a(x)$$

be a strongly elliptic second order differential operator with symmetric smooth coefficients. Now fix $k \in \mathbb{N}$ and arbitrary integers $q_1, \ldots, q_k$ such that $1 \leq q_l \leq l$ for $l = 1, \ldots, k$. Let $\mathcal{E} = \mathcal{E}(q_1, \ldots, q_k)$ be the set of all functions $f \colon \mathbb{R}^4 \to \mathbb{R}$ of the form

$$f(x, y, s, w) = \sum_{l=1}^{k} a_l(x, y) s^{l-q_l} w^{q_l}, \quad (x, y, s, w) \in \mathbb{R}^4, \tag{3.1}$$

where $a_l \in H^2(\Omega)$ for $l = 1, \ldots, k$. For $f \in \mathcal{E}$ and $\varpi \in \mathbb{R}^2$, consider the equations

$$\begin{aligned} u_t &= Lu + f(x, y, u, u_\varpi), && t > 0,\ (x, y) \in \Omega \\ u &= 0, && t > 0,\ (x, y) \in \partial\Omega \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} u_t &= Lu + f(x, y, u, u_\varpi), && t > 0,\ (x, y) \in \Omega \\ \frac{\partial u}{\partial \nu} &= 0, && t > 0,\ (x, y) \in \partial\Omega, \end{aligned} \tag{3.3}$$

where $u_\varpi := \varpi \cdot \nabla u$.

Set $X = L^p(\Omega)$, $p > 2$, and let $A\colon D_A \subset X \to X$ be the sectorial operator induced by L with Dirichlet or Neumann boundary condtion on $\partial\Omega$, where $D_A = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ in the first case, and $D_A = W_N^{2,p}(\Omega)$ in the second case. The operator A generates the corresponding family X^α of fractional power spaces and fixing α with

$$(2+p)/(2p) < \alpha < 1$$

we have that

$$X^\alpha \subset C^1(\overline{\Omega})$$

with continuous inclusion. As usual we rewrite (3.2) and (3.3) as abstract equations in X:

$$\dot{u} + Au = \hat{f}^\varpi(u), \tag{3.4}$$

where $\hat{f}^\varpi(u)(x) := f(x, u(x), u_\varpi(x))$. Note $\hat{f}^\varpi \in C^\infty(X^\alpha, X)$. Define

$$X_0 := \ker A$$

and suppose

$$n = \dim X_0 \geq 1.$$

Let P_0 be the $L^2(\Omega)$-orthogonal projection of X onto X_0. Fix an arbitrary L^2-orthonormal basis $\phi_1, \dots, \phi_n$ of X_0; then

$$P_0 u(x) = \sum_{i=1}^n \phi_i(x) \int_\Omega u(y)\phi_i(y)dy, \quad u \in X. \tag{3.5}$$

Write

$$\phi(x) := (\phi_1(x), \dots, \phi_n(x));$$

note that the assignement

$$Q\colon \mathbb{R}^n \to X_0, \quad Q\xi := \xi \cdot \phi = \sum_{i=1}^n \xi_i \phi_i$$

is a linear isomorphism.

We can identify $\mathcal{E}$ with $\left(H^2(\Omega)\right)^k$; with the norm induced by this identification, $\mathcal{E}$ becomes a Banach space whose topology is stronger than the topology of locally uniform convergence of all derivatives $D^h_{(s,w)} f(x, y, s, w)$, $h = 0, \dots, k+1$ on $\overline{\Omega} \times \mathbb{R}^2$. We can apply the standard theory of center manifolds: we can find an open neighborhood $\mathcal{U}$ in $\mathcal{E}$, $0 \in \mathcal{U}$, and a map

$$\Lambda\colon \mathcal{U} \times B_1^n(0) \subset \mathcal{E} \times \mathbb{R}^n \to X^\alpha$$

with the following properties:

(1) $P_0 \Lambda(f, \xi) \equiv Q\xi$ and $\Lambda(0, \xi) \equiv Q\xi$;

(2) Λ is of class C^{k+1};
(3) the map $\Lambda_f(\cdot) := \Lambda(f, \cdot)$ is an imbedding and the set

$$\mathcal{M}_f^{\mathrm{loc}} := \{\Lambda_f(\xi) \mid \xi \in B_1^n(0)\}$$

is a local invariant manifold of (3.4). Moreover, if $v_f\colon B_1^n(0) \to \mathbb{R}^n$ is defined by

$$v_f(\xi) := Q^{-1} P_0 \hat{f}^{\varpi}(\Lambda_f(\xi)), \quad \xi \in B_1^n(0), \tag{3.6}$$

then the ODE defined by v_f imbeds, via Λ_f, in (3.4).

Define the map

$$\Psi\colon \mathcal{U} \subset \mathcal{E} \to C_b^k(B_1^n(0), \mathbb{R}^n)$$
$$\Psi(f)(\xi) := Q^{-1} \circ P_0 \circ \hat{f}^{\varpi} \circ \Lambda_f(\xi), \quad \xi \in B_1^n(0).$$

Simple computation shows that Ψ is of class C^1 and

$$D\Psi(0) f(\xi) = (Q^{-1} \circ P_0 \circ \hat{f}^{\varpi})(Q\xi).$$

Let $J_0^k(\mathbb{R}^n)$ denote the set of all k-jets on $\mathbb{R}^n$ mapping 0 into itself. Equivalently, $h \in J_0^k(\mathbb{R}^n)$ if and only if h is a polynomial on $\mathbb{R}^n$ of order $\leq k$ with $h(0) = 0$. We say that a jet h can be realized in (3.2) (or (3.3)) by the non-linearity f if the k-th order Taylor polynomial of the vector field v_f defined by (3.6) is equal to h. We introduce the linear bounded operator

$$T^k\colon C_b^k(B_1^n(0), \mathbb{R}^n) \to J_0^k$$
$$(T^k v)(\xi) = \sum_{i=0}^{k} \frac{1}{i!} D^i v(0) \xi^i, \quad v \in C_b^k(B_1^n(0), \mathbb{R}^n),\ \xi \in \mathbb{R}^n.$$

We want to find a condition which guarantees that $D(T^k \circ \Psi)(0)$ is surjective onto J_0^k. Our starting point is the abstract surjectivity condition

(SC) For every polynomial function $h\colon \mathbb{R}^n \to \mathbb{R}^n$ of degree $\leq k$, $h(0) = 0$, there is an $f \in \mathcal{E}(q_1, \ldots, q_k)$ such that

$$T^k(D\Psi(0) f) = h. \tag{3.7}$$

By (3.1) and (3.5), after straightforward manipulations, the Riesz representation theorem gives us an equivalent, but more convenient form of this condition. We introduce the following notations: given $\gamma, \beta \in \mathbb{N}_0^n$, we say that $\gamma \leq \beta$ iff $\gamma_i \leq \beta_i$, $i = 1, \ldots, n$. If $\gamma \in \mathbb{R}_0^n$, $\varpi \in \mathbb{R}^2$, set $\phi^\gamma := \phi_1^{\gamma_1} \cdots \phi_n^{\gamma_n}$ and $\phi_\varpi^\gamma := \phi_{1\varpi}^{\gamma_1} \cdots \phi_{n\varpi}^{\gamma_n}$. Moreover, set

$$\epsilon_j := (\underbrace{0, ..., 0, 1}_{j}, 0, ..., 0) \in \mathbb{N}_0^n.$$

With these notations, the surjectivity condition is equivalent to the following independence condition (cf. [16]):

(IC) For every $l = 1, \dots, k$ and for every q, $1 \le q \le l$, the functions

$$\left\{ \sum_{\substack{\gamma \le \beta \\ |\gamma| = q}} \frac{1}{\gamma!(\beta-\gamma)!} \phi^{\beta-\gamma+\epsilon_j} \phi_{\varpi}^{\gamma} \right\}_{\substack{j=1,\dots,n \\ |\beta|=l}}$$

are linearly independent.

Theorem 3.1. *Let n and $k \in \mathbb{N}$. Assume $\dim\ker A = n$ and assume there is an $L^2(\Omega)$-orthonormal basis $\phi_1, \dots, \phi_n$ of $\ker A$ and a vector $\varpi \in \mathbb{R}^2$ such that (IC) is satisfied up to the order k. Then there is an open neighborhood $\mathcal{B}$ of 0 in $J_0^k(\mathbb{R}^n)$ such that every jet $h \in \mathcal{B}$ can be realized in (3.7) by a a nonlinearity $f \in \mathcal{E}$.*

Remark. As in [12], theorem 3.1 can be srenghtened to obtain realizability of C^m-families of jets; this implies that a dense set of vector fields in $\mathbb{R}^n$ can be realized, up to flow equivalence, in equation (3.2) (or (3.3)) by nonlinearities of the form (3.1).

Remark. Choosing $q_l = 1$ for all $l = 1, \dots, k$ we obtain a jet realization result for nonlinearities which are polynomials in u and which are linear functions of ∇u.

