Data Science and Scientific Computing

Advanced Mathematical Methods

Part of the course by **Prof. Franco Obersnel**

The linear transport equation. An example: pollution in a channel. Derivation of the linear transport equation. The Initial Value Problem. The homogeneous and the non-homogeneous problem; Duhamel's principle. Well-posedeness of the transport problem. The problem with decay. An example with a jump discontinuity. Characteristic lines; inflow and outflow characteristics for the transport problem on a strip. Exercises.

The method of characteristics for quasilinear first order equations. The method of characteristics for a quasilinear equation in dimension 2. The quasilinear problem in dimension N. Characteristic equations and projected characteristics. The structural theorem. Examples: the linear and the semilinear cases. The initial value problem: local existence and uniqueness. The inverse function theorem in $\mathbb{R}^N$. The transversality condition. Failure of the transversality condition: characteristicity of the curve supporting the initial data. Examples: the non-homogeneous Burgers equation, possible existence of non-regular solutions, the transport equation revisited. Exercises.

Scalar conservation laws. Characteristics for conservation laws. Crossing characteristics. Implicit formula for the solution. Review of the Implicit Function Theorem. Appearance of the shock. The need of non-regular solutions. Vanishing viscosity solutions. Mild solutions and weak solutions for the conservation law. Classical solutions are mild solutions. Mild solutions are weak solutions. Smooth weak solutions are classical solutions. Integration by parts in $\mathbb{R}^N$. Solutions with a jump singularity: the Rankine-Hugoniot condition. Speed of the shock wave. Rarefaction waves. Existence of non-physical shock solutions. Solutions via a regularised problem: the formula in case q is uniformly convex or concave. Compression waves and expansion waves: an entropy condition in case of shock. The general entropy condition. A smooth solution is an entropy solution if q is uniformly convex. An entropy solution satisfies the entropy requirement for shocks. Lax-Oleinik well-posedness theorem. An example of vanishing-viscosity solutions vs entropy solutions. The Riemann problem. A traffic flow model: the Lighthill-Whitham-Richards model. The red traffic light and the green traffic light. Exercises.

The wave equation. The vibrating string. Derivation of the 1-dimensional wave equation. D'Alembert's equation. Well-posedeness of the homogeneous Cauchy problem. Weak solutions. Smooth solutions are weak solutions. Propagation of singularities. Domain of dependence and region of influence. The non-homogeneous case: direct derivation and Duhamel's method. D'Alembert's formula. Well-posedeness theorem. Even, odd and periodic solutions. The problem on the half line (a reflection method): the Dirichlet and the Neumann cases. Peculiarities of the $1-$dimensional problem: failure of decay of waves. The radial problem in $\mathbb{R}^3$. Radially symmetric solutions of the homogeneous equation. Decay of waves and Huygens principle. Spherical means. Derivative of spherical means. Darboux equation. Kirchoff's formula. Huygens principle. The existence and uniqueness theorem. The problem in 2 dimensions: Hadamard's descent method. Poisson's formula. The existence and uniqueness theorem in $\mathbb{R}^2$. The wave equation on a bounded interval (separation of variables); the Dirichlet and the Neumann problems. Conservation of energy. Uniqueness and stability.

Reference Textbooks

L. C. Evans, Partial Differential Equations, American Mathematical Society, Providence RI, 1999.

S. Salsa, Partial Differential Equations in Action: From Modelling to Theory Springer, Milan, 2008.

Y. Pinchover, J. Rubinstein An Introduction to Partial Differential Equations, Cambridge Univ. Press