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First part:

Conservation laws and first order equations

Second part:

The wave equation

References I used:

Partial Differential Equations
L.C. Evans, American Mathematical Society

Partial Differential Equations in Action
S. Salsa, Springer

An Introduction to Partial Differential Equations
Y. Pinchover, J. Rubinstein (Cambridge Univ. Press)

A model example: pollution in a channel

A water stream of constant speed v transports the pollutant along the positive direction of the $x-$axis;

we neglect the depth of the water (floating pollutant)

we neglect the transversal dimension (narrow channel)

$u(x,t)$ = concentration of the pollutant

$$\int_x^{x+\Delta x} u(y,t)\,dy$$

is the mass of pollutant inside the interval $[x, x+\Delta x]$ at time t.

Mass conservation law (no sinks, no sources)

$$\frac{d}{dt}\int_x^{x+\Delta x} u(y,t)\,dy = \int_x^{x+\Delta x} u_t(y,t)\,dy = q(x,t) - q(x+\Delta x,t)$$

$\displaystyle\frac{d}{dt}\int_x^{x+\Delta x} u(s,t)\,ds$ is the growth rate of the mass contained in the interval $[x, x+\Delta x]$

$q(x,t) - q(x+\Delta x,t)$ is the net mass flux into $[x, x+\Delta x]$ through the end-points

$$\frac{1}{\Delta x}\int_x^{x+\Delta x} u_t(s,t)\,ds = \frac{q(x,t)-q(x+\Delta x,t)}{\Delta x}$$

$$u_t(x,t) = -q_x(x,t).$$

In higher dimensione ($N > 1$): e.g. $N = 3$, $\Omega \subset \mathbb{R}^3$ bounded smooth basin,

$$\iiint_\Omega u(x,t)\,dx$$

is the mass of pollutant inside the basin Ω at time t.

Mass conservation law:

$$\frac{d}{dt}\iiint_\Omega u(x,t)\,dx = -\iint_{\partial\Omega} q\cdot\nu\,d\sigma.$$

q flux function
ν outward normal
$\partial\Omega$ boundary of the basin Ω
$\iint_{\dots} \dots d\sigma$ denotes a surface integral

By the divergence theorem

$$\iint_{\partial\Omega} q\cdot\nu\,d\sigma = \iiint_\Omega \operatorname{div} q\,dx$$

hence

$$\iiint_\Omega u_t(x,t)\,dx = -\iint_{\partial\Omega} q\cdot\nu\,d\sigma = -\iiint_\Omega \operatorname{div} q\,dx.$$

and we derive

$$u_t(x,t) = -\operatorname{div} q_x(x,t).$$

Constitutive relation for q:

• convection (flux determined by the water stream only; v = constant stream speed)

$$q(x,t) = v\cdot u(x,t)$$

• diffusion (pollution expands from higher concentration regions to lower ones; Fick's law)

$$q(x,t) = -k\nabla_x u(x,t)$$

In general $q(x,t) = v\cdot u(x,t) - k\nabla_x u(x,t)$.

$$\operatorname{div} q(x,t) = \nabla_x u\cdot v - k\Delta_x u(x,t)$$

$$u_t = k\Delta_x u - v\cdot\nabla_x u \qquad \Big(\text{if } N=1:\ u_t(x,t) = k u_{xx}(x,t) - v u_x(x,t)\Big).$$

Suppose q depends only on convection, i.e. $k = 0$, then we obtain

The transport equation in $\mathbb{R}^N$ with constant coefficients

$x \in \mathbb{R}^N$, $t \in [0, +\infty[$, $u : \mathbb{R}^N \times [0, +\infty[\to \mathbb{R}$, $u = u(x, t)$, $v \in \mathbb{R}^N$ constant.

$$u_t(x, t) + v \cdot \nabla_x u(x, t) = 0$$

i.e., if $w = (v, 1)$, $w \cdot \nabla u = 0$, that is

$$\frac{\partial u}{\partial w} = 0.$$

u is constant along the direction w.

$$\gamma(s) = (x, t) + s(v, 1)$$

is the characteristic line passing through (x, t), along which the value of u is constant.
$u(x + sv, t + s) = u(x, t)$ for all $s \in \mathbb{R}$, $t + s \geq 0$.

The initial value problem

$g : \mathbb{R}^N \to \mathbb{R}$

$$\begin{cases} u_t(x, t) + v \cdot \nabla_x u(x, t) = 0 & \text{in } \mathbb{R}^N \times [0, +\infty[, \\ u(x, 0) = g(x) & \text{on } \Gamma = \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

$$u(x, t) = g(x - tv)$$

Travelling wave moving with velocity v.

If the initial datum is $u(x, t_0) = g(x)$ we have $u(x, t) =?$

$$u(x, t) = g(x + (t_0 - t)v).$$

The non-homogeneous problem (distributed source)

$$\begin{cases} u_t(x, t) + v \cdot \nabla_x u(x, t) = f(x, t) & \text{in } \mathbb{R}^N \times [0, +\infty[, \\ u(x, 0) = g(x) & \text{on } \Gamma = \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

$$z(s) = u(x + sv, t + s)$$

$$\frac{dz}{ds} = v\nabla_x u(x + sv, t + s) + \frac{\partial u}{\partial t}(x + sv, t + s) = f(x + sv, t + s)$$

$$u(x, t) = g(x - tv) + \int_0^t f(x - (t - \eta)v, \eta)\, d\eta.$$

Observe that

$$u(x, t) = g(x - tv)$$

is a solution of the homogeneous problem

$$\begin{cases} u_t(x, t) + v \cdot \nabla_x u(x, t) & \text{in } \mathbb{R}^N \times [0, +\infty[, \\ u(x, 0) = g(x) & \text{on } \Gamma = \mathbb{R}^N \times \{t = 0\}, \end{cases}$$

while, for each s,

$$w(x,t) = f(x-(t-s)v, s)$$

is a solution of the problem

$$\begin{cases} w_t(x,t) + v\cdot\nabla_x w(x,t) = 0 & \text{in } \mathbb{R}^N \times [0,+\infty[, \\ w(x,s) = f(x,s) & \text{on } \Gamma = \mathbb{R}^N \times \{t=0\}. \end{cases}$$

Duhamel's Principle.

