A tour through some classical theorems on algebraic surfaces

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July 10, 2002

Abstract

We consider three classical theorems on surfaces of $\mathbf{P}^4$, the projective space of dimension 4 over an algebraically closed field of characteristic 0: the Severi theorem concerning the Veronese surface, the C. Segre theorem which classifies surfaces containing a 2-dimensional family of plane curves, the Franchetta theorem on the irreducibility of the double curve of a general projection of such a surface in $\mathbf{P}^3$. We study relations among these theorems; in particular, we give a modern proof of the third theorem, based on a precise formulation of the "general projection theorem".

1 Introduction.

Let us start with the following definition.

Definition 1 *Let $S \subset \mathbf{P}^4$ be a non degenerate, irreducible surface. We shall call S a general surface of $\mathbf{P}^4$ if the singularities of S are at most a finite number of improper double points (i.e. double points which are the origin of two linear branches of S and whose tangent cone consists of two planes intersecting transversally at the point).*

By Veronese surface S of $\mathbf{P}^4$ we shall mean any projection in $\mathbf{P}^4$ of the usual Veronese surface $V \subset \mathbf{P}^5$, from a point which does not belong to the secant variety of V. Note, in particular, that S is smooth.

Veronese surfaces in $\mathbf{P}^4$ and $\mathbf{P}^5$ play a special role in many classical theorems referring to surfaces in $\mathbf{P}^4$:

Theorem 2 (Kronecker-Castelnuovo) (1894, [C])
Let $S \subset \mathbf{P}^4$ be an irreducible surface having at most isolated singularities. If the general tangent hyperplane to S intersects S in a reducible curve, then S is either a scroll or a Veronese surface in $\mathbf{P}^4$.

*Both authors have been partially supported by MURST, and are members of GNSAGA and EUROPROJ.

Theorem 3 (Severi) (1901,[S])

The only smooth surface in $\mathbf{P}^5$ *whose secant variety is not the whole* $\mathbf{P}^5$ *is the Veronese surface. In other words, the only smooth surface* $S \subset \mathbf{P}^4$ *which is not linearly normal is the Veronese surface.*

Theorem 4 (Corrado Segre) (1921, [cS],[CS],[M])

If an irreducible non-degenerate surface $S \subset \mathbf{P}^4$ *contains a family of dimension two of plane curves, then these curves are conics and* S *is a Veronese surface or a cubic scroll or a cone.*

In particular, note that if S is general, then the conics are irreducible and S is a projection of $V \subset \mathbf{P}^5$.

Theorem 5 (Franchetta) (1941, [F1], [F2])

Let $S \subset \mathbf{P}^4$ *be a general surface and let* $O \in \mathbf{P}^4$ *be a general point. Then the singular locus of the projection* F *of* S *from* O *in* $\mathbf{P}^3$ *is an irreducible curve, unless* S *is a Veronese surface.*

If S is the Veronese surface, then F has 3 non-coplanar lines concurrent in a point as singular locus.

The last three theorems can be related as follows.

Theorem 6 *Let* $S \subset \mathbf{P}^4$ *be an irreducible, smooth, non degenerate surface. Then the following properties are equivalent:*

(a) S *is a Veronese surface;*

(b) S *contains a two-dimensional family of irreducible conics and* S *is not a cubic scroll;*

(c) the double curve of the generic projection of S *in* $\mathbf{P}^3$ *is reducible;*

(d) S *is not linearly normal.*

It is well known that property (a) implies $(b), (c)$ and (d). The theorems of Segre, Franchetta and Severi are respectively the converse implications. Moreover, the first proof Franchetta gave of his theorem was, actually, $(c) \Rightarrow (b)$.

In this paper we will give a proof of $(d) \Rightarrow (c)$ based on the well known "monoidal construction" (see e.g. [Sz], [L]). Moreover, we will present a modern proof of $(c) \Rightarrow (b)$ which follows closely the beautiful geometric argument of Franchetta's first proof. The motivation for doing this comes from an attempt to apply the same technique to the characterization of threefolds in $\mathbf{P}^5$ which are not 2-normal (see [MP]).

Both proofs rely heavily on the next theorem. To state it we need the following definition.

Definition 7 *The irreducible surface* $F \subset \mathbf{P}^3$ *has ordinary singularities if its singular locus is either empty or is a curve* γ *such that* *(i) γ is either non singular or has at most a finite number of ordinary triple points (i.e. points which are origin of three linear branches of the curve with three distinct and non-coplanar tangent lines);*

(ii) every non singular point of γ is either a nodal point of F (i.e. the origin of two linear branches of F with two distinct tangent planes) or a pinch-point of F (i.e. a point which is analytically equivalent to the singularity at the origin of the surface $x^2 - yz^2 = 0$);

(iii) for every irreducible component γ' of γ, the general point of γ' is a nodal point for F (in other words, there are only finitely many pinch-points);

(iv) every triple point of γ is an ordinary triple point of F (i.e. a point which is analytically equivalent to the singularity at the origin of the surface $xyz = 0$).

The curve γ will be called the *double curve* of F.

Theorem 8 (General projection.) *Let S be a general surface in $\mathbf{P}^4$ and let $O \in \mathbf{P}^4$ be a general point. Let $\pi : S \to \mathbf{P}^3$ be the projection from O and set $F := \pi(S)$. Then F has ordinary singularities.*

We remark here that the projection of a smooth surface of $\mathbf{P}^5$ from a general point is a general surface of $\mathbf{P}^4$. Therefore the above theorem yields a step by step proof of the theorem of general projection in $\mathbf{P}^3$ for smooth surfaces in $\mathbf{P}^5$.

By the above theorem the double curve γ and $Sing(\gamma)$ are respectively the double and the triple locus of the map π. There is a canonical way to define a scheme structure on these multiple loci, namely by the Fitting ideals of the $\mathcal{O}_{\mathbf{P}^3}$-module $\pi_*\mathcal{O}_S$ ([KLU]). Moreover, a scheme structure on γ can be defined also by the conductor of π ([R2]). In our case the two methods give the same result, namely the reduced structure on γ. We shall need this more refined point of view only from § 8 on.

When giving his proof Franchetta took the "Theorem of general projection" for granted. In our opinion, this is the only serious criticism one could move to his proof. Nevertheless, Franchetta was right, because his point of view was clearly that of birational geometry. In particular, this point of view allows to solve the singularities of the surface S (if any) and to change its embedding. Then, we can assume that $X \subset \mathbf{P}^n$ is a smooth surface, birationally equivalent to S. Let $\sigma_2 : \mathbf{P}^n \hookrightarrow \mathbf{P}^N$ be the 2−uple imbedding defined by the global sections of $\mathcal{O}_{\mathbf{P}^n}(2)$. Then, there exist open, dense subsets $U_1 \subset \mathbf{G}(N-5, N)$ and $U_2 \subset \mathbf{G}(N-4, N)$ such that, for every $L \in U_1$ (resp. $L \in U_2$) the projection of $\sigma_2(X)$ in $\mathbf{P}^4$ (resp. in $\mathbf{P}^3$) from L is a general surface (resp. is a surface with ordinary singularities) (see [R1], Thm.2 and Thm.3).

A last remark on Franchetta's theorem. A simple proof of the analogous theorem dealing with the projection in $\mathbf{P}^3$ of a smooth, non-degenerate surface $S \subset \mathbf{P}^5$ was given by Mumford (see [Mo]). But the possibility to modify Mumford's proof to the case of a surface $S \subset \mathbf{P}^4$ seems hopeless.

The paper is organized as follows. In Section 2 we will prove the theorem of Kronecker-Castelnuovo and some useful consequence of it. Section 3 contains the proof of the very basic Lemma 13 which refers to the reciprocal position of pairs of tangent planes.

Ziv Ran has shown that the variety of $(n+2)-$secant lines of a smooth variety $X^n \subset \mathbf{P}^{n+2}$ does not fill the whole space (see [R]). In Section 4 we give a proof of this result in the case $n = 2$, under the assumption that S is a general surface, not necessarily a smooth one. Sections 5 and 6 are devoted to the proof of Theorem 8, the "theorem of general projection". The proofs of $(c) \Rightarrow (b)$, namely Franchetta's theorem, and $(d) \Rightarrow (c)$ are the content of Sections 7 and 8, respectively. Finally, the last Section contains the proof of another basic tecnical result, Lemma 27.

We will always work over an algebraically closed field k of characteristic zero. By variety we shall always mean an integral (i.e. reduced and irreducible), separated scheme, of finite type over k. We shall make in what follows a systematic use of the notations introduced in this section.

2 The theorem of Kronecker-Castelnuovo.

Let $S \subset \mathbf{P}^4$ be a general surface. First we mention some facts which will be often used later on.

Theorem 9 (Duality) *A general tangent plane to S is only tangent at one point.*

Proof. In fact, the Gauß map for S is birational onto its image ([Z]), unless S is a developable surface. In this case we would have $dim\ Sing(S) = 1$, but this is ruled out by our assumption that S is a general surface of $\mathbf{P}^4$. □

Corollary 10 *A general tangent hyperplane to S is only tangent at one point.*

Proof. Let p be a general point of S and let T_p be the tangent plane to S at p. By the theorem of Bertini, the general tangent hyperplane to S at p may be tangent to S at a point $q \neq p$ only if $q \in T_p$. So, if the corollary were not true, then the general tangent plane to S would be tangent in more then one point, a contradiction. □

Proof of the Kronecker-Castelnuovo Theorem.