In [16] Rybakowski and the author of the present paper proved that, given $n, k \in \mathbb{N}$, there exists a smooth bounded domain Ω and a potential $a\colon \Omega \to \mathbb{R}$ such that the operator $L = \Delta + a$ with Dirichlet boundary condition has an n-dimensional kernel spanned by eigenfunctions $\phi_1, \dots, \phi_n$ satisfying (IC) up to the order k with $\varpi = (0, 1)$. In Section 7 we will prove that, given arbitrary $a_{ij} \in C^{1,\gamma}(\overline{\Omega})$, with $a_{ij} \equiv a_{ji}$ for all $i, j = 1, 2$ and

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \ge c|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^2,$$

for some $c > 0$, and given $n, k \in \mathbb{N}$, then both for Dirichlet and Neumann boundary conditions on $\partial\Omega$ it is possible to construct a potential $a\colon \overline{\Omega} \to \mathbb{R}$ such that $\dim\ker A = n$ and (IC) is satisfied up to the order k by an appropriate basis of $\ker A$. Here A is the abstract operator associated to the differential operator

$$Lu := \sum_{i,j=1}^{2} \partial_i(a_{ij}\partial_j u) + au$$

with Dirichlet (or Neumann) boundary condition.

4. Perturbation and convergence of eigenfunctions

This and the next sections are devoted to the construction of potential functions with the properties described in sections 2 and 3. First, we recall two general results on perturbation and convergence of eigenvalues and eigenfunctions of selfadjoint operators in Hilbert spaces. The reader is referred to [17] for a detailed discussion.

We use the following notation: If X is a normed space and $r > 0$, then $\mathrm{B}_r(c)$ denotes the open ball in X of radius r centered at c. Moreover, $\mathrm{B}_r := \mathrm{B}_r(0)$. Given normed spaces X and Y, we denote by $\mathcal{L}(X, Y)$ (resp. by $\mathcal{L}^p(X^p, Y)$) the space of all bounded linear (resp. p-linear) operators from X (resp. from X^p) to Y, endowed with the operator norm. Given a real Hilbert space H, $\mathcal{L}_{\mathrm{sym}}(H, H)$ is the (closed) linear subspace of $\mathcal{L}(H, H)$ consisting of all symmetric operators.

By $\mathcal{S}_p$ we denote the (finite dimensional) space of all real symmetric $p \times p$-matrices, endowed with an arbitrary norm. The spectrum of A is denoted by $\operatorname{spec} A$.

Let H be an infinite dimensional real Hilbert space, and let $A\colon \operatorname{dom} A \to H$ be linear, symmetric, bounded below and with compact resolvent. Then it follows that the spectrum of A is a countable set of real eigenvalues of finite multiplicity. This set is bounded below. We can therefore uniquely define a nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which contains exactly the eigenvalues of A, each one repeated according to its multiplicity. We call $(\lambda_n)_{n\in\mathbb{N}}$ *the repeated sequence of eigenvalues of A*.

Definition 4.1. We say that the triple $(H, \mathcal{G}, A)$ *is of type* $[p, M, \eta, \theta]$ if and only if the following properties hold:

(1) $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\mathrm{sym}}(H, H)$.
(2) p is a positive integer, M, η and θ are positive reals.
(3) Let $(\lambda_n)_{n\in\mathbb{N}}$ be the repeated sequence of the eigenvalues of A. There exist real numbers γ_1 and γ_2 and $l \in \mathbb{N}_0$ such that, setting $\lambda_0 = -\infty$,
$$0 < \gamma_2 - \gamma_1 < M,$$
$$\lambda_l < \gamma_1 - 4\eta < \gamma_1 < \lambda_{l+1} \le \lambda_{l+p} < \gamma_2 < \gamma_2 + 4\eta < \lambda_{l+p+1}.$$
(4) There exists an H-orthonormal set of vectors ϕ_j, $j = 1, \ldots, p$, in $\operatorname{dom} A$ such that $A\phi_j = \lambda_{l+j}\phi_j$, $j = 1, \ldots, p$, and such that the operator $T\colon \mathcal{G} \to \mathcal{S}_p$
$$B \mapsto (\langle B\phi_i, \phi_j\rangle)_{ij}$$
is such that
$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta,$$
i.e. the image of the unit ball (at zero) in $\mathcal{G}$ contains the θ-ball (at zero) in $\mathcal{S}_p$.

The following theorem was proved in [17]:

Theorem 4.2. *For every $(p, M, \eta, \theta) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ there exists a positive number $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ with the following property:*

whenever the triple $(H, \mathcal{G}, A)$ is of type $[p, M, \eta, \theta]$, l, γ_1 and γ_2 are as in Definition 4.1 (with respect to the triple $(H, \mathcal{G}, A)$), $0 < \alpha \le \alpha_0$ and $(\mu_1, \ldots, \mu_p) \in \mathbb{R}^p$ is nondecreasing with $|\mu_j - \lambda_{l+j}| < \alpha$ for $j = 1, \ldots, p$, and if $\mathcal{D}$ is an arbitrary linear dense subspace of $\mathcal{G}$, then there exists a $B \in \mathcal{D}$ with $|B| < (1/2)\theta\alpha$, such that, if $(\lambda_n(B))_{n\in\mathbb{N}}$ denotes the repeated sequence of eigenvalues of $A + B$ and $\lambda_0(B) := -\infty$, then

$$\lambda_l(B) < \gamma_1 - 3\eta < \gamma_1 - \eta < \lambda_{l+1}(B) \le \lambda_{l+p}(B) < \gamma_2 + \eta < \gamma_2 + 3\eta < \lambda_{l+p+1}(B) \tag{4.1}$$

and

$$\lambda_{l+j}(B) = \mu_j, \quad j = 1, \ldots, p.$$

Now let Ω and D be bounded domains in $\mathbb{R}^N$ with $\overline{D} \subset \Omega$. Consider a second order elliptic differential operator L on Ω. Define the following sequence of differential operators on Ω:

$$\begin{aligned} L_k u &= Lu + \beta_k b_k(x) u, && x \in \Omega \\ u(x) &= 0, && x \in \partial\Omega \end{aligned}$$

or

$$\begin{aligned} L_k u &= Lu + \beta_k b_k(x) u, && x \in \Omega \\ \frac{\partial u}{\partial \nu}(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

Here, L is a second order elliptic differential operator, β_k, $k \in \mathbb{N}$, are positive real numbers and b_k, $k \in \mathbb{N}$, are (coefficient) functions. It was proved in [17] that under appropriate hypotheses on β_k and b_k the eigenvalues of L_k converge, as $k \to \infty$, to the eigenvalues of the following 'limit' differential operator L_∞ on D:

$$\begin{aligned} L_\infty u &= Lu, && x \in D \\ u(x) &= 0, && x \in \partial D. \end{aligned}$$

It was also proved H^1 convergence of the corresponding eigenfunctions. The hypotheses are, essentially, that $\beta_k b_k(x)$ is very small on D but very large outside of D. To give a unified treatment for the different boundary conditions, it is more convenient to work not with differential operators but rather with the corresponding bilinear forms or even with certain abstract bilinear forms as we shall now explain.

In what follows, all vector spaces are over the reals.

Definition 4.3. Let V be a vector space and $a : V \times V \to \mathbb{R}$ be symmetric bilinear form on V. If $\lambda \in \mathbb{R}$, $u \in V \setminus \{0\}$ satisfy

$$a(u, v) = \lambda \langle u, v \rangle \quad \text{for all } v \in V$$

then we say that λ is *a proper value of* a and u is *a proper vector of* a, *corresponding to* λ. The dimension of the span of all proper vectors of a corresponding to λ is called *the multiplicity of* λ. If the set of proper values of a is countably infinite and if each proper value has finite multiplicity then *the repeated sequence of the proper values of* a is the uniquely determined nondecreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ which contains exactly the proper values of a and the number of occurrences of each proper value in this sequence is equal to its multiplicity.

The following theorem was proved in [17]:

Theorem 4.4. *Assume the following hypotheses:*

1. $\Omega \subset \mathbb{R}^N$ *is a bounded domain and* $D \subset \mathbb{R}^N$ *is a Lipschitz domain with* $\overline{D} \subset \Omega$. *Given a function* u *defined on* D, $u^\sim$ *denotes the trivial extension of* u *to* Ω.