For all $s>0$ let $w(\cdot,\cdot;s)$ be a solution of

$$\begin{cases} w_t + v\cdot\nabla_x w = 0 & \text{in } \mathbb{R}^N \times [0,+\infty[, \\ w(x,s;s) = f(x,s) & \text{on } \Gamma = \mathbb{R}^N \times \{t=0\}. \end{cases}$$

Then $u(x,t) = \int_0^t w(x,t;s)\,ds$ is a solution of

$$\begin{cases} u_t(x,t) + v\cdot\nabla_x u(x,t) = f(x,t) & \text{in } \mathbb{R}^N \times [0,+\infty[, \\ u(x,0) = 0 & \text{on } \Gamma = \mathbb{R}^N \times \{t=0\}. \end{cases}$$

A problem is well-posed (according to J. Hadamard) if

1. The problem has a solution.
2. The solution is unique.
3. The solution is stable (a small change in the equation and in the side conditions gives rise to a small change in the solutions)

Theorem

Let $g \in C^1(\mathbb{R}^N)$, $f \in C^1(\mathbb{R}^N \times [0,+\infty[)$. Then, problem

$$\begin{cases} u_t(x,t) + v\cdot\nabla_x u(x,t) = f(x,t) & \text{in } \mathbb{R}^N \times [0,+\infty[, \\ u(x,0) = g(x) & \text{on } \Gamma = \mathbb{R}^N \times \{t=0\}, \end{cases}$$

has a unique solution. Moreover, it is stable on finite-time intervals, i.e., for all $T>0$, small changes of $f_{|\mathbb{R}^N\times[0,T]}$ in $\|\cdot\|_{L^\infty(\mathbb{R}^N\times[0,T])}$ norm and of g in $\|\cdot\|_{L^\infty(\mathbb{R}^N)}$ norm yield small changes of the solutions in $\|\cdot\|_{L^\infty(\mathbb{R}^N\times[0,T])}$ norm.

The problem with decay (exercise)

Due to biological decomposition the pollutant decays at the rate $-\gamma u(x,t)$, $\gamma>0$;

$$\begin{cases} u_t(x,t) + v\cdot\nabla_x u(x,t) + \gamma u(x,t) = f(x,t) & \text{in } \mathbb{R}^N \times [0,+\infty[, \\ u(x,0) = g(x) & \text{on } \Gamma = \mathbb{R}^N \times \{t=0\}. \end{cases}$$

Multiply the equation by $e^{\gamma t}$

$$w(x,t) = e^{\gamma t}u(x,t)$$

$$u(x,t) = e^{-\gamma t}g(x-tv) + e^{-\gamma t}\int_0^t e^{\gamma\eta} f(x+(\eta-t)v,\eta)\,d\eta.$$

(if $f=0$ damped travelling wave)

An example with a discontinuity

A source of pollutant at $x = 0$ starts working at time $t = 0$.

$$\text{Heaviside function } H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

$$\begin{cases} u_t(x,t) + v \cdot \nabla_x u(x,t) = 0 & (x,t) \in [0,+\infty[\times\mathbb{R}, \\ u(0,t) = \beta H(t) & t \in \mathbb{R} \\ u(x,0) = 0 & x \in [0,+\infty[. \end{cases}$$

Here we have both a boundary condition and a initial condition.

$$u(x,t) = \beta H(vt - x).$$

The jump discontinuity in $(0,0)$ is transported along the characterisctic $x = vt$.

Compare with the heat equation. In that case the solution is smooth even if the initial datum is discontinuous.

Inflow characteristics: the characteristics carry the information from the boundary to the interior of the domain.

Outflow characteristics: no data have to be assigned.

Exercise

Suppose, for $i = 1, 2$, u_i is the solution of the problem

$$\begin{cases} u_t(x,t) + v u_x(x,t) = 0 & \text{in }]0,R[\times]0,+\infty[, \\ u(0,t) = f_i(t) & t > 0 \\ u(x,0) = g_i(x) & \text{in }]0,R[. \end{cases}$$

Prove the least square stability formula

$$\int_0^R (u_1(x,t) - u_2(x,t))^2\, dx \leq \int_0^R (g_1(x) - g_2(x))^2\, dx + v \int_0^t (f_1(s) - f_2(s))^2\, ds.$$

The method of characteristics

$F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$,

$$\begin{cases} F(\nabla u, u, x) = 0 & \text{in } U \subset \mathbb{R}^N, \\ u = g & \text{on } \Gamma \subseteq \partial U. \end{cases}$$

To convert the PDE into an appropriate system of ODEs.

A quasilinear problem

$$\begin{cases} a(x,u) \cdot \nabla u + c(x,u) = 0 & \text{in } U \subset \mathbb{R}^N, \\ u(x) = g(x) & \text{on } \Gamma \subseteq \partial U. \end{cases}$$

Theorem

Let $U \subset \mathbb{R}^N$ be an open set, $u \in C^1(U)$ a solution of the equation

$$a(x,u) \cdot \nabla u + c(x,u) = 0.$$

Set $z(s) = u(x(s))$ where $x(s)$ is a solution of the system

$$x'(s) = a(x(s), z(s)).$$

Then $z(s)$ solves the ODE

$$z'(s) = -c(x(s), z(s)),$$

for those s such that $x(s) \in U$.

Example: $N = 2$.

$$a_1(x_1, x_2, u)u_{x_1} + a_2(x_1, x_2, u)u_{x_2} + c(x_1, x_2, u) = 0.$$

$a_1, a_2, c : \mathbb{R}^3 \to \mathbb{R}$.

Suppose $u = u(x_1, x_2)$ is a solution.

Suppose we know the solution u on the curve Γ. We want to span the graph of u starting from Γ.

Parametrize a curve in $\mathbb{R}^3$ by $(x_1(s), x_2(s), z(s))$ with $z(s) = u(x_1(s), x_2(s))$.

Then

$$z'(s) = \frac{d}{ds}u(x_1(s), x_2(s)) = u_{x_1}(x_1(s), x_2(s))x_1'(s) + u_{x_2}(x_1(s), x_2(s))x_2'(s).$$

If we set

$$\begin{cases} x_1'(s) = a_1(x_1, x_2, z) \\ x_2'(s) = a_2(x_1, x_2, z) \end{cases}$$

then

$$z' = -c(x_1, x_2, z).$$

By solving the ODE system we obtain the value of the solution u along the characteristic. Imposing the initial conditions we hope to recover the whole solution.

Example: the transport equation.

$v \in \mathbb{R}$, $f : \mathbb{R} \times]0, +\infty[\to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$.

$$\begin{cases} v\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = f(x_1, x_2) & \text{in } \mathbb{R} \times]0, +\infty[, \\ u(x_1, 0) = g(x_1) & \text{on } \Gamma = \{(x_1, 0) \in \mathbb{R}^2 : x_1 \in \mathbb{R}\}. \end{cases}$$

Characteristic equations:

$$\begin{cases} x_1'(s) = v \\ x_2'(s) = 1 \\ z'(s) = f(x_1, x_2) \end{cases}$$

$x_1 = vs + x_1^0$, $x_2 = s$. Fix $(x_1, x_2) \in U$ and find the characteristic passing through (x_1, x_2): invert to find $x_1^0 = x_1 - vx_2$ to obtain $z(s) - z^0 = \int_0^s f\big(x_1(\xi), x_2(\xi)\big)\, d\xi$; i.e.