Let $p \in S$ be a general point and let $\pi : \tilde{S} \longrightarrow S$ be the blow-up of S at p; we shall denote by E the exceptional divisor on $\tilde{S}$. Let H be a section of S with a hyperplane which is tangent to S at p. Then H is $2-$connected, unless S is the Veronese surface or S is a scroll ([VdV], Thm.1; the proof given there for smooth surfaces can be easily modified to work also in our situation). Therefore any divisor in the linear system $|\pi^*H - 2E|$ is $1-$connected, hence connected ([VdV], Prop.2). Since the general member of $|\pi^*H - 2E|$ is smooth by Corollary 10, the assertion easily follows. □

As a consequence of Kronecker-Castelnuovo theorem, we prove the following interesting facts.

Proposition 11 *If the general tangent plane to S intersects S along a curve, then S is a scroll and, for p general, $S \cap T_p$ is the line of the ruling passing through p plus a (possibly empty) finite set of points.*

Proof. Since S is non degenerate, the generic hyperplane section of S is not a plane curve. So, if S verifies the hypothesis, then the general tangent hyperplane section is reducible. Thus S is either the Veronese surface or a scroll.

Let S be a Veronese surface of $\mathbf{P}^4$: its hyperplane sections correspond under the Veronese map $v: \mathbf{P}^2 \to S$ to the conics of a linear system λ of dimension 4, and in particular its tangent hyperplane sections at p correspond to the linear system λ_q of conics of λ which are singular at $q := v^{-1}(p)$. It is easy to see that λ_q has no base points out of q, for λ and q general, and this means that $S \cap T_p = \{p\}$.

Let now S be a scroll: it is clear that, for every point p, $S \cap T_p$ contains the line of the ruling through p; if for general p it contains also another curve, then S is a surface containing a family of dimension 2 of reducible plane curves (not lines). By theorem 4 (of C.Segre) these curves are reducible conics and S is a cone. But this possibility is excluded because the unique singularities of S are improper double points. □

Corollary 12 *Let L be a general tangent line to S, i.e. a general line of the tangent plane to S at a general point p. Then L does not intersect S outside p.* □

3 A lemma on the transversality of the tangent planes

In this section we will prove one of the main technical results of the paper, namely Lemma 13. This lemma is due to Franchetta ([F1], [E]). To prove it we will use two other lemmas which are of some interest on their own.

Let S be a general surface in $\mathbf{P}^4$. Let $\widetilde{S \times S}$ be the blow-up of $S \times S$ along the diagonal. There is a birational correspondence between the exceptional divisor E of $\widetilde{S \times S}$ and $Tan(S) \subset \mathbf{G}(1,4)$, the subvariety of the Grassmann variety parametrizing the lines in $\mathbf{P}^4$ which are tangent to S. Note that, since S is a general surface of $\mathbf{P}^4$, it is certainly not a developable surface, and therefore it has a two dimensional family of tangent planes, thus a three dimensional family of tangent lines. Namely, in our situation we have $dim\ Tan(S) = 3$.

Lemma 13 *Let S be a general surface in $\mathbf{P}^4$. Let $\Sigma \subset \widetilde{S \times S}$ be a closed, irreducible subvariety such that $dim(\Sigma) = 3$ and $\Sigma \not\subseteq E$. Finally, assume that*

$$\overline{\bigcup_{\substack{(p,q)\in\Sigma \\ p\neq q}} pq} = \mathbf{P}^4 \qquad (1)$$

If (p,q) is a general element of Σ and T_p, T_q are the tangent planes to S at p, q respectively, then T_p and T_q intersect transversally, i.e. at a single point.

Proof. We shall give the proof in several steps.

Step 1. There are two canonical projections $p_i : \Sigma \longrightarrow S$, $i = 1, 2$. Then *either p_1 or p_2 is surjective.*

Assume the contrary. Then $dim\ p_i(\Sigma) \leq 1$, $i = 1, 2$. Let $f : \widetilde{S \times S} \longrightarrow S \times S$ be the canonical map. Then $f(\Sigma) \subseteq p_1(\Sigma) \times p_2(\Sigma)$, hence $dim\ f(\Sigma) \leq 2$. By hypothesis Σ is not contained in the exceptional divisor of $\widetilde{S \times S}$, therefore $dim\ f(\Sigma) = dim\ \Sigma = 3$, a contradiction.

From now on we will assume that $p_1 : \Sigma \longrightarrow S$ is surjective. Therefore, for a general element $(p,q) \in \Sigma$ we have $dim\ p_1^{-1}(p) = 1$; moreover, $p \neq q$ since, by hypothesis, Σ is not contained in the exceptional divisor of $\widetilde{S \times S}$. Then $p_2(p_1^{-1}(p))$ is a curve on S, which we shall denote by $\sigma(p)$, and which we consider in what follows with its reduced structure.

Step 2. Suppose that T_p and T_q do not intersect transversally. First we remark that T_p has to be distinct from T_q, otherwise the tangent plane to S at a general point would be also tangent elsewhere, and this is impossible by Theorem 9. So T_p and T_q intersect along a line R. We claim this line does not coincide with pq. Assume in fact this happens. Then, for every $q \in \sigma(p)$ the line pq is in T_p, which in turn would imply that T_p intersects S along a curve, hence S is a scroll by Proposition 11. But then, a simple dimensional count shows that the lines pq for $(p,q) \in \Sigma$, $p \neq q$ are the lines of the scroll and this contradicts (1).

Step 3. Being (p,q) a general element of Σ, we can assume that the point q is smooth both for S and for the curve $\sigma(p)$. Let σ be the irreducible component of $\sigma(p)$ through q. Therefore, from $\sigma \subset S$ we get $T_{\sigma,q} \subset T_{S,q}$ and the line $T_{\sigma,q}$ meets R, hence it meets the plane T_p. Now we will apply the following lemma.

Lemma 14 *Let $C \subset \mathbf{P}^n$ be an irreducibe curve such that the general tangent line to C meets a fixed linear subvariety $L \subset \mathbf{P}^n$ of codimension 2. Then C is contained in a hyperplane of $\mathbf{P}^n$ which contains also L.*

Therefore, the curve σ and the plane T_p are both contained in a hyperplane $H \subset \mathbf{P}^4$. Note that, for a general point $q \in \sigma$, the hyperplane H contains q and the line $T_p \cap T_q$. Then, from $q \notin T_p$ it follows that H contains $T_{S,q}$. Therefore, *the hyperplane H is tangent to S along σ.* We repeat the construction of the hyperplane H for any irreducible component of $\sigma(p)$. In this way we get for any $p \in S$ a finite set of hyperplanes in $\mathbf{P}^4$ and, when p varies on S, these hyperplanes describe an algebraic variety in $\check{\mathbf{P}}^4$.

Step 4. Let us denote by $\mathcal{P} \subset \check{\mathbf{P}}^4$ an irreducible component of the family of such tangent hyperplanes. Since S is a non degenerate surface in $\mathbf{P}^4$, we have $dim\ \mathcal{P} > 0$. On the other hand $dim\ \mathcal{P} < 2$. In fact, assume $dim\ \mathcal{P} \geq 2$. Then, a counting constants argument shows that in this case we would have through the general point of S a family of hyperplanes of $\mathcal{P}$ of dimension at least 1. This, in turn, would imply that the general tangent hyperplane to S is tangent along a curve, a contradiction to Corollary 10. Therefore $dim\ \mathcal{P} = 1$.

Let us consider the family of curves

$$\mathcal{F} := \{\sigma \mid \sigma \text{ is an irreducible component of } \sigma(p) \text{ for some } p \in S\}.$$

We claim that *the general curve of $\mathcal{F}$ is a plane curve.* The claim follows at once from the following lemma, which captures, in our opinion, the geometric insight of Enriques "... , *ognuna di queste curve C, dovendo stare nello S_3 da essa determinato, e nello S_3 determinato dalla curva infinitamente vicina, è contenuta in un piano* ..." ([E], pag.9).

Lemma 15 *Let $S \subset \mathbf{P}^n$ be a non-degenerate projective variety of arbitrary dimension. Let $\mathcal{P} \subset \check{\mathbf{P}}^n$ be a closed variety such that, for any $P \in \mathcal{P}$ the corresponding hyperplane H_P is tangent to S along some of the irreducible components of $S \cap H_P$. We shall denote by σ_P the union of the irreducible components of $S \cap H_P$ along which H_P is tangent to S. If $O \in \mathcal{P}$ is a general smooth point for $\mathcal{P}$, then the variety σ_O is contained into the base locus of the linear system of hyperplanes parametrized by the points of $T_{\mathcal{P},O} \subset \check{\mathbf{P}}^n$.*

Since Σ is completely described by the pairs (q, q'), where $q, q' \in \sigma$ and σ varies in $\mathcal{F}$, every line qq' is contained in the subvariety A of $\mathbf{P}^4$ described by the planes of the curves of $\mathcal{F}$. But $\mathcal{F}$ has dimension 1. In fact, any curve of $\mathcal{F}$ is contained in a hyperplane of $\mathcal{P}$ and, for any hyperplane of $\mathcal{P}$ there are only finitely many curves of $\mathcal{F}$ contained in it. From this it follows that A is a proper subvariety of $\mathbf{P}^4$, a contradiction by the assumption 1.

The proof of Lemma 13 is now complete. □

Proof of Lemma 14.