(2) *b, $b_k\colon\overline{\Omega}\to\mathbb{R}$, $k\in\mathbb{N}$, are continuous functions and β_k, $k\in\mathbb{N}$ are positive real numbers. Moreover, $b(x)>0$ for $x\in\Omega\setminus D$, $b_k\to b$ uniformly on $\overline{\Omega}$, $\beta_k\to\infty$, $\inf_{\substack{x\in\Omega\\ k\in\mathbb{N}}}\{\beta_k b_k(x)\}>-\infty$ and $\sup_{x\in D}\{\beta_k|b_k(x)|\}\to 0$.*

(3) *V is a closed linear subspace of $H^1(\Omega)$ such that whenever $u\in H_0^1(D)$ then $u^\sim\in V$. V is endowed with the scalar product of $H^1(\Omega)$.*

(4) *$\|\cdot\|_D$ (resp. $\|\cdot\|$) denotes the $H^1(D)$- (resp. the $H^1(\Omega)$-) norm, $|\cdot|_D$ (resp. $|\cdot|$) denotes the $L^2(D)$- (resp. the $L^2(\Omega)$-) norm and $\langle\cdot,\cdot\rangle_D$ (resp. $\langle\cdot,\cdot\rangle$) denotes the $L^2(D)$- (resp. the $L^2(\Omega)$-) scalar product.*

(5) *$a\colon V\times V\to\mathbb{R}$ is a symmetric bilinear form and there are constants d, C, $\alpha\in\mathbb{R}$, $\alpha>0$, such that, for all u, $v\in V$,*

$$|a(u,v)|\le C\|u\|\|v\|$$
$$a(u,u)\ge\alpha\|u\|^2-d|u|^2.$$

Let $a_\infty\colon H_0^1(D)\times H_0^1(D)\to\mathbb{R}$ be the restriction of a to $H_0^1(D)$. For $k\in\mathbb{N}$ let $(\lambda_n^k)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of the symmetric bilinear form $a_k\colon V\times V\to\mathbb{R}$ defined by

$$a_k(u,v)=a(u,v)+\beta_k\int_\Omega b_k(x)u(x)v(x)\,\mathrm{d}x$$

and $(u_n^k)_{n\in\mathbb{N}}$ be an $L^2(\Omega)$-orthonormal sequence of corresponding proper vectors of a_k. Moreover, let $(\mu_n)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of a_∞.

Then there is an increasing function $\phi\colon\mathbb{N}\to\mathbb{N}$ and a sequence $(v_n)_{n\in\mathbb{N}}$ in $H_0^1(D)$ such that for every $n\in\mathbb{N}$, v_n is a proper vector of a_∞ corresponding to μ_n, the subsequence $(\lambda_n^{\phi(k)})_{k\in\mathbb{N}}$ of $(\lambda_n^k)_{k\in\mathbb{N}}$ converges to μ_n and the subsequence $(u_n^{\phi(k)})_{k\in\mathbb{N}}$ of $(u_n^k)_{k\in\mathbb{N}}$ converges to $v_n{}^\sim$ in V, as $k\to\infty$.

For our pourposes, we need more precise information about convergence of eigenfunctions when the bilinear form in Theorem 4.4 arises from the variational formulation of a linear elliptic equation; if this is the case, then, for all n, $u_n^{\phi(k)}|D\to v_n$ in $C^1_{\mathrm{loc}}(D)$ as $k\to\infty$:

Theorem 4.5. *Assume the same hypotheses of Theorem 4.4. Moreover, assume*

$$a(u,v)=\int_\Omega A(x)\nabla u(x)\cdot\nabla v(x)dx+\int_\Omega a(x)u(x)v(x)dx,$$

where $A(x):=(a_{ij}(x))_{i,j}$ is a symmetric $N\times N$-matrix, $A(x)\xi\cdot\xi\ge c|\xi|^2$ for all $x\in\overline{\Omega}$ and all $\xi\in\mathbb{R}^N$ for some $c>0$, $a_{ij}\colon\overline{\Omega}\to\mathbb{R}$ are of class $C^{1,\gamma}(\overline{\Omega})$ and $a,b,b_k\colon\overline{\Omega}\to\mathbb{R}$ are of class $C^\gamma(\overline{\Omega})$. Then, for all n,

$$u_n^{\phi(k)}\to v_n\quad\text{as } k\to\infty$$

in $C^1_{\mathrm{loc}}(D)$.

Before proving Theorem 4.5 we need to introduce some notation and to prove a technical lemma. For $N\in\mathbb{N}$, $p\in\mathbb{R}$, $1<p<+\infty$, $p<N$, the Sobolev exponent p^* is defined by

$$p^*:=\frac{pN}{N-p}.$$

For $p < N$, we define inductively

$$\begin{aligned} p^{0*} &:= p \\ p^{l*} &:= (p^{(l-1)*})^*, \quad l > 0. \end{aligned}$$

for all l such that $p^{(l-1)*} < N$.

Lemma 4.6. *There are $l \in \mathbb{N}$ and $p \in \mathbb{R}$, $1 < p \leq 2$, such that*

$$p^{l*} < N < p^{(l+1)*}.$$

Proof. First we prove that there exists an l such that $2^{l*} \geq N$. Assume by contradiction that $2^{l*} < N$ for all l; since

$$\frac{1}{2^{l*}} - \frac{1}{2^{(l+1)*}} = \frac{1}{N},$$

then for all l

$$\frac{1}{N} \leq \frac{1}{2^{l*}} = \frac{1}{2^{(l-1)*}} - \frac{1}{N} = \cdots = \frac{1}{2^*} - \frac{l}{N},$$

a contradiction. So we have proved that there is some l such that $2^{l*} \geq N$. If $2^{l*} > N$, we have concluded with $p = 2$. Otherwise we define, for all $\epsilon > 0$, $p_\epsilon := 2 - \epsilon$. It is clear that, for all $\epsilon > 0$,

$$(2-\epsilon)^{l*} < 2^{l*} = N$$

and that

$$(2-\epsilon)^{l*} \nearrow N \quad \text{as } \epsilon \to 0;$$

This implies that $(2-\epsilon)^{(l+1)*}$ is defined for all $\epsilon > 0$ and

$$(2-\epsilon)^{(l+1)*} = \frac{(2-\epsilon)^{l*} N}{N - (2-\epsilon)^{l*}} \to +\infty$$

as $\epsilon \to 0^+$; we take $p := p_\epsilon$ with $\epsilon > 0$ sufficiently small and we have concluded. $\square$

Proof of Theorem 4.5. Fix n and $D' \subset\subset D$; it is not a restriction to assume that D' has smooth boundary. Let M be a positive constant such that

$$\begin{aligned} \sup_{i,j=1,\ldots,N} \sup_{x \in \overline{\Omega}} |a_{ij}(x)| &< M, \\ \sup_{i,j=1,\ldots,N} \sup_{x,y \in \overline{\Omega}} |a_{ij}(x) - a_{ij}(y)|/|x-y|^\gamma &< M, \\ \sup_{x \in \overline{\Omega}} |a(x)| &< M, \\ \sup_{k \in \mathbb{N}} \sup_{x \in D} \beta_k |b_k(x)| &< M, \\ \sup_{k \in \mathbb{N}} |\lambda_n^k| &< M. \end{aligned}$$

For all k, $u_n^{\phi(k)}$ satisfies:

$$\int_\Omega A(x)\nabla u_n^{\phi(k)}(x)\cdot\nabla v(x)dx+\int_\Omega\left(a(x)+\beta_{\phi(k)}b_{\phi(k)}(x)-\lambda_n^{\phi(k)}\right)u_n^{\phi(k)}(x)v(x)dx=0 \quad \text{for all } v\in V.$$

In particular,

$$\int_D A(x)\nabla u_n^{\phi(k)}(x)\cdot\nabla v(x)dx+\int_D\left(a(x)+\beta_{\phi(k)}b_{\phi(k)}(x)-\lambda_n^{\phi(k)}\right)u_n^{\phi(k)}(x)v(x)dx=0 \quad \text{for all } v\in H_0^1(D).$$

By classical regularity results for elliptic equations (see e.g. [6, Thms. 8.8, 9.16]), it follows that, for all k,

$$u_n^{\phi(k)}(x)\in W^{2,p}_{\mathrm{loc}}(D)\quad\text{for all } p$$

and

$$-\operatorname{div}(A(x)\nabla u_n^{\phi(k)}(x))+\left(a(x)+\beta_{\phi(k)}b_{\phi(k)}(x)-\lambda_n^{\phi(k)}\right)u_n^{\phi(k)}(x)=0\quad\text{a.e. in } D.$$

Now take $p\in\mathbb{R}$, $1<p\le 2$, and $l\in\mathbb{N}$ such that

$$p^{l*}<N<p^{(l+1)*}$$

(this is possible thanks to Lemma 4.6). Fix open sets D_j, $j=0,\ldots,l$, with smooth boundaries and such that

$$D':=D_{l+1}\subset\subset D_l\subset\subset\cdots\subset\subset D_1\subset\subset D_0\subset\subset D=:D_{-1}.$$