$$u(x,t) = g(x - vt) + \int_0^t f(v\xi + x - vt, \xi)\, d\xi.$$

Example: a linear problem

$$\begin{cases} x_1 \frac{\partial u}{\partial x_2} - x_2 \frac{\partial u}{\partial x_1} = u \text{ in } U =]0,+\infty[\, \times\,]0,+\infty[, \\ u(x_1,0) = g(x_1) \qquad \text{on } \Gamma = \{(x_1,0) \in \mathbb{R}^2 : x_1 > 0\}, \end{cases}$$

where $g :]0,+\infty[\to \mathbb{R}$.

$$\begin{cases} x_1'(s) = -x_2(s) \\ x_2'(s) = x_1(s) \\ z'(s) = z(s) \end{cases}$$

$z(s) = z^0 e^s$; $z^0 = z(0) = u(x_1^0, 0) = g(x_1^0)$

$\big(x_1(s), x_2(s)\big) = \big(x_1^0 \cos s, x_1^0 \sin s\big)$

Fix $(x_1, x_2) \in U$ and find the characteristic passing through (x_1, x_2).
We invert the system

$$\begin{cases} x_1 = x_1^0 \cos s \\ x_2 = x_1^0 \sin s \end{cases}$$

to obtain

$$x_1^0 = \sqrt{x_1^2 + x_2^2} \qquad \text{and} \qquad s = \operatorname{atan}\left(\tfrac{x_2}{x_1}\right).$$

Therefore

$$u(x_1, x_2) = g\Big(\sqrt{x_1^2 + x_2^2}\Big) \exp\big(\operatorname{atan}(x_2/x_1)\big).$$

Example: a semilinear problem

$$\begin{cases} \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = u^2 \text{ in } U = \mathbb{R} \times]0,+\infty[, \\ u(x_1,0) = g(x_1) \text{ on } \Gamma = \{(x_1,0) \in \mathbb{R}^2\}, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$.

$$u(x_1, x_2) = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}.$$

The solution is defined only locally!

The initial value problem

$$\begin{cases} a(x,u) \cdot \nabla u + c(x,u) = 0 \text{ in } U, \\ u(x) = g(x) \qquad \text{on } \Gamma, \end{cases}$$

$N = 2$

Γ parametrized by $\gamma(t) = (y_1(t), y_2(t))$, $t \in I$ interval, $\gamma(0) = (y_1(0), y_2(0)) = (y_1^0, y_2^0)$.

$$\begin{cases} a_1(x_1,x_2,u) \frac{\partial u}{\partial x_1} + a_2(x_1,x_2,u) \frac{\partial u}{\partial x_2} + c(x_1,x_2,u) = 0 \text{ in } U, \\ u(y_1(t), y_2(t)) = g(y_1(t), y_2(t)) \qquad t \in I, \end{cases}$$

$$\begin{cases} x_1'(s) = a_1(x_1(s), x_2(s), z(s)) \\ x_2'(s) = a_2(x_1(s), x_2(s), z(s)) \\ z'(s) = -c(x_1(s), x_2(s), z(s)) \\ x_1(0) = y_1^0, \quad x_2(0) = y_2^0, \quad z(0) = z^0 = g(y_1^0, y_2^0). \end{cases} \qquad \text{characteristic equations}$$

$$\begin{cases} x_1'(s,t) = a_1(x_1(s,t), x_2(s,t), z(s,t)) \\ x_2'(s,t) = a_2(x_1(s,t), x_2(s,t), z(s,t)) \\ z'(s,t) = -c(x_1(s,t), x_2(s,t), z(s,t)) \\ x_1(0,t) = y_1(t), \quad x_2(0,t) = y_2(t), \quad z(0,t) = g(y_1(t), y_2(t)). \end{cases}$$

Inverse function theorem

Assume $\psi : U \subset \mathbb{R}^N \to \mathbb{R}^N$, $\psi \in C^1(U;\mathbb{R}^N)$, $x^0 \in U$.

Assume $\det J\psi(x^0) \neq 0$, then ψ is a local C^1-diffeomorphism, i.e. there exist neighbourhoods $U_1 \subseteq U$ of x^0, V_1 of $\psi(x^0)$, and a function $\phi \in C^1(V_1, U_1)$ which is the inverse of $\psi_{|U_1}$.

$$\psi(s,t) = (x_1(s,t), x_2(s,t)) \qquad \phi(x_1,x_2) = (s(x_1,x_2), t(x_1,x_2))$$

$$\frac{\partial x_1}{\partial s}(0,0) = a_1(y_1^0, y_2^0, z^0) \qquad \frac{\partial x_2}{\partial s}(0,0) = a_2(y_1^0, y_2^0, z^0)$$

$$\frac{\partial x_1}{\partial t}(0,0) = y_1'(0) \qquad \frac{\partial x_2}{\partial t}(0,0) = y_2'(0)$$

Transversality condition:

$$a_1(y_1^0, y_2^0, z^0) y_2'(0) - a_2(y_1^0, y_2^0, z^0) y_1'(0) \neq 0$$

Call ν the unit normal to Γ:

$$\nu(y_1^0, y_2^0) = \frac{1}{\sqrt{y_1'(0)^2 + y_2'(0)^2}} (y_2'(0), -y_1'(0))$$

$$a(y^0, z_0) \cdot \nu(y^0) \neq 0.$$

Theorem (Local existence and uniqueness)

Suppose $U \subset \mathbb{R}^N$, $I \subset \mathbb{R}$ interval,
$\gamma \in C^1(I;\mathbb{R}^N)$, $\Gamma = \gamma(I)$, $\Gamma \subseteq \partial U$,
$g : \Gamma \to \mathbb{R}$, $g \circ \gamma \in C^1(I;\mathbb{R})$,
$y^0 = \gamma(0)$, $J \subset \mathbb{R}$ is a neighbourhood of $g(y^0)$, $a, c \in C^1(U \times J)$.

Assume the transversality condition holds in a neighbourhood W of y^0 in Γ, i.e. for all $y \in W$

$$a(y, g(y)) \cdot \nu(y) \neq 0.$$

Then, there exists a neighbourhood V of y^0 in $\mathbb{R}^N$ and a unique function $u \in C^1(V;\mathbb{R})$ which solves

$$\begin{cases} a(x,u) \cdot \nabla u + c(x,u) = 0 & \text{in } V, \\ u(x) = g(x) & \text{on } \Gamma \cap V. \end{cases}$$

If for some neighbourhood W of y^0 in Γ the transversality condition is not satisfied for all $y \in W$, then either the problem has no C^1 solutions or it has infinitely many solutions.