Let C' be the normalization of C, and let $\rho: C' \to C$ be the normalization map. Let $\mathbf{P}^1 \subset \mathbf{P}^n$ be a fixed line, disjoint from L, and $f: C' \to \mathbf{P}^1$ be the composition of ρ with the projection from L onto $\mathbf{P}^1$. From our hypothesis on the tangent lines to C it follows that f is ramified at every point of C'. Since $char(k) = 0$, this implies that f is not dominant, hence f is constant. If we set $f(C') = A \in \mathbf{P}^1$, then C is contained in the hyperplane of $\mathbf{P}^n$ joining A to L.

□

Remark 16 Lemma 14 is no longer true if $char(k) \neq 0$. The simplest counterexample is given by the well known strange curves, i.e. conics in characteristic 2. But there are also generalizations of this example in the projective space of dimension n, for every $n \geq 2$.

Proof of Lemma 15.

It is clearly sufficient to prove the case $dim\ \mathcal{P} = 1$. We shall give the proof in several steps.

Step 1. We define closed subvarieties $\mathcal{S}_0 \subset \mathcal{P} \times S$ and $\mathcal{H} \subset \mathcal{P} \times \mathbf{P}^n$ as follows:

$$\mathcal{S}_0 := \{(P,x) \mid x \in \sigma_P\} \qquad \mathcal{H} := \{(P,x) \mid x \in H_P\}$$

From $\sigma_P \subset H_P$ for any $P \in \mathcal{P}$, it follows that $\mathcal{S}_0 \subset \mathcal{H}$. There is a dominant, canonical map $p\colon \mathcal{S}_0 \to \mathcal{P}$. We define $\mathcal{S}$ to be the reduced subvariety of $\mathcal{P} \times S$ whose irreducible components are the irreducible components of $\mathcal{S}_0$ which dominate $\mathcal{P}$ through p.
Let $(O,x) \in \mathcal{S}$ be such that x is a smooth point for S. Then $\mathcal{P} \times S$ is smooth at (O,x) and $\mathcal{O}_{\mathcal{P}\times S,(O,x)}$ is a regular local ring. Since $\mathcal{S}$ is equidimensional and of codimension 1 in $\mathcal{P} \times S$, then the ideal of $\mathcal{S}$ in $\mathcal{O}_{\mathcal{P}\times S,(O,x)}$ is principal, generated by the element s. Moreover, s is a product of prime elements, being $\mathcal{S}$ reduced. There is an affine open neighbourhood $U = Spec(A)$ of (O,x) in $\mathcal{P} \times S$ such that $I(\mathcal{S} \cap U) = sA$.
$\mathcal{H}$ is an effective Cartier divisor of $\mathcal{P} \times \mathbf{P}^n$. Then $\mathcal{P} \times S \not\subseteq \mathcal{H}$ implies that $(\mathcal{P} \times S) \cap \mathcal{H}$ is an effective Cartier divisor of $\mathcal{P} \times S$ whose support contains $\mathcal{S}$. It is harmless to assume that $(\mathcal{P} \times S) \cap \mathcal{H}$ is defined on U by a single equation $h \in A$. From $\mathcal{S} \subseteq (\mathcal{P} \times S) \cap \mathcal{H}$ it follows

$$h = gs^r \qquad \text{where} \qquad g \notin sA \qquad \text{and} \qquad r \geq 1 \tag{2}$$

Step 2. We claim $r \geq 2$. First of all we pick a point $(Q,y) \in \mathcal{S}$ such that g is invertible in $\mathcal{O}_{U,(Q,y)}$. If g is not already invertible in A, let $\mathcal{T}$ be the divisor on U defined by g. Only a finite number of σ_P are contained in $\mathcal{S} \cap \mathcal{T}$. In fact, from $g \notin sA$ it follows $\mathcal{S} \not\subseteq \mathcal{T}$. Let $Q \in \mathcal{P}$ be such that $\sigma_Q \not\subseteq \mathcal{S} \cap \mathcal{T}$. It is harmless to assume that $\mathcal{P}$ is smooth at Q. Then, there is $y \in S$ such that $(Q,y) \in \mathcal{S}$ but $(Q,y) \notin \mathcal{T}$. We can assume that S and σ_Q are smooth at y.
By definition of $\mathcal{S}$ we have that H_Q is tangent to S at y, i.e.

$$T_{S,y} \subset T_{H_Q,y} \subset T_{\mathbf{P}^n,y}.$$

By taking duals we get the inclusion in $\mathcal{O}_{\mathbf{P}^n,y}$

$$I(H_Q)_y + \mathcal{M}^2_{\mathbf{P}^n,y} \subseteq I(S)_y + \mathcal{M}^2_{\mathbf{P}^n,y}$$

hence the inclusion in $\mathcal{O}_{S,y}$

$$I(H_Q \cap S)_y \subseteq \mathcal{M}^2_{S,y} \tag{3}$$

Let $\mathcal{M} \subset A$ be the maximal ideal corresponding to the point (Q,y). We shall denote by $\overline{h}, \overline{g}, \overline{s}$ the images of h, g, s respectively in the canonical map $A_{\mathcal{M}} \to \mathcal{O}_{S,y}$. Finally, from $I(H_Q \cap S)_y = \overline{h}\ \mathcal{O}_{S,y}$, (2) and (3) we get

$$\overline{g} \cdot \overline{s}^r = \overline{h} \in \mathcal{M}^2_{S,y}.$$

By construction of (Q,y) we have $g \notin \mathcal{M}$, hence $\overline{g}$ is invertible in $\mathcal{O}_{S,y}$. Therefore $\overline{s}^r \in \mathcal{M}^2_{S,y}$. But σ_Q is smooth at y, hence $\overline{s} \in \mathcal{M}_{S,y} \backslash \mathcal{M}^2_{S,y}$, whence $r \geq 2$.

Step 3. We assume that x is smooth for σ_O; the unique irreducible component of σ_O through x will be denoted σ in the sequel.
We introduce non-homogeneous coordinates $U_1, \dots, U_n$ in $\check{\mathbf{P}}^n$ and $X_0, \dots, X_{n-1}$ in $\mathbf{P}^n$ such that $O = (0, \dots, 0)$ and $x = (0, \dots, 0)$. The incidence correspondence "point-hyperplane" is defined by the equation

$$X_0 + U_1 X_1 + \ldots + U_{n-1} X_{n-1} + U_n = 0$$

(note that the image of the polynomial at the L.H.S. in the ring A is precisely h). Moreover, we can assume that the tangent line to $\mathcal{P}$ at O is defined by the equations $U_2 = \ldots = U_n = 0$. Therefore, the ideal of $\mathcal{P}$ at O can be generated by suitable polynomials

$$f_i = U_i + \text{terms of higher degree} \qquad \text{for} \quad i = 2, \dots, n.$$

Now, from the equation (2) it follows that in the ring $k[U_1, \dots, U_n, X_0, \dots, X_{n-1}]$ we have a relation

$$T \cdot (X_0 + U_1 X_1 + \ldots + U_{n-1} X_{n-1} + U_n) = \alpha s^r + \sum_{i=2}^{n} \beta_i f_i + \sum_j \gamma_j g_j$$

where $T \in k[U_1, \dots, U_n, X_0, \dots, X_{n-1}]$ is a suitable element such that $T(0,0) \neq 0$, and $g_j \in I(S) \subset k[X_0, \dots, X_{n-1}]$ for any j. If we derive this expression with respect to U_1 we get

$$T'(X_0 + \ldots + U_n) + T X_1 = \alpha' s^r + r \alpha s^{r-1} s' + \sum_{i=2}^{n} \beta_i' f_i + \sum_{i=2}^{n} \beta_i f_i' + \sum_j \gamma_j' g_j.$$

If we specialize this relation in O by setting $U_1 = \ldots = U_n = 0$, we get

$$\overline{T}' X_0 + \overline{T} X_1 = \delta \overline{s} + \sum_j \overline{\gamma}_j' g_j \in \overline{s} \cdot k[\underline{X}] + I(S) \subseteq I(\sigma). \tag{4}$$

Of course, here we use the fact $r \geq 2$ proved in Step 2 and the particular structure of the polynomials f_i.
The hyperplane H_0 is defined by the equation $X_0 = 0$. Therefore, $\sigma \subset H_0$ implies $X_0 \in I(\sigma)$ and from (4) we get $\overline{T} X_1 \in I(\sigma)$. But $T \notin (U_1, \dots, U_n, X_0, \dots, X_{n-1})$, whence $X_1 \in I(\sigma)$, since $I(\sigma)$ is prime. Summing up, σ is contained in any hyperplane of $\mathbf{P}^n$ which belongs to the pencil containing H_0 and the hyperplane of equation $X_1 = 0$. Finally, it is clear that the hyperplanes of this pencil are parametrized by the points of the tangent line to $\mathcal{P}$ at O. □

4 Trisecant lines

Let $S \subset \mathbf{P}^4$ be a general surface. Let us remind that a line L is called *n-secant* to S if the scheme-theoretic intersection $S \cap L$ has length at least n.