By [6, Th. 9.11], there are constants $C_j=C(N,M,D,D_j,p^{j*})$, $j=0,\ldots,l+1$, such that, for all $k\in\mathbb{N}$ and all $j=0,\ldots,l+1$,

$$\left\|u_n^{\phi(k)}\right\|_{W^{2,p^{j*}}(D_j)}\le C_j\left\|u_n^{\phi(k)}\right\|_{L^{p^{j*}}(D_{j-1})};$$

moreover, by the Sobolev imbedding theorems, there exist constants $K_j=K(N,D_j,p^{j*})$, $j=0,\ldots,l+1$, such that, for all $k\in\mathbb{N}$ and all $j=0,\ldots,l+1$,

$$\left\|u_n^{\phi(k)}\right\|_{L^{p^{j*}}(D_{j-1})}\le K_j\left\|u_n^{\phi(k)}\right\|_{W^{2,p^{(j-1)*}}(D_{j-1})}.$$

These inequalities together imply that there exists a constant C such that, for all k,

$$\left\|u_n^{\phi(k)}\right\|_{W^{2,p^{(l+1)*}}(D')}\le C\left\|u_n^{\phi(k)}\right\|_{L^p(D)}. \tag{4.2}$$

Now, since $u_n^{\phi(k)}\to v_n{}^{\sim}$ in $H^1(\Omega)$, we have that the sequence $u_n^{\phi(k)}$ is bounded in $L^p(D)$; then, by (4.2), we deduce that the sequence $u_n^{\phi(k)}|D'$ is bounded also in $W^{2,p^{l+1}*}(D')$. Since $p^{(l+1)*}>N$, the Sobolev imbedding theorem implies that

$$W^{2,p^{(l+1)*}}(D')\hookrightarrow C^1(\overline{D}')$$

with compact inclusion. Then we conclude that

$$u_n^{\phi(k)}\to v_n\quad\text{as } k\to\infty$$

in $C^1(\overline{D}')$. Since D' was arbitrary, we finally conclude that

$$u_n^{\phi(k)}\to v_n\quad\text{as } k\to\infty$$

in $C^1_{\mathrm{loc}}(D)$. $\square$

5. Localization

Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let L' be a differential operator of the form

$$L'u = \sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j u).$$

We assume that L' is uniformly elliptic and symmetric and its coefficients are smooth. As we have seen, the problem of realization of vector fields and jets in scalar parabolic PDEs reduces to the problem of constructing a potential function a in such a way that the operator

$$Lu := L'u + au = \sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j u) + au$$

with Dirichlet or Neumann boundary conditions has a high dimension kernel, spanned by eigenfunctions satisfying certain nondegeneracy conditions. As a first step towards this direction we will prove a sort of "localization lemma". The content of this lemma is essentially the following: we can always find some small subdomain $D \subset \Omega$ and some potential a on D in such a way that the operator $L = L' + a$ on D with Dirichlet condition on ∂D satisfies the above properties, provided we are able to construct a potential a_0 on some other open bounded domain S in such a way that the operator $\Delta + a_0$ on S with Dirichlet condition on S satisfies the same properties.

Lemma 5.1. *Let $\Omega, S \subset \mathbb{R}^N$ be open bounded domains; assume S has $C^{2,\gamma}$ boundary. Let $a_{ij}\colon \Omega \to \mathbb{R}$ be of class $C^{1,\gamma}$, $i, j = 1, \dots, N$, $a_{ij} \equiv a_{ji}$, $i, j = 1, \dots, N$, and*

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq c|\xi|^2, \quad x \in \Omega,\ \xi \in \mathbb{R}^N$$

for some $c > 0$. Consider the differential operator

$$L' = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j).$$

Let us suppose there exists a $C^\gamma(\overline{S})$ potential $a_0\colon \overline{S} \to \mathbb{R}$ such that the operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S has an n-dimensional kernel, spanned by $L^2(S)$-orthonormal eigenfunctions $\phi_1, \dots, \phi_n$, and that the set of functions

$$\{\phi_i\phi_j,\ 1 \leq i \leq j \leq n\}$$

is linearly independent. Then for every $\epsilon > 0$ there exist an invertible affine transformation $W\colon \mathbb{R}^N \to \mathbb{R}^N$, an open bounded domain $D \subset\subset \Omega$ and a potential $a\colon \overline{\Omega} \to \mathbb{R}$, $a \in C^\gamma(\overline{\Omega})$, with the following properties:

1. *$D = W(S)$;*
2. *$L' + a(x)$ on D with Dirichlet boundary condition on ∂D has an n-dimensional kernel spanned by $L^2(D)$-orthonormal functions $\psi_1, \dots, \psi_n$;*
3. *$\left\|(\det DW)^{1/2}\psi_i(W(\cdot)) - \phi_i(\cdot)\right\|_{C^1(\overline{D})} < \epsilon$, $i = 1, \dots, n$.*

Moreover, if there exists a $C^\gamma(\overline{S})$ function $b_0\colon \overline{S} \to \mathbb{R}$ and a positive constant κ such that the operator $\Delta + a_0(x) + b_0(x)$ on S with Dirichlet boundary condition on ∂S has all eigenvalues $< -\kappa$, then W, D and $a(x)$ above can be chosen in such a way that, setting $b(x) := \rho^{-2} b_0(W^{-1}(x))$ for an appropriate $\rho > 0$, the operator $L' + a(x) + b(x)$ on D with Dirichlet boundary condition on ∂D has all eigenvalues $< -\kappa$.

Proof. First we introduce some notation; we indicate by λ_i, $i \in \mathbb{N}$, the repeated sequence of the eigenvalues of the operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S; in the hypothesis, we have assumed that this operator has an n-dimensional kernel, so there is an $l > 1$ such that $\lambda_l < \lambda_{l+1} = \ldots = \lambda_{l+n} = 0 < \lambda_{l+n+1}$.

We procede in several steps:

1^{st} *step*: Take $\bar{x} \in S$ and $x_0 \in \Omega$; let $G_0 := G(x_0)$, where $G(x) := (a_{ij}(x))_{ij}$; G_0 is a symmetric positive definite $N \times N$-matrix, so we can take an invertible $N \times N$-matrix Q such that $G_0 = QQ^T$. We define the affine transformation

$$\begin{aligned} Z\colon \mathbb{R}^N &\to \mathbb{R}^N \\ x &\mapsto x_0 + Q(x - \bar{x}) \end{aligned}$$

and we set $D_1 := Z(S)$; finally, we define

$$\begin{aligned} \tilde{a}\colon D_1 &\to \mathbb{R} \\ \tilde{a}(x) &:= a_0(Z^{-1}(x)). \end{aligned}$$

The operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S has the same repeated sequence of eigenvalues of the operator $\operatorname{div}(G_0 \nabla) + \tilde{a}(x)$ on D_1 with Dirichlet boundary condition on ∂D_1. In particular, this last operator has an n-dimensional kernel spanned by the $L^2(D_1)$-orthonormal functions

$$\tilde{\phi}_i(x) := (\det Q)^{-1/2} \phi_i(Z^{-1}(x)), \quad i = 1, \ldots, n.$$

Obviously, the set of functions

$$\left\{ \tilde{\phi}_i \tilde{\phi}_j,\ 1 \leq i \leq j \leq n \right\}$$

is linearly independent.

2^{nd} *step*: For $\rho \geq 0$ sufficiently small, we consider the differential operators

$$L_\rho := \operatorname{div}(G(x_0 + \rho(x - x_0))\nabla) + \tilde{a}(x)$$

on D_1 with Dirichlet boundary condition on ∂D_1; note $L_0 = \operatorname{div}(G_0 \nabla) + \tilde{a}(x)$. We indicate by λ_i^ρ, $i \in \mathbb{N}$, the repeated sequence of eigenvalues of L_ρ.

Let A_ρ be the sectorial operator in $L^2(D_1)$ corresponding to L_ρ; since the boundary of D_1 is of class $C^{2,\gamma}$ and the coefficients a_{ij} are in $C^{1,\gamma}$, it follows that, for all ρ, the domain of A_ρ is $H^2(D_1) \cap H_0^1(D_1)$. Morever, the map

$$\begin{gathered} \rho \mapsto A_\rho \\ [0, \rho_0[\to \mathcal{L}(H^2(D_1) \cap H_0^1(D_1), L^2(D_1)) \end{gathered}$$

is continuous. This implies that $\lambda_i^\rho \to \lambda_i$ as $\rho \to 0$ for all i; then we can find some $\eta > 0$ such that, for all sufficiently small ρ,

$$\lambda_l^\rho < -4\eta < -\eta < \lambda_{l+1}^\rho \leq \cdots \leq \lambda_{l+n}^\rho < \eta < 4\eta < \lambda_{l+n+1}^\rho;$$

in particular, the set

$$\{\lambda_{l+1}^\rho, \dots, \lambda_{l+n}^\rho\}$$

is a spectral set of A_ρ and we can consider the corresponding spectral projection P_ρ and the corresponding spectral invariant subspace X_ρ. By the general formula

$$P_\rho = \frac{1}{2\pi i} \int_\Gamma (\zeta - A_\rho)^{-1} d\zeta,$$

it follows that the map

$$\begin{gathered} \rho \mapsto P_\rho \\ [0, \rho_0[\to \mathcal{L}(L^2(D_1), H^2(D_1) \cap H_0^1(D_1)) \end{gathered}$$