What happens if the transversality condition is not satisfied?

$$\begin{cases} a_1 u_{x_1} + a_2 u_{x_2} = -c, \\ u\big(y_1(t), y_2(t)\big) = g\big(y_1(t), y_2(t)\big). \end{cases}$$

Assume u is a solution. Set $h(t) = u\big(y_1(t), y_2(t)\big)$. Then the vector $\nabla u(y_1^0, y_2^0)$ solves the algebraic system

$$\begin{cases} \nabla u(y_1^0, y_2^0) \cdot \big(a_1(y_1^0, y_2^0, u(y_1^0, y_2^0)), a_2(y_1^0, y_2^0, u(y_1^0, y_2^0))\big) = -c\big(y_1^0, y_2^0, u(y_1^0, y_2^0)\big), \\ \nabla u(y_1^0, y_2^0) \cdot (y_1'(0), y_2'(0) = h'(0). \end{cases}$$

By Rouché-Capelli Theorem the vectors $(a_1, a_2, -c)$ and $(y_1'(0), y_2'(0), h'(0))$ must be parallel.

In this case a necessary condition to get a solution is that the curve $\gamma(t)$ must be parallel to the characteristic curve at (y_1^0, y_2^0, z^0).

Example: non-homogeneous Burgers equation

$$\begin{cases} u\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 1, \\ u(x_1, 0) = h(x_1) \end{cases}$$

where $h \in C^1(\mathbb{R})$, for example if $h(x) = x$.

$$u(x_1, x_2) = x_2 + \frac{2x_1 - x_2^2}{2 + 2x_2}.$$

Example

$$\begin{cases} u\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 1, \\ u(x_1, x_2) = \frac{x_2}{2} \quad \text{on } \Gamma = \{(\mathrm{t}^2, 2\mathrm{t}) : \mathrm{t} \in \mathbb{R}\}. \end{cases}$$

$$u(x_1, x_2) = 1/2x_2 - 1/2\sqrt{4x_1 - x_2^2} \qquad \text{or} \qquad u(x_1, x_2) = 1/2x_2 + 1/2\sqrt{4x_1 - x_2^2}$$

non-regular solutions.

Example

$$\begin{cases} \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 1, \\ u(t, t) = t \qquad t \in \mathbb{R}. \end{cases}$$

$$u(x_1, x_2) = x_2 + f(x_1 - x_2)$$

for any f such that $f(0) = 0$.
(infinitely many solutions)

Exercise

$$\begin{cases} x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} = 4u, \ (x_1, x_2) \in \mathbb{R}^2 \\ u(x_1, x_2) = 1 \qquad x_1^2 + x_2^2 = 1. \end{cases}$$

Scalar Conservation Laws

We consider equations of the form

$$u_t + \operatorname{div}_x q\big(u(x,t)\big) = 0, \qquad x \in \mathbb{R}^N, t > 0.$$

(If $N = 1$ we write $u_t + q(u)_x = 0, \qquad x \in \mathbb{R}, t > 0.$)

Denote by $u(x,t)$ the concentration of a physical quantity Q inside a set Ω at time t. The amount of Q inside the set Ω at time t is given by (assume e.g. $N = 3$)

$$\iiint_\Omega u(x,t)\,dx$$

The conservation law says that

$$\frac{d}{dt}\iiint_\Omega u(x,t)\,dx = -\iint_{\partial\Omega} q\cdot\nu\,d\sigma,$$

where $\frac{d}{dt}\iiint_\Omega u(x,t)\,dx$ is the rate of change of Q in Ω, and $-\iint_{\partial\Omega} q\cdot\nu\,d\sigma$ is the net flux through the boundary of Ω.

(If $N = 1$ and $\Omega = [x_1, x_2]$ we have $\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = q(u(x_1,t)) - q(u(x_2,t))$.)

By the divergence theorem

$$\iint_{\partial\Omega} q\cdot\nu\,d\sigma = \iiint_\Omega \operatorname{div}_x q\,dx$$

$$\text{hence} \qquad \iiint_\Omega \left(u_t(x,t) + \operatorname{div}_x q\right)\,dx = 0,$$

and we derive

$$u_t + \operatorname{div}_x q\big(u(x,t)\big) = 0.$$

(If $N = 1$ we write $u_t + q(u)_x = 0$.)

Let us consider the following problem:

$$\begin{cases} u_t + q'(u)u_x = 0, \\ u(x,0) = g(x) \qquad x \in \mathbb{R}. \end{cases}$$

We shall use the method of the characteritics for the equation $a_1 u_{x_1} + a_2 u_{x_2} = c$ with $x_1 = x$, $x_2 = t$, $a_1 = q'(u)$, $a_2 = 1$.

The characteristic equations are

$$\begin{cases} x'(s) = q'(z(s)) \\ t'(s) = 1 \\ z' = 0 \end{cases}$$

The characteristics are straight lines, here $s = t$ hence we can write the cartesian equation of the lines instead of the parametric equation:

$$x(t) = q'(g(x^0))t + x^0, \qquad u(x,t) = g(x^0).$$

The transversality condition is always satisfied, indeed $a(y, g(y)) = (q, 1), \quad \nu(y) = (0, 1)$.

Notice however that the characteristics may possibly intersect!

How can we write the solution? Recall the solution of the transport equation:

$$u(x,t) = g(x^0) = g(x - tv).$$

Now we still have $u(x,t) = g(x^0)$. Since $x(t) - q'(g(x^0))t = x^0$ we can write

$$u(x,t) = g\big(x(t) - q'(g(x^0))t\big)$$

We obtain an implicit formula for the solution: $u = g(x - tq'(u))$

Implicit Function Theorem:
Consider the level set

$$F(x,t,z) = 0.$$

Suppose (x^0, t^0, z^0) belongs to the level set, i.e. $F(x^0, t^0, z^0) = 0$.
Then, if $\frac{\partial F}{\partial z}(x^0, t^0, z^0) \neq 0$, there exists locally a funcion $u = u(x,t)$ such that

$$F\big(x, t, u(x,t)\big) = 0 \quad \text{for all} \quad (x,t).$$

Moreover

$$\frac{\partial u}{\partial x}(x,t) = -\frac{\frac{\partial F\left(x,t,u(x,t)\right)}{\partial x}}{\frac{\partial F\left(x,t,u(x,t)\right)}{\partial z}}$$

and

$$\frac{\partial u}{\partial t}(x,t) = -\frac{\frac{\partial F\left(x,t,u(x,t)\right)}{\partial t}}{\frac{\partial F\left(x,t,u(x,t)\right)}{\partial z}}$$

Here we have

$$u - g(x - tq'(u)) = 0$$

Therefore it is possible to write $u = u(x,t)$ if

$$1 + tq''(u)g'(x - tq'(u)) \neq 0.$$

What if $q''(u) > 0$ and $g' < 0$?
Smooth solutions may fail to exist.
"However, the fluid described by the equation keeps flowing unaware of our mathematical troubles..."
What kind of solutions can we expect?