Lemma 17 *Let $\Omega \subset \mathbf{G}(1,4)$ be a variety parametrizing a family of n-secant lines of S, with $n \geq 3$. Then dim $\Omega \leq 3$ and, if equality holds, then a general line of Ω intersects S at n distinct points.*

Proof. The first assertion is just a consequence of the "trisecant lemma" for curves in $\mathbf{P}^3$. In fact, let us consider the incidence correspondence $I = \{(r, H) \in \Omega \times \check{\mathbf{P}}^4 \mid r \subset H\}$ and the two projections p_1, p_2 from I to Ω and $\check{\mathbf{P}}^4$ respectively. The fibres of p_1 are all irreducible of dimension 2, therefore I is irreducible and $dim\ I = dim\ \Omega + 2$. Each fibre of p_2 is formed by n-secant lines of a hyperplane section of S, so it has dimension ≤ 2. Therefore, $dim\ \Omega \leq 4$. If $dim\ \Omega = 4$, we should have $p_2(I) = \check{\mathbf{P}}^4$ and any secant line of a general hyperplane section of S should be a trisecant, against the "trisecant lemma". Finally, assume that $dim\ \Omega = 3$; the possibility that $\Omega \subset Tan(S)$ is excluded by Corollary 12.

□

Theorem 18 *Let $\Omega \subset \mathbf{G}(1,4)$ be a variety of dimension 3 parametrizing a family of n-secant lines of S, $n \geq 3$. Assume that the lines of Ω fill up $\mathbf{P}^4$. Then:*

a) $n = 3$;

b) if $L \in \Omega$ is general and $L \cap S = \{p_1, p_2, p_3\}$, then the intersection of the three hyperplanes $H_i := L + T_{p_i}, i = 1, 2, 3$, is precisely L.

Proof. By contradiction suppose that $n \geq 4$. As in Section 3, we will denote by $\widetilde{S \times S}$ the blow-up of $S \times S$ along the diagonal. Let $\phi : \widetilde{S \times S} \to \mathbf{G}(1,4)$ be the natural morphism and set $\Sigma := \phi^{-1}(\Omega)$. By Lemma 17, Σ satisfies the assumptions of Lemma 13. Therefore, if L is a general line of Ω and $L \cap S = \{p_1, \ldots, p_n\}$, then the tangent planes to S at the points $p_1, \ldots, p_n$ intersect two by two transversally; set $p_{ij} := T_{p_i} \cap T_{p_j}$. Let us consider the curves $\sigma(p_i), i = 1, \ldots, n$, as in Lemma 13, and let $\Sigma(p_i)$ be the cone of vertex p_i over $\sigma(p_i)$. Since L is general, $\sigma(p_1)$ passes through $p_2, \ldots, p_n$ with multiplicity one. The lines $T_{\sigma(p_1), p_2}, \ldots, T_{\sigma(p_1), p_n}$ are contained in the planes $T_{p_2}, \ldots, T_{p_n}$ respectively, but also in the plane α_1 which is tangent to (a branch of) $\Sigma(p_1)$ along L. So $\alpha_1 \cap T_{p_i} \cap T_{p_j} = p_{ij}, i, j = 2, \ldots, n$, thus α_1 is uniquely determined by L and one of the points $p_{ij}, i, j = 2, \ldots, n$. Similarly one sees that there is a plane α_n which intersects in lines the planes $T_{p_1}, \ldots, T_{p_{n-1}}$ and passes through L, and α_n is again determined by L and one of the points $p_{ij}, i, j = 1, \ldots, n-1$. Since $n \geq 4$, $\alpha_1 = \alpha_n$ and it is therefore the unique plane, call it α, containing L and intersecting $T_{p_1}, \ldots, T_{p_n}$ in lines.

Look now at Ω as a threefold in $\mathbf{G}(1,4)$ and at L as a point on it. We also consider the plane $A \subset \mathbf{G}(1,4)$ described by all lines of the plane α. We claim that A is tangent to Ω at L. In fact consider the n cones $\Sigma(p_i)$ as curves on Ω. The tangent lines to (branches of) these curves clearly are (distinct) lines in A which are also contained in the tangent space to Ω at L. Since $n \geq 4$ (and in fact $n \geq 2$ suffices), these lines span A, proving our claim.

Now we show that this gives a contradiction with the hypothesis that Ω fills up the space. Consider in fact a general hyperplane H in $\mathbf{P}^4$, and look at the rational map $f : r \in \Omega \to r \cap H \in H$, which we may certainly consider to be defined in a neighbourhood U of L. Since Ω fills up the space, we may also assume f to be dominant, and in fact we may suppose the differential df of f to be an isomorphism at L. But this clearly contradicts the existence of the tangent plane A to Ω at L. In fact all the tangent vectors to the lines of this plane passing through L are of course sent by df to the tangent vectors of the line $\alpha \cap H$. This contradiction shows that $n = 3$. Moreover we may observe that there are three distinct planes $\alpha_1, \alpha_2, \alpha_3$, generated by L and by p_{23}, p_{13}, p_{12} respectively.

To prove assertion b), let us remark that $H_i \cap H_j = L + p_{ij},\ i, j = 1, 2, 3$; so $H_1 \cap H_2 \cap H_3$ cannot be a plane because it should coincide with each of the α_i.

□

Remark 19 The natural problem of classifying the surfaces S such that the $3-$secant lines do not fill $\mathbf{P}^4$ has been studied by Severi in [S]. In [A] Aure proved by modern techniques that, if such a surface is smooth, then either it is contained in a hyperquadric or it is an elliptic scroll of degree 5.

5 A geometric construction

Let P be a point of S. We want to study now the family of projections of S from a point O, as O tends to P along a line L. In particular, we are interested in identifying the limit position of the double curve of the projections. To simplify notations in what follows, whenever we will project a variety A from a point $P \in A$ we will write $\pi_P(A)$ instead of $\overline{\pi_P(A - \{P\})}$ to denote the image of A under the projection π_P. We will also denote by $\deg(X)$ the degree of a projective scheme X.

Consider a neighbourhood U of P on L, such that $U \cap S = \{P\}$. For each point $O \in U - \{P\}$, the projection of S from O is a morphism. Putting all this projections together yields a rational map from $S \times U$ to $\mathbf{P}^3 \times U$. The singularities of this rational map are solved by the map g in the following diagram

$$\begin{array}{ccc} \mathcal{S} & & \\ \downarrow \rho & \searrow^{g} & \\ S \times U & \dashrightarrow & \mathbf{P}^3 \times U \end{array}$$

where $\rho : \mathcal{S} \longrightarrow S \times U$ denotes the blow-up of $S \times U$ at the point (P, P). We will denote by f the composition of g with the canonical map $\mathbf{P}^3 \times U \longrightarrow U$. For a point $O \in U - \{P\}$, the fibre of f over O is just the surface S. The fibre of f over P is $\mathbf{P} \cup S'$, where $\mathbf{P} \simeq \mathbf{P}^2$ is the exceptional divisor of the blow-up of $S \times U$ and the surface S' is the blow-up of S at the point P. If $\mathbf{E}$ is the exceptional divisor of the blow-up of S at P, then $\mathbf{P} \cap S' = \mathbf{E}$.

It is important to understand how g acts on $\mathbf{P} \cup S'$. Of course, on S' the map g is nothing but the projection of S from P, which becomes a morphism on

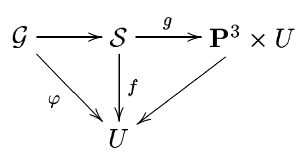
S'. On the other hand, $\mathbf{P}$ can be thought of as the $\mathbf{P}^2$ of lines passing through P and lying in the hyperplane H, which is tangent to S at P and contains L. Therefore, it is easy to see that $\mathbf{P}$ is sent by g into a plane in $\mathbf{P}^3$ which is the projection of H from P. Finally, we want to study the curve $g(\mathbf{P}) \cap g(S')$. Let $C := S \cap H$; we have $\deg(C) = \deg(S)$. The curve C has a node in P, because H is tangent to S at P, hence $m_P(C) = 2$ and

$$\deg \pi_P(C) = \deg(C) - m_P(C) = \deg(S) - 2$$

But $g(S') = \pi_P(S)$, whence $\deg(g(S')) = \deg(S) - 1$. Therefore, $\deg(g(\mathbf{P}) \cap g(S')) = \deg(S) - 1$, and the curve $g(\mathbf{P}) \cap g(S')$ breaks into $\pi_P(C)$ and the line in $\mathbf{P}^3$ which is the projection of the tangent plane of S at P.

We will denote the double locus of g by $\mathcal{G}$. It is easily seen that $\mathcal{G}$ is reduced and Cohen-Macaulay ([KLU]; see also § 9 for a similar proof). As a set we have also $\mathcal{G} = p_1(\mathcal{S} \times_{\mathbf{P}^3 \times U} \mathcal{S})$, where $p_1 : \mathcal{S} \times_{\mathbf{P}^3 \times U} \mathcal{S} \longrightarrow \mathcal{S}$ is one of the canonical maps.

Let φ be the restriction of f to $\mathcal{G}$, as shown by the diagram:



For a point $O \in U - \{P\}$, the fibre of φ over O is nothing but the pull- back Γ on S of the double curve of the projection from O. By contrast, the fibre of φ over P consists of two parts: the first one is the pull-back on S' of the double curve of the projection from P, the second one consists of the union of the pull-backs of C on $\mathbf{P}$ (including $\mathbf{E}$) and on S'. We conclude (with the notations previously introduced in this section):

Lemma 20 *The image on S, via ρ, of the fibre of φ over P is $\Gamma_P \cup C$, where Γ_P is the double curve (on S) of the projection π_P (it comes out from trisecant lines of S through P).*

□

If the line L of the previous construction is in some sense special, from Lemma 20 we will deduce some information on the double curve of the projection from a general point of L.

