Tangent lines and the inverse function differentiation rule

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Abstract

The usual elementary definition of tangent line to the graph of a function leads to the following unlikely situation: in the $xy-$plane there exists a set $\mathcal{G}$ which admits tangent at some point if it is seen as a graph with respect the $x-$axis, whereas it does not have tangent at the same point if it is thought as a graph with respect to the $y-$axis. This is equivalent to say that there exists an invertible function f, which is differentiable at x_0 with $f'(x_0) \neq 0$, but whose inverse f^{-1} is not differentiable at $f(x_0)$.

Let I, J be real intervals and let $f : I \to J$ be a function, whose graph is denoted by $\mathcal{G}$. In almost any textbook of calculus[1] the following definition is encountered: *if f is differentiable at $x_0 \in I$, the equation*

$$y = f(x_0) + f'(x_0)(x - x_0)$$

represents, in the $xy-$plane, a line r which is called the tangent to $\mathcal{G}$ at $P = (x_0, f(x_0))$.

This analytical definition is the natural translation of a geometrical intuition. Yet, it may engender the (wrong) impression that having tangent at some point is a property of $\mathcal{G}$, as a set of points in the plane, rather than a property of $\mathcal{G}$, as a graph of a function. At an elementary teaching level, this point is sometimes left a little bit vague. This would not be a serious drawback in itself. But, unfortunately, some inconvenience occurs when the following naïve explanation of the inverse function differentiation rule is proposed. Let $f : I \to J$ be an invertible function, with inverse $f^{-1} : J \to I$. Denote by $\mathcal{H}$ the graph of f^{-1}. $\mathcal{G}$ and $\mathcal{H}$ are symmetric with respect to the principal diagonal. Assume that f is differentiable at some point x_0 with $f'(x_0) \neq 0$. If $Q = (f(x_0), x_0) \in \mathcal{H}$ is the symmetric of $P = (x_0, f(x_0)) \in \mathcal{G}$, then the tangent s to $\mathcal{H}$ at Q is the

[1]see, e.g., R. Courant, F. John, Introduction to Calculus and Analysis, Springer-Verlag, New York, 1989 (Vol. I, pp. 156–157)

symmetric with respect to the principal diagonal of the tangent r to $\mathcal{G}$ at P. Hence, one concludes that $(f^{-1})'(f(x_0)) = 1/f'(x_0)$. Of course, the crucial question here is whether the tangent s to $\mathcal{H}$ at Q does exist, i.e., whether f^{-1} is differentiable at $f(x_0)$.

On account of its geometrical evidence, one could guess that the following statement holds true: *if $f : I \to J$ is invertible and differentiable at $x_0 \in I$, with $f'(x_0) \neq 0$, then the inverse function $f^{-1} : J \to I$ is differentiable at $f(x_0)$.*

Actually, this is false. Indeed, under the above assumptions one cannot even guarantee the continuity of f^{-1} at $f(x_0)$. To the best of our knowledge, this fact seems to have been not yet explicitly pointed out.

The example we produce goes precisely in this direction: we construct an invertible function $f : I \to J$, which is differentiable at $x_0 \in I$ with $f'(x_0) \neq 0$, such that the inverse function f^{-1} is discontinuous, and therefore not differentiable, at $f(x_0)$. Accordingly, the graph $\mathcal{G}$ of f is a rather peculiar set in the $xy-$plane: $\mathcal{G}$ admits tangent at some point if it is seen as a graph with respect the $x-$axis, but conversely it does not have any tangent at that point if it is thought as a graph with respect to the $y-$axis.

Example For all positive integers n and k, let us set

$$a_k^{(n)} = \tfrac{1}{n} - \tfrac{1}{(n+k)^2}$$

and

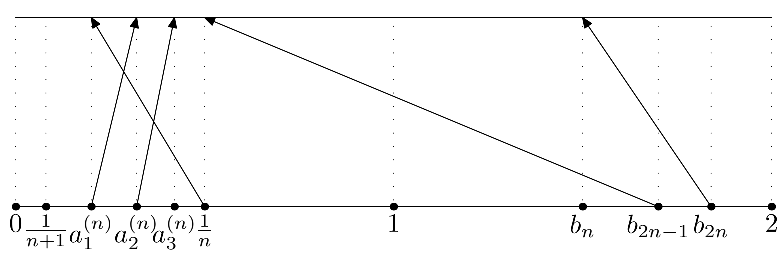
$$b_n = 2 - \tfrac{1}{n+1}.$$

It is obvious that

$$\tfrac{1}{n+1} < a_k^{(n)} < \tfrac{1}{n} \quad \text{and} \quad b_n > 1.$$

Define a function $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} a_1^{(n)} & \text{if } x = \frac{1}{n}, \\ a_{k+1}^{(n)} & \text{if } x = a_k^{(n)}, \\ \frac{1}{n} & \text{if } x = b_{2n-1}, \\ b_n & \text{if } x = b_{2n}, \\ x & \text{otherwise.} \end{cases}$$