Example: Burgers equation (shockwave)

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \text{ in } \mathbb{R} \times]0, +\infty[\\ u(x,0) = g(x) \quad x \in \mathbb{R} \end{cases}$$

where

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1 - x & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$u(x,t)=\begin{cases}1 & \text{if } x\le t,\ 0\le t\le 1 \text{ or } t>1,\ x<\frac{1}{2}t+\frac{1}{2}\\ \frac{1-x}{1-t} & \text{if } 0\le t\le x\le 1\\ 0 & \text{if } x\ge 1, 0\le t\le 1 \text{ or } t>1, x>\frac{1}{2}t+\frac{1}{2}\end{cases}$$

Mild solution

An integrable function $u:\mathbb{R}\times]0,+\infty[\to\mathbb{R}$ is a mild solution of

$$\begin{cases} u_t+q(u)_x=0,\\ u(x,0)=g(x) \quad x\in\mathbb{R}.\end{cases}$$

if $u(x,0)=g(x)$ for all $x\in\mathbb{R}$ and, for all $x_1<x_2$,

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = q(u(x_1,t))-q(u(x_2,t)).$$

Notice that mild solutions may be discontinuous.

Weak solution

A function $u\in L^\infty(\mathbb{R}\times]0,+\infty[)$ is a weak solution of

$$\begin{cases} u_t+q(u)_x=0,\\ u(x,0)=g(x) \quad x\in\mathbb{R}.\end{cases}$$

if, for all test functions $\phi\in C^\infty(\mathbb{R}\times[0,+\infty[)$, with compact support, we have

$$\int_0^{+\infty}\left(\int_{-\infty}^{+\infty} u(x,t)\phi_t(x,t)+q(u(x,t))\phi_x(x,t)\,dx\right)dt+\int_{-\infty}^{+\infty} g(x)\phi(x,0)\,dx=0.$$

Observation: A classical solution is a mild solution, a mild solution is a weak solution. A function $u\in C^1(\mathbb{R}\times]0,+\infty[)$ is a classical solution of the problem if and only if u is a weak solution of the problem.

What information about the u is hidden in the formula for a weak solution if u is, for example, singular along a shock curve (jump discontinuity)?

The Rankine-Hugoniot condition

We suppose now that u is a weak solution which is C^1 in some open region $V\subset\mathbb{R}\times]0,+\infty[$ except on a smooth curve C which separates V into two parts: V^l and V^r.

Then the speed of the shock wave is the quotient of the flux jump over the density jump:

$$q(u^+)-q(u^-)=(u^+-u^-)\varphi'(t),$$

where $\gamma(t)=(\varphi(t),t)$, γ being a parametrization of C.

Example: Burgers equation again

$$\begin{cases} u_t+\left(\frac{u^2}{2}\right)_x=0 \text{ in } \mathbb{R}\times]0,+\infty[\\ u(x,0)=g(x) \quad x\in\mathbb{R}\end{cases} \qquad g(x)=\begin{cases}1 & \text{if } x\le 0\\ 1-x & \text{if } 0<x\le 1\\ 0 & \text{if } x>1\end{cases}$$

$$u(x,t)=\begin{cases}1 & \text{if } x\le t,\ 0\le t\le 1 \text{ or } t>1,\ x<\frac{1}{2}t+\frac{1}{2}\\ \frac{1-x}{1-t} & \text{if } 0\le t\le x\le 1\\ 0 & \text{if } x\ge 1, 0\le t\le 1 \text{ or } t>1, x>\frac{1}{2}t+\frac{1}{2}\end{cases}$$

Example: rarefaction wave

$$\begin{cases}u_t+\left(\frac{u^2}{2}\right)_x=0 \text{ in } \mathbb{R}\times]0,+\infty[\\ u(x,0)=g(x)\quad x\in\mathbb{R}\end{cases}\qquad g(x)=\begin{cases}0 \text{ if } x\le 0\\ 1 \text{ if } x>0\end{cases}$$

What is u in the wedge $x>0$, $t\ge x$?
We set

$$u(x,t)=\begin{cases}0 \text{ if } x<\frac{t}{2}\\ 1 \text{ if } x>\frac{t}{2}\end{cases}$$

u is a shock solution and the Rankine-Hugoniot condition is satisfied.
Is this an acceptable solution?

We expect a shock in presence of a compression wave, not in presence of an expansion wave.

Looking for a second solution:
Regularised problem:

$$\begin{cases}u_t+\left(\frac{u^2}{2}\right)_x=0 \text{ in } \mathbb{R}\times]0,+\infty[\\ u(x,0)=g_\varepsilon(x)\quad x\in\mathbb{R}\end{cases}$$

$$g_\varepsilon(x)=\begin{cases}0 & \text{if } x\le 0\\ \frac{1}{\varepsilon}x & \text{if } 0<x<\varepsilon\\ 1 & \text{if } x\ge\varepsilon\end{cases}$$

$$u(x,t)=\begin{cases}0 & \text{if } x<0\\ \frac{x}{t+\varepsilon} & \text{if } 0<x<t+\varepsilon\\ 1 & \text{if } x>t+\varepsilon\end{cases}$$

When $\varepsilon\longrightarrow 0$:

$$u(x,t)=\begin{cases}0 & \text{if } x<0\\ \frac{x}{t} & \text{if } 0<x<t\\ 1 & \text{if } x>t>0\end{cases}$$

More in general, assuming q' is invertible, if g has a jump at $x=a$, in the wedge we can define

$$u(x,t)=(q')^{-1}(\frac{x-a}{t}).$$

How can we chose the "right" solution?

The Entropy Condition

We require an "entropy condition"

$$q'(u^-)>\sigma>q'(u^+)$$

Characteristics must enter the shock curve and are not allowed to emanate from it.

Assume $q'' > 0$. A weak solution is said to be an entropy solution if there exists $C \geq 0$ such that, for every $x, \Delta x \in \mathbb{R}$, $\Delta x > 0$, and every $t > 0$, we have

$$u(x + \Delta x) - u(x,t) \leq \frac{C}{t}\Delta x.$$

Assume $q'' \geq K > 0$ and $g' > 0$. If u is smooth, then u is an entropy solution.

Assume u is an entropy solution. Then, for all fixed $t > 0$ the function

$$\psi_{[t]}(x) := u(x,t) - \frac{C}{t}x$$

is decreasing.

Assume $q'' \geq K > 0$, u is an entropy solution presenting a shock curve $\varphi(t)$. Then the slope of the shock curve is smaller than the slope of the left characteristics and larger than the slope of the right characteristics:

$$q'(u^+) < \varphi'(t) < q'(u^-).$$

Lax-Oleinik theorem.

Assume $q \in C^2(\mathbb{R})$ is strictly convex (or strictly concave) and $g \in L^\infty(\mathbb{R})$. Then problem

$$\begin{cases} u_t + q(u)_x = 0 \; x \in \mathbb{R},\, t > 0, \\ u(x,0) = g(x) \;\; x \in \mathbb{R}. \end{cases}$$

has a unique entropy solution.