Proposition 21 *Let L be a proper chord of S, i.e. a line such that $L \cap S = \{P_1, P_2\}, P_1 \neq P_2$. Let O be a general point of L. Then the pull-back Γ on S of the double curve of the projection π_O passes with multiplicity one through P_1 and P_2. Therefore the tangent lines to Γ at P_1 and P_2 coincide with the lines joining P_1 and P_2 with $T_{P_1} \cap T_{P_2}$ respectively.*

Proof. Keeping notations as in the previous construction, let us consider the linear system on S of the fibres of the map φ. A general member of the system is Γ and the particular member corresponding to P_1 is $\Gamma_{P_1} \cup C$, by Lemma 20. It is clear that $P_2 \notin \Gamma_{P_1}$ because L is not a trisecant. If we assume by contradiction that P_2 belongs to C with multiplicity at least 2, from this we deduce that the hyperplane $H = T_{P_1} + L$ is tangent to S also at P_2: but this is against Lemma 13. So P_2 is a simple point for $\Gamma_{P_1} \cup C$, and by consequence it is a simple point also for Γ. The last assertion easily follows. $\square$

Proposition 22 *Let S be a surface of $\mathbf{P}^4$ having a family of dimension 3 of trisecant lines filling $\mathbf{P}^4$. Let L be a proper trisecant line of S, i.e. a line L such that $L \cap S = \{P_1, P_2, P_3\}, P_i \neq P_j$, for $i \neq j$. Let O be a general point of L. Then the pull-back Γ on S of the double curve of the projection π_O has nodes at each of the points P_1, P_2, P_3.*

Proof. We proceed as in Proposition 21, by considering the linear system of the fibres of the map φ on S, whose particular member is $\Gamma_{P_1} \cup C$. Now we claim that P_2 is a simple point both for Γ_{P_1} and for C. In fact Γ_{P_1} passes through P_2 with multiplicity one because L is a proper trisecant line. If we assume by contradiction that P_2 belongs to C with multiplicity at least 2, then the hyperplane $H = T_{P_1} + L$ is tangent to S also at P_2: so T_{P_1} and T_{P_2} generate a hyperplane; but this contradicts Lemma 13. The tangent line to C at P_2 is $(L + T_{P_1}) \cap T_{P_2}$ i.e. the line joining P_2 with $T_{P_1} \cap T_{P_2}$, while it is clear that the tangent to Γ_{P_1} is the line joining P_2 with $T_{P_3} \cap T_{P_2}$, hence they are distinct.

$\square$

Remark 23 Under the assumptions of Proposition 22, the tangent lines to Γ at P_1 are the lines joining P_1 with $T_{P_1} \cap T_{P_2}$ and $T_{P_1} \cap T_{P_3}$ respectively. Analogously for T_{Γ, P_2} and T_{Γ, P_3}.

6 The "theorem of general projection"

We prove now Theorem 8.

Let O be a general point of $\mathbf{P}^4$. Let F be the surface of $\mathbf{P}^3$ image of S under the projection from O. Theorem 18 a) tells us that F does not have points of multiplicity > 3. By Corollary 12, a general tangent line to S does not meet again S. So the triple points of F, if there are any, come only from proper trisecant lines of S, and they are ordinary, by Theorem 18 b). The number of triple points of F is finite (possibly zero) because the family of trisecant lines has dimension at most 3, by Lemma 17. Similarly F has a finite number of pinch-points, which come from the tangent lines to S passing through O. The analytical expression of such singularities may be computed, for example, as in [GH] (pages 617-618).

By Lemma 13, since O is general, each proper secant line L through O gives rise to a nodal double point of F, because the two tangent planes to S at the intersection points with L are transversal, so they project on distinct planes in $\mathbf{P}^3$. Let now γ be the double curve on F and Γ be its pull-back on S. The degree of γ is clearly the number $\frac{1}{2}(d-1)(d-2) - \pi$, where d is the degree of

S and π is its sectional genus; in particular γ can degenerate to a finite set of points only if S is degenerate.

Finally if P is a double point of S, it projects on a nodal double point P' of F by the generality of O. Moreover, P' being origin of two branches of F, it is known that it cannot be an isolated singularity of F, so it lies on γ. This concludes the list of the singularities of F.

To conclude we analyze the singularities of γ. By Proposition 21 if L is a proper secant to S at P_1 and P_2, then Γ has P_1 and P_2 as simple points with tangent lines intersecting at a point; so γ has $\pi_O(P_1) = \pi_O(P_2)$ as a simple point. Similarly, a tangent line through O is a simple tangent, hence it projects on a simple point of γ (see also [GH]). So the only singularities of γ come from trisecant lines. But from Proposition 22 and its proof it follows that, at a point of F coming from a trisecant line, γ has an ordinary triple point with no coplanar tangent lines. This concludes the proof of the theorem. □

Remark 24 If S is a surface as in Remark 19, then γ is smooth.

7 The theorem of Franchetta

In this section S will always denote a general surface of $\mathbf{P}^4$, F its projection in $\mathbf{P}^3$ from a general point O, γ the double curve of F, Γ the double curve on S, i.e. the pull-back of γ on S.

Lemma 25 *The curves Γ and γ are connected.*

Proof. It is sufficient to show that Γ is connected. As the center of projection O varies in $\mathbf{P}^4$, the double curves Γ on S describe a unirational system Ψ of divisors, hence a linear system. We claim that Ψ is not composed with a pencil. In fact, such a pencil Φ should have a (possibly empty) base locus B with the property that two curves of Φ do not intersect out of B. But this does not happen, because, for any point A of S, there exist at least two curves of Ψ passing through A, corresponding to the projections centered at points of two different secant lines through A.

Hence a general curve of the complete linear system of divisors generated by Ψ is irreducible, and the curves of the system are all connected. By consequence, also Γ is connected.

□

Proof of the Franchetta Theorem.
Assume by contradiction that γ is reducible. By Lemma 25, if γ is smooth it is irreducible. So γ has at least a singular point p such that at least two irreducible components intersect at p. By the "Theorem of general projection" p is an ordinary triple point, image via π_O of three distinct ordinary double points P_1, P_2, P_3 of Γ, lying on the line L which joins p to O. The tangent lines to Γ at P_i, $i = 1, 2, 3$, are determined only by S and L and not by the particular center of projection $O \in L$ (Remark 23). By consequence also the tangent lines

to γ at p are uniquely determined by S and L (whereas F depends on O). We recall also that, being the projection general, the three tangents to γ are not coplanar.

Concerning γ two possibilities may occur:

(i) there are three irreducible components $\gamma_1, \gamma_2, \gamma_3$ of γ passing through p;

(ii) there are two irreducible components γ_1, γ_2 of γ passing through p , one of them having p as a double point.

Case (i). $\gamma_1, \gamma_2, \gamma_3$ are projections of three curves (non necessarily irreducible!) $\Gamma_1, \Gamma_2, \Gamma_3$ contained in Γ, and we may assume that $P_2, P_3 \in \Gamma_1$, $P_1, P_3 \in \Gamma_2$, $P_1, P_2 \in \Gamma_3$. Moreover the tangent lines to Γ_1 at P_2 and P_3 project on $T_{\gamma_1,p}$, etc. A priori, if we let O vary on L, then $\Gamma_1, \Gamma_2, \Gamma_3$ vary in three linear systems on S, formed by curves that project all on $\gamma_1, \gamma_2, \gamma_3$ respectively.

Assume that each curve Γ_1, Γ_2 and Γ_3 varies in a pencil. Then it is possible to select a point $\bar{O}$ of L such that the curve Γ_1 passes also through P_1. Hence the plane T_{S,P_1} contains also the line T_{Γ_1,P_1}. Therefore, $T_{\gamma_1,p}$ is contained in the linear span of $T_{\gamma_2,p}$ and $T_{\gamma_3,p}$: contradiction. The same argument applies to Γ_2 and Γ_3.

Therefore, each curve Γ_1, Γ_2, Γ_3 is fixed as O varies on L. Let us consider one of them, say Γ_1. Let $Q \in \Gamma_1$ and $Q \notin L$. Since Γ_1 projects in a double curve for F, then for a general point $O \in L$ the line OQ intersects Γ_1 in a point different from Q. Therefore, the plane in $\mathbf{P}^4$ joining L to Q contains infinitely many points of Γ_1, hence an irreducible component of Γ_1. This plane curve is a conic, because its image on F has multiplicity two. It follows that S contains a 2-dimensional family of conics. In fact, through any point of S there is a finite number of trisecant lines, each determining three conics. By the theorem of C. Segre, S being general, these conics are irreducible and S is a projection of the Veronese surface of $\mathbf{P}^5$. S cannot be a rational normal scroll (in that case γ is smooth), so it is a Veronese surface of $\mathbf{P}^4$.

Case (ii). Assume that γ_2 is singular at p. Let Γ_1 and Γ_2 be curves contained in Γ and projecting on γ_1 and γ_2 respectively. We may suppose that Γ_1 and Γ_2 both pass through P_2 and P_3 with multiplicity one and that Γ_2 passes moreover through P_1 with multiplicity 2. Letting O vary on L, if Γ_1 moves, arguing as in case (i) we reach a contradiction. Then Γ_1 stays fixed, and we conclude as above that it is a conic and that S a Veronese surface. □

8 "Irreducible double curve" implies "linearly normal"

In this section we introduce the monoidal construction for a smooth surface $S \subset \mathbf{P}^4$ (see [Sz] for curves in $\mathbf{P}^3$, [L] for surfaces in general). The application of the Eagon-Northcott complex ([GP]) to a particular map coming from this construction reveals fruitful. This point of view has been introduced and developed by Gruson and Peskine ([P]).