is continuous. By using the spectral projection P_ρ together with the Grahm-Schmidt orthonormalization algorithm, we can find, for all ρ, an $L^2(D_1)$-orthonormal basis $\tau_1^\rho, \dots,$ τ_n^ρ of X_ρ, with

$$\tau_i^\rho \to \tilde{\phi}_i \quad \text{as } \rho \to 0$$

in $H^2(D_1) \cap H_0^1(D_1)$ for all $i = 1, \dots, n$. In order to apply Theorem 4.2, we need a *basis of eigenfunctions*; to overcome this difficulty, we procede in the following way: for all $\rho > 0$ we can find an orthogonal $n \times n$-matrix $R_\rho = (r_{ij}^\rho)_{ij}$ such that the functions

$$\chi_i^\rho := \sum_{j=1}^n r_{ij}^\rho \tau_j^\rho, \quad i = 1, \dots n,$$

are an $L^2(D_1)$-orthonormal basis of eigenfunctions of X_ρ, with

$$A_\rho \chi_i^\rho = \lambda_i^\rho \chi_i^\rho, \quad i = 1, \dots, n.$$

By compactness, we can find a sequence $(\rho_k)_{k \in \mathbb{N}}$, with $\rho_k \to 0$ as $k \to \infty$, and an orthogonal matrix $R = (r_{ij})_{ij}$, such that

$$R^{\rho_k} \to R \quad \text{as } k \to \infty.$$

It follows that, for all $i = 1, \dots, n$,

$$\chi_i^{\rho_k} \to \sum_{j=1}^{n} r_{ij} \tilde{\phi}_j =: \chi_i \quad \text{as } k \to \infty$$

in $H^2(D_1) \cap H_0^1(D_1)$. Of course $\chi_1, \dots, \chi_n$ are an orthonormal basis of the n-dimensional kernel of $L_0 = \operatorname{div}(G_0 \nabla) + \tilde{a}(x)$. Moreover the set of functions

$$\{\chi_i \chi_j,\ 1 \le i \le j \le n\}$$

is still linearly independent.

For $c \in C^0(\overline{D_1})$ let $B_c \in \mathcal{L}_{\mathrm{sym}}(L^2(D_1), L^2(D_1))$ be the map

$$(Bu)(x) = c(x)u(x), \qquad u \in L^2(D_1),\, x \in \mathrm{D}_1.$$

Note that

$$|B_c|_{\mathcal{L}(L^2(D_1), L^2(D_1))} = |c|_{C^0(\overline{\mathrm{D}_1})}. \tag{5.1}$$

Let $\mathcal{G}$ be the set of all B_c with $c \in C^0(\overline{D_1})$. It follows that $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\mathrm{sym}}(L^2(D_1), L^2(D_1))$. Now, since the functions $\{\chi_i \chi_j,\ 1 \le i \le j \le n\}$ are linearly independent, it is easy to see that the operator $T\colon \mathcal{G} \to \mathcal{S}_p$

$$B \mapsto (\langle B\chi_i, \chi_j \rangle)_{ij}$$

is surjective. By the open mapping theorem there is a $\theta > 0$ such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta.$$

For $k \in \mathbb{N}$ let $T_k\colon \mathcal{G} \to \mathcal{S}_p$ be the map

$$B \mapsto (\langle B\chi_i^{\rho_k}, \chi_j^{\rho_k} \rangle)_{ij}.$$

Then $T_k \to T$ in $\mathcal{L}(\mathcal{G}, \mathcal{S}_p)$ so it is easy to see that

$$T_k(\mathrm{B}_1) \supset \mathrm{B}_\theta \qquad \text{for } k \text{ large enough.}$$

Moreover we have

$$\lambda_l^{\rho_k} < -4\eta < -\eta < \lambda_{l+1}^{\rho_k} \le \lambda_{l+n}^{\rho_k} < \eta < 4\eta < \lambda_{l+n+1}^{\rho_k},\ k \text{ large enough.} \tag{5.2}$$

Let $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ be as in Theorem 4.2. For all large k, there is an $\alpha_k > 0$ such that $|\lambda_{l+j}^{\rho_k}| < \alpha_k < \alpha_0$ for $j = 1, \dots, n$ and $\alpha_k \to 0$ as $k \to 0$. Thus by Theorem 4.2 (with $A := A_{\rho_k}$, $\mu_j := 0$, $\lambda_{l+j} := \lambda_{l+j}^{\rho_k}$ for $j = 1, \dots, n$ and $\mathcal{D}$ equal to the set of all B_c where c is a $C^\gamma(\mathbb{R}^N)$ function) there exists, for each large k, a $C^\gamma(\mathbb{R}^N)$ function $c_k\colon \mathbb{R}^N \to \mathbb{R}$

such that $|c_k|_{C^0(\overline{D}_1)} < (1/2)\theta\alpha_k$ and such that if $(\hat{\lambda}_n^{\rho_k})_{n\in\mathbb{N}}$ denotes the repeated sequence of eigenvalues of $L_{\rho_k} + c_k$, then

$$\hat{\lambda}_l^{\rho_k} < -3\eta < -\eta < \hat{\lambda}_{l+1}^{\rho_k} \le \hat{\lambda}_{l+n}^{\rho_k} < \eta < 3\eta < \hat{\lambda}_{l+n}^{\rho_k} \tag{5.3}$$

and

$$\hat{\lambda}_{l+j}^{\rho_k} = 0, \quad j = 1, \ldots, n. \tag{5.4}$$

So we have found a sequence of potentials $c_k\colon \mathbb{R}^N \to \mathbb{R}$, $c_k \in C^\gamma(\mathbb{R}^N)$, $c_k \to 0$ in $C^0(\overline{D_1})$ as $k \to \infty$, such that, for all (sufficiently large) k, the operator $L_{\rho_k} + c_k(x)$ on D_1 with Dirichlet boundary condition on ∂D_1 has an n-dimensional kernel.

3^{rd} *step*: For $c \in C^0(\overline{D_1})$ let $B_c \in \mathcal{L}(L^p(D_1), L^p(D_1))$ be the map

$$(Bu)(x) = c(x)u(x), \qquad u \in L^p(D_1),\, x \in \mathrm{D}_1.$$

Note that

$$|B_c|_{\mathcal{L}(L^p(D_1),L^p(D_1))} = |c|_{C^0(\overline{\mathrm{D}_1})}. \tag{5.5}$$

Let A_ρ be the sectorial operator in $L^p(D_1)$ corresponding to L_ρ; since the boundary of D_1 is of class $C^{2,\gamma}$ and the coefficients a_{ij} are in $C^{1,\gamma}$, it follows that, for all ρ, for all $c \in C^0(\overline{D}_1)$ and for all $p > 1$, the domain of $A_\rho + B_c$ is $W^{2,p}(D_1) \cap W_0^{1,p}(D_1)$. Moreover

$$A_{\rho_k} + B_{c_k} \to A_0 \quad \text{as } k \to \infty$$

in $\mathcal{L}(W^{2,p}(D_1) \cap W_0^{1,p}(D_1), L^p(D_1))$. We choose $p > N$, so that $W^{2,p}(D_1) \subset C^1(\overline{D_1})$. Again by using the spectral projection P_{ρ_k} on the kernel of $A_{\rho_k} + B_{c_k}$ in $L^p(D_1)$ together with the Grahm-Schmidt $L^2(D_1)$-orthonormalization algorithm, we can find an $L^2(D_1)$-orthonormal basis $\tilde{\phi}_1^{\rho_k}, \ldots, \tilde{\phi}_n^{\rho_k}$ of $\ker(A_{\rho_k} + B_{c_k})$ with

$$\tilde{\phi}_i^{\rho_k} \to \tilde{\phi}_i \quad \text{as } k \to \infty$$

in $C^1(\overline{D_1})$ for all $i = 1, \ldots, n$.

Summarising, we have found a sequence of positive numbers ρ_k, $\rho_k \to 0$ as $k \to \infty$, and a sequence of $C^\gamma(\mathbb{R}^N)$ functions $c_k\colon \mathbb{R}^N \to \mathbb{R}$, $c_k \to 0$ in $C^0(\overline{D_1})$ as $k \to \infty$, such that, for all (sufficiently large) k, the operator

$$\operatorname{div}(G(x_0 + \rho_k(x - x_0))\nabla) + \tilde{a}(x) + c_k(x)$$

on D_1 with Dirichlet boundary condition on ∂D_1 has an n-dimensional kernel spanned by $L^2(D_1)$-orthonormal functions $\tilde{\phi}_1^{\rho_k}, \ldots, \tilde{\phi}_n^{\rho_k}$, with

$$\tilde{\phi}_i^{\rho_k} \to \tilde{\phi}_i \quad \text{as } k \to \infty$$

in $C^1(\overline{D_1})$ for $i = 1, \dots, n$.