Furthermore, the solution u is stable and depends continuously on the initial data, in the following sense: there exists a constant A such that, if $h \in L^\infty(\mathbb{R})$ and v is the entropy solution for the problem with initial datum h, then, for every $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, $t > 0$,

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)|\, dx \leq \int_{x_1 - At}^{x_2 + At} |g(x) - h(x)|\, dx.$$

(For uniqueness the convexity or concavity of q is not necessary, but the entropy must be suitably defined.)

The Riemann problem

Assume $q \in C^2(\mathbb{R})$ and $q'' \geq C > 0$. Set

$$g(x) = \begin{cases} u^- \text{ if } x < 0 \\ u^+ \text{ if } x > 0 \end{cases}$$

$u^+ \neq u^-$.

Then, the unique entropy solution of the problem

$$\begin{cases} u_t + q(u)_x = 0 \; x \in \mathbb{R}\, t > 0, \\ u(x,0) = g(x) \;\; x \in \mathbb{R}. \end{cases}$$

is

(i) if $u^+ > u^-$,

$$u(x,t) = \begin{cases} u^- \text{ if } x < \sigma t \\ u^+ \text{ if } x > \sigma t \end{cases}$$

where $\sigma = \frac{q(u^+)-q(u^-)}{u^+-u^-}$;
(ii) if $u^- < u^+$,

$$u(x,t) = \begin{cases} u^- & \text{if } x < q'(u^-)t \\ (q')^{-1}\left(\frac{x}{t}\right) & \text{if } q'(u^-)t < x < q'(u^+)t \\ u^+ & \text{if } x > q'(u^+)t \end{cases}$$

A model example: the traffic flow

Traffic on a highway along the positive direction of the $x-$axis;
no overtaking allowed
no exits or entrances
$u(x,t) =$ density of cars in the point x at the time t.
$v(x,t) =$ average speed.
q flux; $q = vu$
The average speed depends on the density alone: $v = v(u)$.

$$v'(u) = \frac{dv}{du} \le 0.$$

Conservation law

$$u_t + q(u)_x = 0$$

Constitutive relation for v:

$$v(u) = v_m \left(1 - \frac{u}{u_m}\right),$$

$v_m =$ maximal velocity,
$u_m =$ maximal concentration (bumper to bumper).

$$u_t + v_m \left(1 - \frac{2u}{u_m}\right) u_x = 0.$$

$$\begin{cases} u_t + v_m \left(1 - \frac{2u}{u_m}\right) u_x = 0 \\ u(x,0) = g(x) \end{cases} \qquad g(x) = \begin{cases} \frac{1}{8}u_m & \text{if } x < 0 \\ u_m & \text{if } x > 0 \end{cases}$$

Traffic jam ahead ($v = 0$ if $x > 0$).
On the left $v = \frac{7}{8}v_m$.

$$u(x,t) = \begin{cases} \frac{1}{8}u_m & \text{if } x < -\frac{1}{8}v_m t \\ u_m & \text{if } x > -\frac{1}{8}v_m t \end{cases}$$

Shock line: $\varphi(t) = -\frac{1}{8}v_m t$.
The shock is revealed by the breaking lights of the cars, slowing down because of the traffic jam ahead.

Example: the green traffic-light

$$\begin{cases} u_t + v_m \left(1 - \frac{2u}{u_m}\right) u_x = 0 \\ u(x,0) = g(x) \end{cases} \qquad g(x) = \begin{cases} u_m \text{ if } x < 0 \\ 0 \quad \text{if } x > 0 \end{cases}$$

$$u(x,t) = \begin{cases} u_m & \text{if } x \leq -v_m t \\ \frac{1}{2} u_m \left(1 - \frac{1}{v_m} \frac{x}{t}\right) & \text{if } -v_m t < x < v_m t \\ 0 & \text{if } x \geq v_m t \end{cases}$$

Example

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \text{ in } \mathbb{R} \times]0, +\infty[\\ u(x,0) = g(x) \quad x \in \mathbb{R} \end{cases} \qquad g(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } 0 \leq x \leq 1 \\ 0 \text{ if } x > 1 \end{cases}$$

Exercise

Determine a weak solution o the Lighthill-Whitham-Richard model for traffic flow

$$\begin{cases} u_t + (v(u)u)_x = 0 \text{ in } \mathbb{R} \times]0, +\infty[\\ u(x,0) = g(x) \qquad x \in \mathbb{R} \end{cases}$$

with velocity

$$v(u) = 2 - \frac{u}{2}$$

and initial density

$$g(x) = \begin{cases} 2 & \text{if } x < 0 \\ x + 2 & \text{if } 0 \leq x < 1 \\ 3 & \text{if } x \geq 1 \end{cases}$$

Describe the trajectory of a car initially in position $x = -2$.

Exercise

Discuss existence and uniqueness and determine a weak solution of the scalar conservation law

$$\begin{cases} u_t + 4uu_x = 0 \text{ in } \mathbb{R} \times]0, +\infty[\\ u(x,0) = g(x) \ x \in \mathbb{R} \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } 0 \leq x < 1 \\ -1 & \text{if } x \geq 1 \end{cases}$$

The wave equation

The vibrating string

We consider small transversal vibrations of a tightly stretched perfectly flexible horizontal string (the stress at any point can be modelled by a tangential force, the tension)

we neglect friction

vibrations have small amplitude

we assume there is only vertical displacement, and this depends on the position x and time t: $u = u(x,t)$

Consider a string element at a fixed time t, represented by the curve $\gamma(x) = \big(x, u(x,t)\big)$.

The forces acting to the string= external vertical forces f (gravity, loads) + internal forces $\bar{T}$(tension)

The horizontal forces have to balance:

$$\bar{T}(x_2)|_{\text{horizontal}} = \bar{T}(x_1)|_{\text{horizontal}}$$

$$\tau(x_2,t)\cos(\alpha(x_2,t)) - \tau(x_1,t)\cos(\alpha(x_1,t)) = 0$$

$\tau = |\bar{T}|$ magnitude
$\alpha(x,t)$ angle between the x-axis and the tangent of γ at x

$$\frac{\partial}{\partial x}\left(\tau(x,t)\cos(\alpha(x,t))\right) = 0$$

$$\tau(x,t)\cos(\alpha(x,t)) = \tau_0(t)$$

Vertical tension:

$$\tau(x,t)\sin(\alpha(x,t)) = \tau_0(,t)\tan(\alpha(x,t)) = \tau_0(t)u_x(x,t)$$

Conservation of mass:
$\rho_0 = \rho_0(x) =$ linear density of the string at rest
$\rho(x,t) =$ linear density of the string at time t

$$\rho_0(x)\Delta x = \rho(x,t)\Delta s$$

Newton law:

$$\int_\gamma u_{tt}(s,t)\rho(s,t)\,ds = \int_{x_1}^{x_2} u_{tt}(x,t)\rho_0(x)\,dx = \int_{x_1}^{x_2} f(x,t)\rho_0(x)\,dx + \tau_0(t)\left(u_x(x_2,t) - u_x(x_1,t)\right)$$

$$u_{tt}(x,t) - \frac{\tau_0(t)}{\rho_0(x)}u_{xx}(x,t) = f(x,t) \qquad \text{(J. d'Alembert 1752)}$$

Since the string is perfectly elastic τ_0 is constant; since the string is homogeneous ρ_0 is constant.