Let $\pi_P : S \to F \subset \mathbf{P}^3$ be the projection of S in $\mathbf{P}^3$ from a general point P. Fixing homogeneous coordinates $(x_0 : ... : x_4)$ in $\mathbf{P}^4$, we may suppose $P = (0:0:0:0:1)$. Let us consider the map of $\mathcal{O}_{\mathbf{P}^3}-$modules given by $(x_4^2 \;\; x_4 \;\; 1)$

$$w : \mathcal{O}_{\mathbf{P}^3}(-2) \oplus \mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3} \longrightarrow \pi_* \mathcal{O}_S$$

By the "general projection theorem" it follows that every fiber of $\pi_P : S \to F$ has length at most 3. This implies that w is surjective (see [L]).
Let now $E := ker(w)$; the sheaf E is locally free of rank 3, with $c_1(E) = -d-3$, $c_2(E) = \frac{1}{2}(d^2 + 3d + 6) - \pi$ (where d and π denote respectively the degree and the sectional genus of S).
The sheaf $\pi_* \mathcal{O}_S$ enters also in the short exact sequence "of the conductor", involving the dualizing sheaf ω_γ of the double curve γ (see [R2]):

$$O \to \mathcal{O}_F \to \pi_* \mathcal{O}_S \to \omega_\gamma(4-d) \to 0.$$

This exact sequence and the map w fit together in the following commutative diagram of sheaves on $\mathbf{P}^3$ with exact rows and columns, which also defines the sheaf $\mathcal{R}$ and the map $\phi(-1)$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}(-d) & \longrightarrow & \mathcal{O}_{\mathbf{P}^3} & \longrightarrow & \mathcal{O}_F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E & \longrightarrow & \bigoplus_{0 \le i \le 2} \mathcal{O}_{\mathbf{P}^3}(-i) & \xrightarrow{w} & \pi_* \mathcal{O}_S & \longrightarrow & 0 \\
 & & \downarrow & \overset{\phi(-1)}{\searrow} & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{O}_{\mathbf{P}^3}(-2) \bigoplus \mathcal{O}_{\mathbf{P}^3}(-1) & \longrightarrow & \omega_\gamma(4-d) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (5)$$

In the next proposition we give a cohomological interpretation of the linear normality of S together with some auxiliary vanishing results.

Proposition 26 *(i)* $h^0(E) = h^1(E) = 0$;

(ii) if $h^0(\mathcal{I}_S(k)) = 0$, *then* $h^0(E(k)) = 0$; *if* $k = 1$ *(resp. 2), then also* $h^0(E(k)) = 0$ *implies* $h^0(\mathcal{I}_S(k)) = 0$;

(iii) S *is linearly normal if and only if* $h^1(E) = 0$.

Proof. (i) follows from the exact sequence

$$0 \to H^0(E) \to H^0(\mathcal{O}_{\mathbf{P}^3}) \to H^0(\mathcal{O}_S) \to H^1(E) \to 0,$$

and from the assumption that S is integral.

(ii) From the cohomology of the second row of the diagram (5) tensorized with $\mathcal{O}(k)$, we have the exact sequence:

$$0 \to H^0(E(k)) \to H^0(\mathcal{O}(k)) \oplus H^0(\mathcal{O}(k-1)) \oplus H^0(\mathcal{O}(k-2)) \stackrel{\phi_k}{\to}$$
$$\to H^0(\mathcal{O}_S(k)) \to H^1(E(k)) \to 0.$$

The map ϕ_k acts as follows: $\phi_k(F_k, F_{k-1}, F_{k-2}) = f_k + x_4 f_{k-1} + x_4^2 f_{k-2}$, where f_j denotes the restriction to S of the polynomial F_j. By factorizing ϕ_k through $H^0(\mathcal{O}_{\mathbf{P}^4}(k))$, we get the commutative diagram

$$\begin{array}{ccccccc}
H^0(E(k)) & \longrightarrow & \bigoplus_{i=0}^{2} H^0(\mathcal{O}_{\mathbf{P}^3}(k-i)) & \xrightarrow{\phi_k} & H^0(\mathcal{O}_S(k)) & \twoheadrightarrow & H^1(E(k)) \\
\downarrow & & \downarrow \phi'_k & & \| & & \downarrow \\
H^0(\mathcal{I}_S(k)) & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^4}(k)) & \longrightarrow & H^0(\mathcal{O}_S(k)) & \twoheadrightarrow & H^1(\mathcal{I}_S(k))
\end{array} \qquad (6)$$

where $\phi'_k = (1 \ x_4 \ x_4^2)$. To conclude the proof, it is enough to remark that ϕ'_k is always injective and it is surjective if $k \leq 2$.

(iii) If $k = 1$, diagram (6) above becomes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^3}(1)) \oplus H^0(\mathcal{O}_{\mathbf{P}^3}) & \xrightarrow{\phi_1} & H^0(\mathcal{O}_S(1)) & \longrightarrow & H^1(E(1)) & \longrightarrow & 0 \\
 & & \downarrow \phi'_1 & & \| & & \downarrow \psi & & \\
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbf{P}^4}(1)) & \longrightarrow & H^0(\mathcal{O}_S(1)) & \longrightarrow & H^1(\mathcal{I}_S(1)) & \longrightarrow & 0
\end{array}$$

where ϕ'_1 and ψ are isomorphisms. Therefore, the surface S is linearly normal if and only if $h^1(\mathcal{I}_S(1)) = h^1(E(1)) = 0$.

□

From diagram (5) we deduce, in particular, the following exact sequence:

$$E(1) \stackrel{\phi}{\to} \mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3} \to \omega_\gamma(5-d) \to 0. \qquad (7)$$

From now on we shall denote $\omega_\gamma(5-d)$ simply by $\mathcal{M}$. We will consider, now, the second Eagon-Northcott complex associated to the map ϕ (see [GP]):

$$0 \to \bigwedge^2 (E(1)) \to E(1) \otimes (\mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3}) \to S_2(\mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3}) \to S_2(\mathcal{M}) \to 0 \qquad (8)$$

The crucial Lemma 27 below relates the cohomology of γ with the cohomology of this Eagon-Northcott complex. This will enable us to exploit Proposition 26 in order to prove the linear normality of S. In § 9 we will prove that the sheaf $\mathcal{F}_1(\pi_* \mathcal{O}_S)$ defines on γ the reduced structure (see Remark 33); from now on we will always consider γ with its reduced structure.

Lemma 27 *Let $\widetilde{\gamma}$ be the normalization of γ. Then there exists an isomorphism of $\mathcal{O}_\gamma$-modules $S_2(\mathcal{M}) \simeq \mathcal{O}_{\tilde{\gamma}}$.*

(The proof of this Lemma is the content of § 9.)

The support of the cohomology of (8) is contained in the degeneracy locus of $\wedge^2 \phi$ ([GP], §2), i.e. in γ. But

$$codim_{\mathbf{P}^3}(\gamma) = 2 = rk(E) - rk(\mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3}) + 1.$$

Hence, *the complex (8) is exact* ([GP], §2).

Note that $\wedge^2 (E(1)) \simeq \check{E}(-d-1)$. In fact, we have $\wedge^2 (E(1)) = (\wedge^2 E)(2)$ and the desired conclusion follows from $\wedge^2 E \simeq \wedge^3 E \otimes \check{E}$ and from $c_1(E) = -d-3$. Therefore, by Lemma 27, the complex (8) becomes

$$0 \to \check{E}(-d-1) \to E \oplus E(1) \to \bigoplus_{0 \le i \le 2} \mathcal{O}_{\mathbf{P}^3}(-i) \to \mathcal{O}_{\widetilde{\gamma}} \to 0. \qquad (9)$$

We are ready to prove now the main result of this section, namely the implication $(d) \Rightarrow (c)$ in Theorem 6.

Theorem 28 *If γ is irreducible, then S is linearly normal.*

Proof. Let us split the Eagon-Northcott (exact) complex (9) in the following way:

$$0 \to \check{E}(-d-1) \to E \oplus E(1) \to \mathcal{K} \to 0$$

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(-1) \oplus \mathcal{O}_{\mathbf{P}^3}(-2) \to \mathcal{O}_{\tilde{\gamma}} \to 0$$

By Proposition 26, the associated long exact sequences of cohomology become respectively

$$0 \to H^0(\mathcal{K}) \to H^1(\check{E}(-d-1)) \to H^1(E(1)) \to H^1(\mathcal{K}) \qquad (10)$$

$$0 \to H^0(\mathcal{K}) \to H^0(\mathcal{O}_{\mathbf{P}^3}) \to H^0(\mathcal{O}_{\tilde{\gamma}}) \to H^1(\mathcal{K}) \to 0 \qquad (11)$$

By Serre duality, $h^1(\check{E}(-d-1)) = h^2(E(d-3))$ and, by diagram (5), we get

$$h^2(E(d-3)) = h^1(\mathcal{O}_S(d-3)) = h^2(\mathcal{I}_S(d-3)) = 0$$

because $\mathcal{I}_S$ and E are $(d-1)$-regular ([L]). Therefore (10) yields $H^0(\mathcal{K}) = 0$, and from (11) we get $h^0(\mathcal{O}_{\tilde{\gamma}}) = h^1(\mathcal{K}) + 1$. If γ is irreducible, then $\widetilde{\gamma}$ is also irreducible, hence $h^0(\mathcal{O}_{\tilde{\gamma}}) = 1$ and $h^1(\mathcal{K}) = 0$. From (10) we get $h^1(E(1)) = 0$ and S is linearly normal by Proposition 26 (iii). □