4^{th} *step*: For all $\rho > 0$ we define the homothety

$$
\begin{aligned}
O_\rho \colon \mathbb{R}^N &\to \mathbb{R}^N \\
x &\mapsto x_0 + \rho(x - x_0)
\end{aligned}
$$

and we define

$$
D_\rho := O_\rho(D_1) = \left\{ y \in \mathbb{R}^N \mid y = x_0 + \rho(x - x_0),\ x \in D_1 \right\}.
$$

If ρ is sufficiently small, then $\overline{D_\rho} \subset \Omega$. So, for sufficiently large k, we can consider the operator

$$
\begin{aligned}
&\operatorname{div}(G(x)\nabla) + (\rho_k)^{-2}\tilde{a}(x_0 + (\rho_k)^{-1}(x - x_0)) + (\rho_k)^{-2}c_k(x_0 + (\rho_k)^{-1}(x - x_0)) = \\
&\qquad = \operatorname{div}(G(x)\nabla) + (\rho_k)^{-2}\tilde{a}((O_{\rho_k})^{-1}(x)) + (\rho_k)^{-2}c_k((O_{\rho_k})^{-1}(x))
\end{aligned}
\tag{5.6}
$$

on D_{ρ_k} with Dirichlet boundary condition on ∂D_{ρ_k}. This operator has the same repeated sequence of eigenvalues of the operator

$$
(\rho_k)^{-2}\operatorname{div}(G(x_0 + \rho_k(x - x_0))\nabla) + (\rho_k)^{-2}\tilde{a}(x) + (\rho_k)^{-2}c_k(x)
\tag{5.7}
$$

on D_1 with Dirichlet boundary condition on ∂D_1. In particular, the operator (5.6) has an n-dimensional kernel spanned by the $L^2(D_{\rho_k})$-orthonormal functions

$$
\begin{aligned}
\psi_i^{\rho_k}(x) :&= (\rho_k)^{-N/2}\tilde{\phi}_i^{\rho_k}(x_0 + (\rho_k)^{-1}(x - x_0)) \\
&= (\rho_k)^{-N/2}\tilde{\phi}_i^{\rho_k}((O_{\rho_k})^{-1}(x)),
\end{aligned}
$$

$i = 1, \dots, n$. Now we define $W_k := O_{\rho_k} \circ Z$, $D_k := W_k(S) = D_{\rho_k}$ and

$$
\begin{aligned}
a_k(x) :=& (\rho^k)^{-2}\tilde{a}((O_{\rho_k})^{-1}(x)) + (\rho^k)^{-2}c_k((O_{\rho_k})^{-1}(x)) \\
=& (\rho^k)^{-2}a_0((W_k)^{-1}(x)) + (\rho^k)^{-2}c_k((O_{\rho_k})^{-1}(x)).
\end{aligned}
$$

We finally estimate, for $i = 1, \dots, n$,

$$
\begin{aligned}
&\left\| (\det DW_k)^{1/2}\psi_i^{\rho_k}(W_k(\cdot)) - \phi_i(\cdot) \right\|_{C^1(\overline{S})} \\
=& \left\| (\det Q)^{1/2}(\rho_k)^{N/2}\psi_i^{\rho_k}(W_k(\cdot)) - \phi_i(\cdot) \right\|_{C^1(\overline{S})} \\
=& \left\| (\det Q)^{1/2}\tilde{\phi}_i^{\rho_k}((O_{\rho_k}^{-1} \circ W_k(\cdot)) - \phi_i(\cdot) \right\|_{C^1(\overline{S})} \\
=& \left\| (\det Q)^{1/2}\tilde{\phi}_i^{\rho_k}(Z(\cdot)) - \phi_i(\cdot) \right\|_{C^1(\overline{S})} \\
=& (\det Q)^{1/2} \left\| \tilde{\phi}_i^{\rho_k}(Z(\cdot)) - \tilde{\phi}_i(Z(\cdot)) \right\|_{C^1(\overline{S})} \to 0
\end{aligned}
$$

as $k \to \infty$.

Now, fixed $\epsilon > 0$, we choose a sufficiently large k and we set $W := W_k$, $D := D_k$ and $a := a_k$ and we have concluded the proof of the first part of the theorem.

5^{th} *step*: In order to conclude the proof of the theorem, we observe that, for all k, the operator

$$\begin{aligned}(5.8)\quad &\operatorname{div}(G(x)\nabla) + (\rho_k)^{-2}\tilde{a}((O_{\rho_k})^{-1}(x)) + \\ &\qquad + (\rho_k)^{-2}c_k((O_{\rho_k})^{-1}(x)) + (\rho_k)^{-2}b_0(Z^{-1} \circ (O_{\rho_k})^{-1}(x))\end{aligned}$$

on D_k with Dirichlet boundary condition on ∂D_k has the same repeated sequence of eigenvalues of the operator

$$(5.9)\quad (\rho_k)^{-2}\operatorname{div}(G(x_0 + \rho_k(x - x_0))\nabla) + (\rho_k)^{-2}\tilde{a}(x) + (\rho_k)^{-2}c_k(x) + (\rho_k)^{-2}b_0(Z^{-1}(x))$$

on D_1 with Dirichlet boundary condition on ∂D_1, which is obtained multiplying by $(\rho_k)^{-2}$ the eigenvalues of the operator

$$\operatorname{div}(G(x_0 + \rho_k(x - x_0))\nabla) + \tilde{a}(x) + c_k(x) + b_0(Z^{-1}(x)) \tag{5.10}$$

on D_1 with Dirichlet boundary condition on ∂D_1. As $k \to \infty$, the first eigenvalue of (5.10) tends to the first eigenvalue of

$$\operatorname{div}(G_0\nabla) + \tilde{a}(x) + b_0(Z^{-1}(x))$$

on D_1 with Dirichlet boundary condition on ∂D_1, that is the same as the first eigenvalue of the operator

$$\Delta + a(x) + b_0(x)$$

on S with Dirichlet boundary condition on ∂S. So, if k is sufficiently large, the first eigenvalue of (5.10) is $< -\kappa$, and since $(\rho_k)^{-2} \to \infty$ as $k \to \infty$, the first eigenvalue of (5.8) is $< -\kappa$ and we have concluded. $\square$

6. The Poláčik condition

Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with $C^{2,\gamma}$ boundary. Let $a_{ij}\colon \overline{\Omega} \to \mathbb{R}$, $i, j = 1, \dots, N$, be of class $C^{1,\gamma}$, $a_{ij} \equiv a_{ji}$, $i, j = 1, \dots, N$, and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^N$$

for some $c > 0$. Let us consider the differential operator

$$L' = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j). \tag{6.1}$$

In this section we want to prove that, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, we can construct a potential $a\colon \overline{\Omega} \to \mathbb{R}$ of class C^γ such that all assumptions in Theorem 2.2 are satisfied with

$$L = \sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j) + a(x). \tag{6.2}$$

We will prove the following:

Theorem 6.1. *Let L' as above and let $\kappa > 1$; then, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, there exists a potential $a\colon \overline{\Omega} \to \mathbb{R}$ of class $C^\gamma(\overline{\Omega})$ with the following properties:*

(1) *the operator L in (6.2) satisfies the Poláčik condition on Ω;*
(2) *$U \subset \Omega$ is an open set;*
(3) *$R(x) \neq 0$ for all $x \in U$;*
(4) *there is a function $b \in C^\infty(\overline{\Omega})$ with $\operatorname{supp} b \subset U$ such that*

$$\lambda < -\kappa$$

for every eigenvalue λ of the operator $L + b$ on Ω with Dirichlet (or Neumann) boundary condition on $\partial\Omega$.

Proof. Our starting point is the existence (extablished in [14]) of two functions a_0, b_0 satisfying properties (1)–(4) of the present theorem when $\Omega = B$ is the unit ball in $\mathbb{R}^N$, $a_{ij}(x) \equiv \delta_{ij}$, i.e. $L' = \Delta$, and we take the Dirichlet condition on ∂B. In this case there is a basis of $\ker L$ given by functions

$$\phi_i(x) = \frac{w(|x|)}{|x|} x_i, \quad x \in B,\ i = 1, \dots, N$$

and

$$\phi_{N+1}(x) = v(|x|), \quad x \in B$$

where $w, v\colon \mathbb{R} \to \mathbb{R}$ are analytic functions such that

$$w(0) = 0,\ w'(0) \neq 0,\ v(0) \neq 0,\ v'(0) = 0. \tag{6.3}$$

We claim that

the functions $\phi_i\phi_j$, $1 \le i \le j \le N+1$, are linearly independent.