Set

$$c^2 = \frac{\tau_0}{\rho_0}$$

The homogeneous equation.

$f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0$$

Set $v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$, then solve the linear transport equation $v_t - cv_x = 0$.
We have $v(x,t) = \varphi(x+ct)$ for some φ.
Solve $u_t + cu_x = \varphi(x+ct)$.

$$u(x,t) = \varphi(x-ct) + \int_0^t \varphi\big(x + (\eta - t)c + c\eta\big)\, d\eta.$$

Observe that $u(x,0) = \psi(x)$ and $u_t(x,0) = \varphi(x) - c\psi'(x)$.

Since $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, we deduce
D'Alembert formula:

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

Theorem The Cauchy problem above has a unique solution, and for all $T > 0$, this is uniformly stable on $\mathbb{R} \times [0,T]$.

Weak solution

Assume $f \in C(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$.
A function $u \in C(\mathbb{R} \times [0,+\infty[)$ is a weak solution of

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & x \in \mathbb{R},\, t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

if, for all test functions $v \in C^2(\mathbb{R} \times [0,+\infty[)$, with compact support, we have

$$\int_0^{+\infty} \left(\int_{-\infty}^{+\infty} u(x,t)\left(v_{tt}(x,t) - c^2 v_{xx}(x,t)\right)\, dx \right) dt$$

$$- \int_{-\infty}^{+\infty} \left(g(x)v(x,0) - f(x)v_t(x,0)\right)\, dx = 0.$$

Observation: the singularities of the solutions of the wave equation are travelling only along characteristics.

Domain of dependence and region of influence

Example (chord of infinite length plucked at the origin)

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

where

$$f(x) = \begin{cases} 0 & \text{if } -\infty < x < -1 \\ x+1 & \text{if } -1 \le x < 0 \\ 1-x & \text{if } 0 \le x < 1 \\ 0 & \text{if } x \ge 1 \end{cases}$$

The non-homogeneous equation

$f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = h(x,t) & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

Theorem The problem is well-posed for $h,\ h_x \in C(\mathbb{R}^2)$, $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$, for each $T > 0$, in $\mathbb{R} \times [0,T]$.

D'Alembert formula:

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi + \frac{1}{2c}\iint_{\Delta(x,t)} h(\xi,\tau)\, d\xi d\tau$$

Here $\Delta(x,t)$ is the characteristic triangle with vertex (x,t).

Observation Let f and g be even (odd, periodic of period P) functions; let, for all $t \ge 0$, $h(\cdot,t)$ be even (odd, periodic of period P). Then, for all $t \ge 0$, the solution $u(\cdot,t)$ is also even (odd, periodic of period P).

The problem on the half line (a reflection method).

$f \in C^2([0,+\infty[)$, $g \in C^1([0,+\infty[)$, $f(0) = f''(0) = g(0) = 0$;

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & 0 < x < +\infty,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 \le x + \infty \\ u_t(x,0) = g(x) & 0 \le x < +\infty \end{cases}$$

Extend f and g as odd functions $\tilde{f}$ and $\tilde{g}$ over $\mathbb{R}$ and consider the problem on $\mathbb{R}$, to obtain

$$u(x,t) = \begin{cases} \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi & \text{if } x > ct \\ \frac{1}{2}\left(f(x+ct) - f(ct-x)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\xi)\, d\xi & \text{if } 0 \le x \le ct \end{cases}$$

Peculiarities of dimensione $N = 1$.

There is no decay of waves.

Once the wave if detected, even if it has a compact support it will never disappear.

Radially symmetric solutions of the wave equation in three dimensions.

$$u_{tt}(x_1, x_2, x_3, t) - c^2 \Delta u(x_1, x_2, x_3, t) = 0$$

Spherical coordinates:

$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $x_1 = r \sin\varphi \cos\theta$, $x_2 = r \sin\varphi \sin\theta$, $x_1 = r\cos\varphi$.

Laplacian in spherical coordinates: (radial part) + (spherical part)

$$\Delta u = \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \varphi^2} + \frac{\cos\varphi}{\sin\varphi}\frac{\partial u}{\partial \varphi}\right)$$

$$u_{tt}(x_1, x_2, x_3, t) - c^2 \Delta u(x_1, x_2, x_3, t) = 0$$

$$\begin{cases} u_{tt} - c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = 0 & 0 < r < +\infty,\ t > 0 \\ u(r,0) = f(r) & 0 \le r < +\infty \\ u_t(r,0) = g(r) & 0 \le r < +\infty \end{cases}$$

$$u(r,t) = \frac{1}{2r}\Big((r+ct)\tilde{f}(r+ct) + (r-ct)\tilde{f}(r-ct)\Big) + \frac{1}{2rc}\int_{r-ct}^{r+ct} \xi \tilde{g}(\xi)\, d\xi$$

In dimension 3 there is a decay of the wave with time at any point.

Examples

$$f(r) = 0,$$

$$g(r) = \begin{cases} 1 \text{ if } 0 \le r \le 1 \\ 0 \text{ if } r > 1 \end{cases}$$

$$\begin{cases} u_{tt} - \Delta u = 0 & 0 \le r < +\infty,\ t \ge 0 \\ u(r,0) = f(r) & 0 \le r < +\infty \\ u_t(r,0) = g(r) & 0 \le r < +\infty \end{cases}$$

$$\begin{cases} u_{tt} - \Delta u = 0 & 0 \le r < +\infty,\ t \ge 0 \\ u(r,0) = g(r) & 0 \le r < +\infty \\ u_t(r,0) = f(r) & 0 \le r < +\infty \end{cases}$$

Spherical means and the general Cauchy problem in $\mathbb{R}^3$.

Spherical mean. $h \in C^1(\mathbb{R}^3)$,

$$M_h(r,x) = \frac{1}{4\pi r^2}\iint_{\partial B(x,r)} h(\sigma)\, d\sigma$$

is the average of h over the sphere $\partial B(x,r)$.

We have

$$\lim_{r\to 0} M_h(r,x) = ?$$

$$\lim_{r\to 0} M_h(r,x) = h(x)$$

$$\frac{\partial}{\partial r} M_h(r,x) = \frac{1}{4\pi r^2} \iiint_{B(x,r)} \Delta h(x)\, dx$$

$$\frac{\partial^2}{\partial r^2} M_h(r,x) = -\frac{1}{2\pi r^3} \iiint_{B(x,r)} \Delta h(x)\, dx + \frac{1}{4\pi r^2} \iint_{\partial B(x,r)} \Delta h(\sigma)\, d\sigma$$

$$\Delta_x M_h(r,x) = \frac{1}{4\pi r^2} \iint_{\partial B(x,r)} \Delta h(\sigma)\, d\sigma$$

Darboux equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) M_h(r,x) = \Delta_x M_h(r,x)$$

Proposition I.