9 Proof of Lemma 27

From diagram (5) in § 8 we extract the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(1) & \longrightarrow & \pi_* \mathcal{O}_S(1) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbf{P}^3} & \longrightarrow & \mathcal{M} \end{array} \tag{12}$$

We shall still denote by "x_4" the global section of $\mathcal{M}$, image of the global section x_4 of $\pi_*\mathcal{O}_S(1)$ in the map $\pi_*\mathcal{O}_S(1) \to \mathcal{M}$. This section induces canonically a map of $\mathcal{O}_\gamma$-modules $v : \mathcal{O}_\gamma \to \mathcal{M}$. Finally, we set

$$U := \{x \in \gamma \mid \gamma \text{ is smooth at } x\}.$$

Proposition 29 *There is an isomorphism of $\mathcal{O}_\gamma|_U$-modules $S_2(\mathcal{M})|_U \simeq \mathcal{O}_{\tilde{\gamma}}|_U$ which sends $x_4 \otimes x_4$ to the unit section.*

Proof. First of all, we will prove that the restriction to U of the map $v : \mathcal{O}_\gamma \to \mathcal{M}$ is an isomorphism. It is clearly sufficient to show this locally, so let $x \in U$. The map $v_x : \mathcal{O}_{\gamma,x} \to \mathcal{M}_x$ is surjective. In fact, being x a smooth point for γ, the map $(\mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(1))_x \to (\pi_*\mathcal{O}_S(1))_x$ is surjective. The commutativity of (5) then yields the surjectivity of the map $\mathcal{O}_{\mathbf{P}^3,x} \to \mathcal{M}_x$, hence the surjectivity of v_x. But v_x is also injective. In fact, from $\mathcal{M} = \omega_\gamma(5-d)$ it follows that $\mathcal{M}_x$ is a free $\mathcal{O}_{\gamma,x}$-module of rank one.
From this isomorphism $\mathcal{O}_\gamma|_U \to \mathcal{M}|_U$ it follows that

$$S_2(\mathcal{M}|_U) \simeq S_2(\mathcal{O}_\gamma|_U) \simeq \mathcal{O}_\gamma|_U.$$

Finally, $\mathcal{O}_\gamma|_U$ and $\mathcal{O}_{\tilde{\gamma}}|_U$ are clearly isomorphic by the usual properties of the blow-up. □

To deal with the singularities of γ we need a careful preparation of the data. As usual, let $\pi: S \to \mathbf{P}^3$ denote the projection map; moreover, let $\{T_1, \ldots, T_\alpha\}$ be the singular locus of γ. For any triple point T_i of γ we set

$$\pi^{-1}(T_i) = \{A_i, B_i, C_i\}$$

Lemma 30 *There exist hypersurfaces $F, G \subset \mathbf{P}^4$ containing S, defined by equations $f = 0$ and $g = 0$ respectively, and there exists an affine open subset $V = Spec(B) \subset \mathbf{P}^3$, such that the following properties hold:*

(i) $f := \varphi_0 x_4^3 + \varphi_1 x_4^2 + \varphi_2 x_4 + \varphi_3$, where $\varphi_i \in k[x_0, \ldots, x_3]$ for any $i = 0, \ldots, 3$; moreover, the surface in $\mathbf{P}^3$ defined by $\varphi_0 = 0$ does not intersect $Sing(\gamma)$;

(ii) F is smooth along $\pi^{-1}(Sing(\gamma))$;

(iii) $Sing(\gamma) \subset V$;

(iv) $\pi^{-1}(V) \simeq Spec(A)$*, where*

$$A := \frac{B[x_4]}{(f,g)};$$

(v) the coefficient φ_0 *of* x_4^3 *in* f *is invertible in* B;

(vi) we can choose G *so that on* $V \times \mathbf{A}^1 = Spec(B[x_4])$ *it is defined by an equation of the form* $\psi_1 x_4^2 + \psi_2 x_4 + \psi_3$ *where* $\psi_i \in B,\ i = 1,2,3.$

Proof. We shall denote by R and D respectively the rings $k[x_0,\ldots,x_3]$ and $k[x_0,\ldots,x_3,x_4]/I(S)$. From diagram (5) in § 8 we extract the short exact sequence

$$0 \longrightarrow E \xrightarrow{u} \bigoplus_{0\le i\le 2} \mathcal{O}_{\mathbf{P}^3}(-i) \longrightarrow \pi_*\mathcal{O}_S \longrightarrow 0 \tag{13}$$

which yields for $\nu \gg 0$ a surjective map

$$w_\nu : R_{\nu-2} \oplus R_{\nu-1} \oplus R_\nu \xrightarrow{(x_4^2 \quad x_4 \quad 1)} D_\nu$$

Let $\varphi_0 \in R_{\nu-3}$, $\varphi_0 \neq 0$, be such that $\varphi_0(T_i) \neq 0$ for any $i = 1,\ldots,\alpha$. The surjectivity of w_ν above implies the existence of polynomials $\varphi_1 \in R_{\nu-2}$, $\varphi_2 \in R_{\nu-1}$ and $\varphi_3 \in R_\nu$ such that the following equality holds in D_ν

$$\varphi_0 x_4^3 = -\varphi_1 x_4^2 - \varphi_2 x_4 - \varphi_3$$

Hence we set $f := \varphi_0 x_4^3 + \varphi_1 x_4^2 + \varphi_2 x_4 + \varphi_3 \in I(S) \subset k[x_0,\ldots,x_3,x_4]$, and we shall denote by F the corresponding hypersurface in $\mathbf{P}^4$.

To show the smoothness of F at the points of $\pi^{-1}(Sing(\gamma))$, let us first recall that we are projecting S from the point $P = (0:0:0:0:1) \in \mathbf{P}^4$. From $\varphi_0(T_i) \neq 0$ it follows that the line $L_i := PT_i$ is not contained in F. More precisely, we have $L_i \cap F = \{A_i, B_i, C_i, P\}$ (we are assuming $\nu > 3$ now). Concerning the intersection multiplicities, the structure of the polynomial f yields $i(P, L_i \cdot F; \mathbf{P}^4) \geq \nu - 3$. Therefore, equality holds and we have the equality of 0-cycles in $\mathbf{P}^4$

$$L_i \cdot F = A_i + B_i + C_i + (\nu-3)P$$

which yields

$$i(A_i, L_i\cdot F; \mathbf{P}^4) = i(B_i, L_i\cdot F; \mathbf{P}^4) = i(C_i, L_i\cdot F; \mathbf{P}^4) = 1$$

Therefore, we can conclude that F is smooth at A_i, B_i, C_i for any $i = 1,\ldots,\alpha$ by the "Criterion of Multiplicity One" ([W], Thm.6, pag.152), and (ii) is proved.

Let $Q \in \mathbf{P}^4$ be a point such that the lines QA_i, QB_i, QC_i are not secant lines for S for any $i = 1,\ldots,\alpha$. Assume, moreover, that Q does not lie on any tangent hyperplane to F at the points A_i, B_i, C_i. Then the join of Q and S is a hypersurface $G \subset \mathbf{P}^4$ which contains S and is transversal to F at any point

A_i, B_i, C_i where $i = 1, \ldots, \alpha$. Let us consider the complete intersection surface $F \cap G \subset \mathbf{P}^4$. The points of S where S is different from $F \cap G$ form a proper closed subset, which is disjoint from $\pi^{-1}(Sing(\gamma))$. The existence of V such that G and V satisfy (iii),(iv) and (v) follows from the fact that π is a closed map and from (i).

Finally, by (v) we can divide g by f in $B[x_4]$, namely $g = fq + r$, where $r = \psi_1 x_4^2 + \psi_2 x_4 + \psi_3$ and $\psi_i \in B, i = 1, 2, 3$. Therefore the ideals (f, g) and (f, r) in $B[x_4]$ are equal, and (vi) holds true. □

Lemma 31 *The restriction $E|_V$ is trivial and the map u in the short exact sequence (13) can be represented on V by a persymmetric matrix.*

Proof. We keep the notations of the previous lemma; moreover we set

$$C := \frac{B[x_4]}{(f)}$$

By Lemma 30, C is a free B-module of rank 3 and $\{f, g\}$ is a regular sequence in $B[x_4]$. Therefore, we have the free presentation for the B-module A

$$0 \longrightarrow C \xrightarrow{\cdot g} C \longrightarrow A \longrightarrow 0$$

Note that $\Gamma(V, \pi_* \mathcal{O}_S) \simeq A$, and by sheafifying the above exact sequence we get

$$0 \longrightarrow E|_V \xrightarrow{u} \bigoplus_{0 \leq i \leq 2} \mathcal{O}_{\mathbf{P}^3}(-i)|_V \longrightarrow \pi_* \mathcal{O}_S|_V \longrightarrow 0$$

In particular, this shows that the restriction $E|_V$ is trivial.
A basis for C over B is $\mathcal{B} = \{1, x_4, x_4^2\}$. Let us consider the basis for $C^* = Hom_B(C, B)$, dual of $\mathcal{B}$. In particular, we set $\tau := (x_4^2)^*$. We have on C^* a canonical structure of C-module by setting

$$c\phi(v) := \phi(cv) \qquad \text{for any} \quad c, v \in C \qquad \text{and any} \quad \phi \in C^*$$

Using this structure of C-module we can construct the basis $\{\tau, x_4\tau, x_4^2\tau\}$ for C^* over B. Finally, if we dualize it, then we get a new basis $\mathcal{B}' = \{v_1, v_2, v_3\}$ for C.
It is easily seen that the matrix which represents the multiplication by g into C with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$ (for the first copy of C and for the target, respectively) is

$$\begin{pmatrix} (g\tau^*)(1) & g(x_4\tau^*)(1) & g(x_4^2\tau^*)(1) \\ (g\tau^*)(x_4) & g(x_4\tau^*)(x_4) & g(x_4^2\tau^*)(x_4) \\ (g\tau^*)(x_4^2) & g(x_4\tau^*)(x_4^2) & g(x_4^2\tau^*)(x_4^2) \end{pmatrix} = \begin{pmatrix} \tau^*(g) & \tau^*(gx_4) & \tau^*(gx_4^2) \\ \tau^*(gx_4) & \tau^*(gx_4^2) & \tau^*(gx_4^3) \\ \tau^*(gx_4^2) & \tau^*(gx_4^3) & \tau^*(gx_4^4) \end{pmatrix},$$

which is persymmetric.