In fact, let ρ_{ij}, $1 \le i \le j \le N+1$, be real numbers with

$$\sum_{1 \le i \le j \le N+1} \rho_{ij}\phi_i\phi_j \equiv 0.$$

Evaluating this expression at $x = 0$ and using (6.3) we obtain $\rho_{N+1,N+1} = 0$. Thus

$$\frac{w(|x|)^2}{|x|^2} \sum_{1 \le i \le j \le N} \rho_{ij} x_i x_j \equiv -\frac{w(|x|)v(|x|)}{|x|} \sum_{1 \le i \le N} \rho_{i,N+1} x_i \quad \text{for } x \neq 0.$$

Since

$$\frac{w(|x|)^2}{|x|^2} \neq 0 \quad \text{and} \quad \frac{w(|x|)v(|x|)}{|x|} \neq 0 \quad \text{for } |x| \text{ small},$$

it follows that

$$|\sum_{1\leq i\leq N}\rho_{i,N+1}x_i| = o(|x|) \quad \text{for } x\to 0.$$

However, this implies that $\rho_{i,N+1}=0$ for $i=1,\ldots,N$. Hence

$$\sum_{1\leq i\leq j\leq N}\rho_{ij}x_ix_j\equiv 0$$

which immediately implies that $\rho_{ij}=0$ for $1\leq i\leq j\leq N$. The claim is proved.

Now we can apply Lemma 5.1 with $S=B$, $n=N+1$ and a_0, b_0 given by the construction in [14]. Following the terminology of Lemma 5.1, we claim that, if we choose a sufficiently small ϵ, then the corresponding operators $L_a=L'+a$ and $L_{a+b}=L'+a+b$ on $D=W(S)$ with Dirichlet boundary condition on ∂D satisfy properties (1)–(4) of the present theorem. First, we observe that, for a fixed invertible affine transformation W, on S we have

$$\begin{pmatrix}\psi_1(W(x)) & \nabla_x\psi_1(W(x))\\ \vdots & \vdots\\ \psi_{N+1}(W(x)) & \nabla_x\psi_{N+1}(W(x))\end{pmatrix}=$$

$$=\begin{pmatrix}\psi_1(W(x)) & (\nabla\psi_1)(W(x))\\ \vdots & \vdots\\ \psi_{N+1}(W(x)) & (\nabla\psi_{N+1})(W(x))\end{pmatrix}\circ\begin{pmatrix}1 & 0\\ 0 & DW(x)\end{pmatrix}.$$

Since $DW(x)$ is constant and invertible, we have that, if $x\in D=W(S)$, then

$$\begin{gathered}R(\psi_1,\ldots,\psi_{N+1})(x)\neq 0\\ \text{if and only if}\\ R(\psi_1(W(\cdot)),\ldots,\psi_{N+1}(W(\cdot)))(W^{-1}x)\neq 0.\end{gathered} \tag{6.4}$$

Let

$$U_0:=\{x\in B\mid R(\phi_1,\ldots,\phi_{N+1})(x)\neq 0\}\,;$$

U_0 is open and by construction $\operatorname{supp} b_0\subset U_0$; take an open set U_0', such that $\operatorname{supp} b_0\subset U_0'\subset\subset U_0$ and set $U':=W(U_0')$; then $U'\subset D$ is open and, since by definition $b(x)=\rho^{-2}b_0(W^{-1}(x))$, it follows that $\operatorname{supp} b\subset U'$. But now property 3) in Lemma 5.1 implies that, if ϵ is sufficiently small, then

$$R(\psi_1(W(\cdot)),\ldots,\psi_{N+1}(W(\cdot)))(x)\neq 0$$

for all $x\in U_0'$, and hence, by (6.4),

$$R(\psi_1,\ldots,\psi_{N+1})\neq 0$$

for all $x\in U'$. This proves the claim. The same argument shows that, if ϵ is sufficiently small, then the functions $\psi_i\psi_j$, $1\leq i\leq j\leq N+1$ are linearly independent.

Summarising, we have obtained the following intermediate result: we have found open sets $U' \subset\subset D \subset \Omega$ and two $C^{\gamma}(\overline{\Omega})$ functions $a, b\colon \overline{D} \to \mathbb{R}$, $\operatorname{supp} b \subset U'$, such that:

(1) the operator $L_a = L' + a$ satisfies the Poláčik condition on D, with Dirichlet boundary condition on ∂D;
(2) $R(\psi_1, \dots, \psi_{N+1}) \neq 0$ for all $x \in U'$, where $\psi_1, \dots, \psi_{N+1}$ is any $L^2(D)$-orthonormal basis of the kernel of L_a on D with Dirichlet boundary condition on ∂D;
(3) $\lambda < -\kappa$ for every eigenvalue λ of the operator $L_{a+b} = L' + a + b$ on D with Dirichlet boundary condition on ∂D.

Moreover,

(6.5) for every basis $\psi_1, \dots, \psi_{N+1}$ of the kernel of L_a on D with Dirichlet boundary condition on ∂D the functions $\psi_i \psi_j$, $1 \leq i \leq j \leq N+1$ are linearly independent.

Now we procede as in the proof of Th. 4.4 in [17]: let $H := L^2(\Omega)$, $V := H_0^1(\Omega)$ if we are working with Dirichlet boundary condition on $\partial\Omega$, $V := H^1(\Omega)$ if we are working with Neumann boundary condition on $\partial\Omega$; if $d\colon \overline{\Omega} \to \mathbb{R}$ with $d \in C^{\gamma}(\overline{\Omega})$, define $g_d\colon V \times V \to \mathbb{R}$ by

$$g_d(u, v) = \int_{\Omega} G(x) \nabla u \cdot \nabla v \, \mathrm{d}x + \int_{\Omega} duv \, \mathrm{d}x,$$

where $G(x) := (a_{ij}(x))_{i,j}$. Regularity theory of PDEs implies that, both for Dirichlet and Neumann boundary condition, λ is an eigenvalue of $L_d = L' + d$ and u is a corresponding eigenvector if and only if λ is a proper value of g_d and u is a corresponding proper vector. (In fact, every proper vector of g_d lies in $C^{2,\gamma}(\overline{\Omega})$.) Let $c\colon \mathbb{R}^N \to \mathbb{R}$ be of class C^{γ} and such that $c(x) = 0$ for $x \in \overline{D}$ and $c(x) > 0$ for $x \notin D$. Furthermore, let $(\beta_k)_{k\in\mathbb{N}}$ be an arbitrary sequence of positive numbers tending to ∞. Finally, for $k \in \mathbb{N}$ let $c_k\colon \mathbb{R}^N \to \mathbb{R}$ be a $C^{\gamma}(\overline{\Omega})$ function such that $\sup_{x\in\Omega} |c_k(x) - c(x)| < 1/k\beta_k$.

Let $L_k := L_{a+\beta_k c_k}$, $g_k := g_{a+\beta_k c_k}$ and let g_∞ be the restriction of g_a to $H_0^1(D)$. We are now in a position to apply Theorem 4.4: for $k \in \mathbb{N}$ let $(\lambda_n^k)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of g_k and $(u_n^k)_{n\in\mathbb{N}}$ be an H-orthonormal sequence of corresponding proper vectors of g_k. Moreover, let $(\mu_n)_{n\in\mathbb{N}}$ be the repeated sequence of proper values of g_∞.

Then, using Theorem 4.4 and passing to a subsequence if necessary we may assume that there is a sequence $(v_n)_{n\in\mathbb{N}}$ in $H_0^1(D)$ such that for every $n \in \mathbb{N}$, v_n is a proper vector of g_∞ corresponding to μ_n, $(\lambda_n^k)_{k\in\mathbb{N}}$ converges to μ_n and $(u_n^k)_{k\in\mathbb{N}}$ converges to $v_n{}^{\sim}$ in V, as $k \to \infty$. Set $p = N + 1$. There are numbers γ_1 $\gamma_2 \in \mathbb{R}$, M, $\eta \in \mathbb{R}_+$ and $l \in \mathbb{N}_0$, such that, setting $\mu_0 = -\infty$, we have

$$0 < \gamma_2 - \gamma_1 < M,$$

$$\mu_l < \gamma_1 - 4\eta < \gamma_1 < 0 = \mu_{l+1} = \mu_{l+p} < \gamma_2 < \gamma_2 + 4\eta < \mu_{l+p+1}.$$

For $h \in C^0(\overline{\Omega})$ let $B_h \in \mathcal{L}_{\mathrm{sym}}(H, H)$ be the map

$$(Bu)(x) = h(x)u(x), \qquad u \in H, \, x \in \Omega.$$

Note that

$$|B_h|_{\mathcal{L}(H,H)} = |h|_{C^0(\overline{\Omega})}. \tag{6.6}$$

Let $\mathcal{G}$ be the set of all B_h with $h \in C^0(\overline{\Omega})$. It follows that $\mathcal{G}$ is a closed linear subspace of $\mathcal{L}_{\mathrm{sym}}(H, H)$. Now (6.5) easily implies that the operator $T: \mathcal{G} \to \mathcal{S}_p$

$$B \mapsto (\langle B(v_i{}^{\sim}), v_j{}^{\sim}\rangle)_{ij}$$

is surjective. By the open mapping theorem there is a $\theta > 0$ such that

$$T(\mathrm{B}_1) \supset \mathrm{B}_\theta.$$

For $k \in \mathbb{N}$ let $T_k: \mathcal{G} \to \mathcal{S}_p$ be the map

$$B \mapsto (\langle Bu_i^k, u_j^k\rangle)_{ij}.$$

Then $T_k \to T$ in $\mathcal{L}(\mathcal{G}, \mathcal{S}_p)$ so it is easy to see that

$$T_k(\mathrm{B}_1) \supset \mathrm{B}_\theta \quad \text{for } k \text{ large enough.}$$