If u is a solution of

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^3,\ t \geq 0 \\ u(x,0) = 0 & x \in \mathbb{R}^3 \\ u_t(x,0) = g(x) & x \in \mathbb{R}^3 \end{cases}$$

then $w = M_u(r,x,t)$ is a solution of

$$\begin{cases} w_{tt} - c^2 \left(\frac{\partial^2 w}{\partial r^2} + \frac{2}{r}\frac{\partial w}{\partial r}\right) = 0 & 0 < r < +\infty,\ t > 0 \\ w(r,0) = 0 & 0 \leq r < +\infty \\ w_t(r,0) = M_g(r,x) & 0 \leq r < +\infty \end{cases}$$

Proposition II.

If u is a solution of

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^3,\ t \geq 0 \\ u(x,0) = 0 & x \in \mathbb{R}^3 \\ u_t(x,0) = g(x) & x \in \mathbb{R}^3 \end{cases}$$

then $v(x,t) := u_t(x,t)$ is a solution of

$$\begin{cases} v_{tt} - c^2 \Delta v = 0 & x \in \mathbb{R}^3,\ t \geq 0 \\ v(x,0) = g(x) & x \in \mathbb{R}^3 \\ v_t(x,0) = 0 & x \in \mathbb{R}^3 \end{cases}$$

Solution: (Kirchhoff's formula)

$$u(x,t) = tM_g(ct,x) + \frac{\partial}{\partial t}\left(tM_f(ct,x)\right)$$

$$u(x,t) = \frac{1}{4\pi c^2 t}\iint_{\partial B(x,ct)} g(\sigma)\,d\sigma + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{\partial B(x,ct)} f(\sigma)\,d\sigma\right)$$

Huygens principle holds.

Theorem Let $f \in C^3(\mathbb{R}^3)$, $h \in C^2(\mathbb{R}^3)$. Then Kirchhoff's formula yields the unique solution $u \in C^2(\mathbb{R}^3 \times [0,+\infty[)$ of the problem

$$\begin{cases} u_{tt} - c^2\Delta u = 0 & x \in \mathbb{R}^3,\ t \geq 0 \\ u(x,0) = f(x) & x \in \mathbb{R}^3 \\ u_t(x,0) = g(x) & x \in \mathbb{R}^3 \end{cases}$$

The problem in $\mathbb{R}^2$ (Hadamard's descent method).

$$\begin{cases} u_{tt} - c^2(u_{x_1x_1} + x_{x_2x_2}) = 0 & (x_1,x_2) \in \mathbb{R}^2,\ t \geq 0 \\ u(x_1,x_2,0) = f(x_1,x_2) & (x_1,x_2) \in \mathbb{R}^2 \\ u_t(x_1,x_2,0) = g(x_1,x_2) & (x_1,x_2) \in \mathbb{R}^2 \end{cases}$$

Poisson's formula:

$$u(x_1,x_2,t) = \frac{1}{2\pi c}\iint_{B(x_1,x_2;ct)} \frac{g(\xi_1,\xi_2)}{\sqrt{c^2t^2 - (x_1-\xi_1)^2 - (x_2-\xi_2)^2}}\,d\xi_1 d\xi_2$$
$$+\frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\iint_{B(x_1,x_2;ct)} \frac{f(\xi_1,\xi_2)}{\sqrt{c^2t^2 - (x_1-\xi_1)^2 - (x_2-\xi_2)^2}}\,d\xi_1 d\xi_2\right)$$

Theorem Let $f \in C^3(\mathbb{R}^2)$, $g \in C^2(\mathbb{R}^2)$. Then Poisson's formula yields the unique solution $u \in C^2(\mathbb{R}^2 \times [0,+\infty[)$ of the problem.

In dimension 2 Huygens principle does not hold. Any perturbation will leave trace for all later times.

The wave equation in a bounded interval (separation of variables)

The Dirichlet problem:

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & x \in [0,L] \\ u_t(x,0) = g(x) & x \in [0,L] \end{cases}$$

$$w''(t) = \lambda c^2 w(t)$$

$$\begin{cases} v''(x) = \lambda v(x) \\ v(0) = 0 \\ v(L) = 0 \end{cases}$$

$$u_k(x,t) = \left(a_k \cos(\frac{\pi k c}{L}t) + b_k \sin(\frac{\pi k c}{L}t)\right)\sin(\frac{\pi k}{L}x)$$

$a_k, b_k \in \mathbb{R}$, $k = 1,2,3,\ldots$.

The Neumann problem: (exercise)

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & 0 < x < L,\ t > 0 \\ u_x(0,t) = u_x(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & x \in [0,L] \\ u_t(x,0) = g(x) & x \in [0,L] \end{cases}$$

where $f'(0) = f'(L) = g'(0) = g'(L) = 0$.

$$u_k(x,t) = \left(a_k \cos(\frac{\pi kc}{L} t) + b_k \sin(\frac{\pi kc}{L} t) \right) \cos(\frac{\pi k}{L} x)$$

$a_k, b_k \in \mathbb{R}$, $k = 1, 2, 3, \ldots$.

Imposing initial conditions:

A formal solution:

$$\sum_{k=1}^{+\infty} \left(\hat{f}_k \cos(\frac{\pi kc}{L} t) + \frac{L}{\pi kc} \hat{g}_k \sin(\frac{\pi kc}{L} t) \right) \sin(\frac{\pi k}{L} x)$$

Energy: $E(t) = \dfrac{1}{2} \displaystyle\int_0^L (w_t^2 + c^2 w_x^2)\, dx$
Energy is conserved $\Rightarrow$ uniqueness.
Uniqueness? Stability?

Exercise

Solve the hyperbolic problem

$$\begin{cases} u_{tt} - 4\, u_{xx} = x, & \text{in }]0,+\infty[\times]0,+\infty[, \\ u(0,t) = 0, & \text{in }]0,+\infty[, \\ u(x,0) = x^4, & \text{in } [0,+\infty[, \\ u_t(x,0) = 0, & \text{in } [0,+\infty[. \end{cases}$$

Exercise

Compute the solution u of the hyperbolic problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R} \\ u(x,y,z,0) = 0 & (x,y,z) \in \mathbb{R}^3 \\ u_t(x,y,z,0) = h(x,y,z) & (x,y,z) \in \mathbb{R}^3 \end{cases}$$

where

$$h(x,y,z) = \begin{cases} 2 \text{ if } x^2 + y^2 + z^2 \leq 1 \\ 0 \text{ if } x^2 + y^2 + z^2 > 1 \end{cases}$$

at the point $P = (2,0,0)$ at the times $t_1 = \frac{1}{2}$, $t_2 = \frac{3}{2}$, $t_3 = 4$