□

From now on, we shall denote the above matrix by

$$\begin{pmatrix} a & b & c \\ b & c & d \\ c & d & e \end{pmatrix} \tag{14}$$

We set $R := \Gamma(V, \mathcal{O}_\gamma)$, $M := \Gamma(V, \mathcal{M})$ and $J := (a, b, c)R$. Moreover, we shall still denote by "a" the residue class of $a \in B$ into R. In the following lemma we derive the basic properties of these objects we shall use in the sequel.

Lemma 32 *(i) The curve γ is defined on V by the ideal $I \subset B$ generated by the 2×2 minors of the first two rows of (14);*

(ii) the singular locus of γ is defined on V by the ideal $(a, b, c) \subset B$;

(iii) $R = B/I$ is a reduced, Cohen-Macaulay ring;

(iv) a is a regular element of R and $aJ = J^2$.

Proof. By the "Theorem of general projection", γ and its singular locus are the double and triple locus for the map π. As recalled in the Introduction, the scheme structures on these loci are given (on V) by the Fitting ideals $\mathcal{F}_1(A)$ and $\mathcal{F}_2(A)$, namely the ideals generated by all the 2×2 minors of (14), and all the entries of (14), respectively. Then, (i) and (ii) follow from [GP], Lemme 1.3, or by a direct, cumbersome calculation based on the explicit form for f and g given in Lemma 30.

The curve γ is generically reduced, hence R enjoys Serre's property (R_0). This follows from [KLU], page 3. Therefore, (iii) will be completely proved if we show that R is Cohen-Macaulay. Two different proofs of this fact are given in [R2] and [KLU]. In our situation this is straightforward because the ideal I is generated by the minors of a 2×3 matrix, and from $grade(I) = ht(I) = 2$ it follows that I is perfect.

Finally, we first prove that $aJ = J^2$. We have $aJ = (a^2, ab, ac)$; therefore, we have to show that $b^2, bc, c^2 \in aJ$. From (ii) it follows $d = \lambda a + \mu b + \nu c$ for some $\lambda, \mu, \nu \in R$, hence $ad \in aJ$. But $ad = bc$ in R and we get $bc \in aJ$. In a similar way one can prove that $b^2, c^2 \in aJ$. To show that a is a regular element of R, we remark first that J is a regular ideal (i.e. J contains a regular element). In fact R is a Cohen-Macaulay ring and $ht(J) = 1$. Then J^2 is also a regular ideal and from $a \in \mathcal{Z}(R)$, the set of zero-divisors of R, we would get $J^2 = aJ \subseteq \mathcal{Z}(R)$, a contradiction. □

Remark 33 In [KLU], page 4, it is proved that γ is Cohen-Macaulay at every point. Joint with the generic reducedness this yields that γ is reduced.

From the sequence (7) in § 8 it follows that the B-module M has the free presentation

$$L \xrightarrow{\begin{pmatrix} a & b & c \\ b & c & d \end{pmatrix}} N \longrightarrow M \longrightarrow 0$$

where on $L = B^3$ we consider the base $\mathcal{B}$ and $N \subset B^3$ is the free submodule generated by $v_1, v_2 \in \mathcal{B}'$.
From the sequence (8) in § 8 we deduce the following free presentation of the B-module $S_2(M)$

$$L \otimes N \xrightarrow{\lambda} S_2(N) \xrightarrow{i} S_2(M) \longrightarrow 0 \tag{15}$$

It is easily seen that the map λ above is represented (with respect to an obvious choice of the bases) by the matrix

$$\begin{pmatrix} a & 0 & b & 0 & c & 0 \\ b & a & c & b & d & c \\ 0 & b & 0 & c & 0 & d \end{pmatrix} \tag{16}$$

Of course, $S_2(M)$ is an R-module, and we tensorize (15) by R to get a free presentation of $S_2(M)$ as an R-module. Let us consider, now, the presentation of the ideal J

$$R^3 \xrightarrow{j} J \longrightarrow 0$$

given by the matrix $(c \quad -b \quad a)$. A direct computation shows that every column of (16) is in the kernel of j, namely $ker(i) \subseteq ker(j)$. Therefore, we get the short exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow S_2(M) \longrightarrow J \longrightarrow 0. \tag{17}$$

Proposition 34 *There is an isomorphism of R-modules $S_2(M) \simeq J$ which sends $x_4 \otimes x_4$ to a.*

Proof. It is sufficient to show that $K = 0$ in the exact sequence (17).
From (15) it follows that the Fitting ideals $\mathcal{F}_0(S_2(M))$ and $\mathcal{F}_1(S_2(M))$ of the R-module $S_2(M)$ are respectively 0 and a regular ideal of R. Therefore, if T denotes the set of regular elements of R, we have $\mathcal{F}_0(S_2(M)_T) = 0$ and $\mathcal{F}_1(S_2(M)_T) = R_T$, which means that $S_2(M)_T$ is a free R_T-module of rank one. But J_T is a free R_T-module of rank one as well, and we conclude $K_T = 0$, namely K is a torsion submodule of $S_2(M)$.
To complete the proof we will show, now, that $S_2(M)$ is torsionfree. This property is equivalent to

$$depth(S_2(M)_{\mathcal{P}}) \geq min\{1\ , grade(\mathcal{P})\} \qquad \text{for every } \ \mathcal{P} \in Spec(R)\ .$$

Let $\mathcal{P}$ be a maximal ideal of R (the minimal prime ideals cause no troubles). Let $\mathcal{Q} \subset B$ be the preimage of $\mathcal{P}$. From the exact sequence (8) in § 8 we get $pd_{B_{\mathcal{Q}}}(S_2(M)_{\mathcal{Q}}) \leq 2$. Therefore

$$depth_{B_{\mathcal{Q}}}(S_2(M)_{\mathcal{Q}}) = depth(B_{\mathcal{Q}}) - pd_{B_{\mathcal{Q}}}(S_2(M)_{\mathcal{Q}}) \geq 3 - 2 = 1$$

We conclude by remarking that the image into $R_{\mathcal{P}}$ of any element of $B_{\mathcal{Q}}$ which is regular for $S_2(M)_{\mathcal{Q}}$ is still regular for $S_2(M)_{\mathcal{P}} = S_2(M)_{\mathcal{Q}}$. □

Proposition 35 *There is an isomorphism of* $\mathcal{O}_\gamma|_V$*-modules* $\widetilde{J} \simeq \mathcal{O}_{\tilde{\gamma}}|_V$ *which sends the section* a *to the unit section.*

Proof. We introduce the following subring of the total quotient ring K of R:

$$R' := R\left[\frac{J}{a}\right].$$

Lemma 32 (iv) allows us to apply [KLU], Lemma (4.5): by part (4), the blow-up of $\gamma \cap V$ along $Sing(\gamma)$ is isomorphic to $Spec(R')$. Moreover, let us consider the injective map of R-modules $R' \stackrel{\cdot a}{\to} K$. The image of this map contains J. By loc. cit. part (2), the image is exactly J. Therefore, we get an isomorphism of R-modules $R' \simeq J$. By sheafifying the inverse of this isomorphism we get the required isomorphism of $\mathcal{O}_\gamma|_V$-modules between $\widetilde{J}$ and $\mathcal{O}_{\tilde{\gamma}}|_V$. □

To conclude the proof of Lemma 27 we sheafify the isomorphism of Proposition 34 and we compose the resulting isomorphism with that of Proposition 35. The composition $S_2(\mathcal{M})|_V \simeq \mathcal{O}_{\tilde{\gamma}}|_V$ sends the section $x_4 \otimes x_4$ to the unit section, hence it glues with the isomorphism constructed in Proposition 29.

□

Acknowledgment.
This report is widely based on a previous, unpublished manuscript by Ciro Ciliberto and Christian Peskine; we are grateful to them for letting us use their work. We also thank them, Laurent Gruson and Wolmer Vasconcelos for discussions and advice.

Moreover, the first author would like to thank Christian Peskine for introducing her to the subject several years ago.

This work has been done in the framework of the activities of Europroj. Both authors have been supported by funds of MURST and are members of GNSAGA.

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