Moreover, setting $\lambda_0^k = -\infty$, we have

$$\lambda_l^k < \gamma_1 - 4\eta < \gamma_1 < \lambda_{l+1}^k \le \lambda_{l+p}^k < \gamma_2 < \gamma_2 + 4\eta < \lambda_{l+p+1}^k, \ k \text{ large enough.} \tag{6.7}$$

Let $\alpha_0 = \alpha_0(p, M, \eta, \theta)$ be as in Theorem 4.2. For all large k, there is an $\alpha_k > 0$ such that $|\lambda_{l+j}^k| < \alpha_k < \alpha_0$ for $j = 1, \ldots, p$ and $\alpha_k \to 0$ as $k \to 0$. Thus by Theorem 4.2 (with $A := L_k$, $\mu_j := 0$, $\lambda_{l+j} := \lambda_{l+j}^k$ for $j = 1, \ldots, p$ and $\mathcal{D}$ equal to the set of all B_h where h is a $C^\gamma(\overline{\Omega})$ function) there exists, for each large k, a $C^\gamma(\overline{\Omega})$ function $h_k: \overline{\Omega} \to \mathbb{R}$ such that $|h_k|_{C^0(\overline{\Omega})} < (1/2)\theta\alpha_k$ and such that if $(\hat\lambda_n^k)_{n\in\mathbb{N}}$ denotes the repeated sequence of eigenvalues of $L_{a+h_k+\beta_k c_k}$, $(\hat u_n^k)_{n\in\mathbb{N}}$ is an H-orthogonal sequence of the corresponding eigenfunctions and $\hat\lambda_0^k := -\infty$, then

$$\hat\lambda_l^k < \gamma_1 - 3\eta < \gamma_1 - \eta < \hat\lambda_{l+1}^k \le \hat\lambda_{l+p}^k < \gamma_2 + \eta < \gamma_2 + 3\eta < \hat\lambda_{l+p+1}^k \tag{6.8}$$

and

$$\hat\lambda_{l+j}^k = 0, \quad j = 1, \ldots, p. \tag{6.9}$$

Now the assumptions of Theorem 4.4 are satisfied with c_k replaced by $(1/\beta_k)h_k + c_k$. Therefore using Theorem 4.4 again and passing to a subsequence if necessary we may assume that there is a sequence $(\hat v_n)_{n\in\mathbb{N}}$ in $H_0^1(D)$ such that for every $n \in \mathbb{N}$, $\hat v_n$ is a proper vector of g_∞ corresponding to μ_n, $(\hat\lambda_n^k)_{k\in\mathbb{N}}$ converges to μ_n and $(\hat u_n^k)_{k\in\mathbb{N}}$ converges to $\hat v_n{}^{\sim}$ in V, as $k \to \infty$.

Finally, by Theorem 4.5, $(\hat u_n^k|D)_{k\in\mathbb{N}}$ converges to $\hat v_n$ in $C^1_{\mathrm{loc}}(D)$ as $k \to \infty$. It follows that, if $U \subset D$ is an open set, $\operatorname{supp} b \subset U \subset\subset U'$, then, for all k large enough, $R(\hat u_{l+1}^k, \ldots, \hat u_{l+p}^k)(x) \ne 0$ for all $x \in U$. In order to complete the proof, we apply again Theorem 4.4: if k is sufficiently large, all the eigenvalues of $L' + a + h_k + \beta_k c_k + b$ are $< -\kappa$; so, for every such k, $a + h_k + \beta_k c_k$ is a $C^\gamma(\overline{\Omega})$ function and the conclusion follows with a replaced by $a + h_k + \beta_k c_k$. This proves the theorem. $\square$

7. The Algebraic independence condition

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^{2,\gamma}$ boundary. Let $a_{ij}\colon \overline{\Omega} \to \mathbb{R}$, $i, j = 1, 2$, be of class $C^{1,\gamma}$, $a_{ij} \equiv a_{ji}$, $i, j = 1, 2$, and

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i i\xi_j \geq |\xi|^2, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^2$$

for some $c > 0$. Let us consider the differential operator

$$L' = \sum_{i,j=1}^{2} \partial_i(a_{ij}(x)\partial_j). \tag{7.1}$$

In this section we want to prove that, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, we can construct a potential $a\colon \overline{\Omega} \to \mathbb{R}$ of class C^∞ such that the operator

$$L = L' + a(x) = \sum_{i,j=1}^{2} \partial_i(a_{ij}(x)\partial_j) + a(x) \tag{7.2}$$

has a kernel of a prescribed dimension n, spanned by eigenfunctions satisfying the algebraic independence condition (IC) in Section 4.1 up to a prescribed order k with an appropriate $\varpi \in \mathbb{R}^2$. We will prove the following:

Theorem 7.1. *Let L' as above and let $n, k \in \mathbb{N}$. Then, both for Dirichlet and Neumann boundary condition on $\partial\Omega$, there exists a potential $a\colon \overline{\Omega} \to \mathbb{R}$ of class $C^\infty(\overline{\Omega})$ with the following properties:*

1. *the operator L in (7.2) has an n-dimensional kernel;*
2. *there exists a vector $\varpi \in \mathbb{R}^2$ and an $L^2(\Omega)$-orthonormal basis $u_1, \ldots, u_n$ of the kernel of L such that the algebraic independence condition (IC) in Section 4.1 is satisfied up to the order k.*

Proof. As in the proof of Theorem 6.1, our starting point is the existence (extablished in [16]) of such a potential for a suitable smooth bounded domain when $a_{ij}(x) \equiv \delta_{ij}$, i.e. $L' = \Delta$, and with $\varpi = (0, 1)$. So we can always take a bounded smooth domain S and a smooth potential $a_0\colon \overline{S} \to \mathbb{R}$ such that:

1. the operator $\Delta + a_0(x)$ on S with Dirichlet boundary condition on ∂S has an n-dimensional kernel;
2. there is an $L^2(S)$-orthonormal basis $\phi_1, \ldots, \phi_n$ of the kernel of $\Delta + a_0(x)$ such that (IC) is satisfied up to the order k with $\varpi = (0, 1)$, i.e. for every $l = 1, \ldots, k$ and every q, $1 \leq q \leq l$, the functions

$$\left\{ \sum_{\substack{\gamma \leq \beta \\ |\gamma| = q}} \frac{1}{\gamma!(\beta-\gamma)!} \phi^{\beta-\gamma+\epsilon_j} \phi_y^\gamma \right\}_{\substack{j=1,\ldots,n \\ |\beta|=l}}$$

are linearly independent.

Moreover, the functions $\phi_i\phi_j$, $1 \le i \le j \le n$ are linearly independent. Now, as in the proof of Theorem 6.1, we apply Lemma 5.1; following the terminology of Lemma 5.1, we obtain that, if we choose a sufficiently small ϵ, then, for some smooth potential a, the kernel of $L + a$ on D with Dirichlet condition on ∂D is spanned by $L^2(D)$-orthonormal functions $\psi_1, \ldots, \psi_n$ such that for every $l = 1, \ldots, k$, and for every q, $1 \le q \le l$, the functions

$$\left\{ \sum_{\substack{\gamma \le \beta \\ |\gamma| = q}} \frac{1}{\gamma!(\beta-\gamma)!} \psi(W(\cdot))^{\beta-\gamma+\epsilon_j} \psi(W(\cdot))_y^{\gamma} \right\}_{\substack{j=1,\ldots,n \\ |\beta|=l}}$$

are linearly independent on S. Since, for $i = 1, \ldots n$,

$$\psi_i(W(\cdot))_y = (\nabla\psi_i)(W(\cdot)) \cdot \varpi,$$

where ϖ is the second column of the (constant) matrix $DW(\cdot)$, we reach that, for every $l = 1, \ldots, k$ and for every q, $1 \le q \le l$, the functions

$$\left\{ \sum_{\substack{\gamma \le \beta \\ |\gamma| = q}} \frac{1}{\gamma!(\beta-\gamma)!} \psi^{\beta-\gamma+\epsilon_j} \psi_{\varpi}^{\gamma} \right\}_{\substack{j=1,\ldots,n \\ |\beta|=l}}$$

are linearly independent. Moreover, the functions $\psi_i\psi_j$, $1 \le i \le j \le n$ are linearly independent on D. Finally, we conclude arguing exactly as in the proof of Theorem 6.1, applying Theorem 4.4, Theorem 4.5 and Theorem 4.2. □

Remark. The present result generalizes naturally to any space dimension $N \ge 2$ (see [12]).

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