It is easy to check that f is one-to-one and onto. Let us verify that f is differentiable at 0, with $f'(0) = 1$. Indeed, if $x \in]\frac{1}{n+1}, \frac{1}{n}]$, we have

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} \leq \frac{f(x)}{x} \leq \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

and therefore

$$\frac{f(x)}{x} \to 1, \qquad \text{as } x \to 0.$$

Finally, since

$$f^{-1}\left(a_1^{(n)}\right) = \frac{1}{n} \to 0 \quad \text{and} \quad f^{-1}\left(\frac{1}{n}\right) = b_{2n-1} \to 2, \quad \text{as } n \to +\infty,$$

we conclude that f^{-1} is not continuous at $f(0) = 0$.

Actually, the continuity of f^{-1} at $f(x_0)$ is the key ingredient in order to get the following statement of the inverse function differentiation rule.

Theorem A *Let I be a real interval and let $f : I \to f(I)$ be invertible and differentiable at $x_0 \in I$, with $f'(x_0) \neq 0$. Then, the inverse function $f^{-1} : f(I) \to I$ is differentiable at $f(x_0)$, with $(f^{-1})'(f(x_0)) = 1/f'(x_0)$, if and only if it is continuous at $f(x_0)$.*

We give a proof of the "if" part of this known statement just for the sake of completeness.

Proof. Since f is invertible on I and differentiable at $x_0 \in I$, with $f'(x_0) \neq 0$, for every $\varepsilon > 0$ there is $\delta > 0$ such that, for every $x \in I$, if $0 < |x - x_0| < \delta$, then

$$\left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon.$$

Hence, f^{-1} being continuous at $f(x_0)$, there is $\gamma > 0$ such that, for every $y \in f(I)$, if $0 < |y - f(x_0)| < \gamma$, then $0 < |f^{-1}(y) - x_0| < \delta$ and therefore

$$\left| \frac{f^{-1}(y) - x_0}{y - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon.$$

This means that f^{-1} is differentiable at $f(x_0)$, with $(f^{-1})'(f(x_0)) = 1/f'(x_0)$. ∎

We conclude observing that the continuity of f^{-1} at $f(x_0)$ can be always achieved, provided f is restricted to a sufficiently small neighbourhood U of x_0. Hence, in particular, Theorem A applies to $f_{|U}$.

Theorem B *Let I be a real interval and $f : I \to f(I)$ be invertible and differentiable at $x_0 \in I$, with $f'(x_0) \neq 0$. Then, there exists a neighbourhood U of x_0 such that the inverse of the restriction of f to U $(f_{|U})^{-1} : f(U) \to U$ is continuous at $f(x_0)$.*

Proof. Assume by contradiction that the conclusion is false. Then, for every positive integer n there are a point $x^{(n)} \in I$, with $0 < |x^{(n)} - x_0| < 1/n$, and a sequence $(x_k^{(n)})_k \subset I$ such that $x_k^{(n)} \to x^{(n)}$ and $f(x_k^{(n)}) \to f(x_0)$, as $k \to +\infty$. Hence, for every n we can find a point $\xi_n = x_{k(n)}^{(n)} \in I$ such that $\frac{1}{2}|x^{(n)} - x_0| \leq |\xi_n - x^{(n)}| \leq \frac{3}{2}|x^{(n)} - x_0|$ and $|f(\xi_n) - f(x_0)| \leq \frac{1}{4}|f'(x_0)||x^{(n)} - x_0|$ and therefore

$$\left|\frac{f(\xi_n) - f(x_0)}{\xi_n - x_0}\right| \leq \frac{1}{2}|f'(x_0)|.$$

This yields a contradiction, letting $n \to +\infty$. ∎

In respect of our previous example, Theorem B also displays its intrinsic non-local character.