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Some Generalizations of the Bernstein Theorem

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INTRODUCTION

It is well-known that if $u \in C^2(\mathbf{R}^2)$ is a solution of the minimal-surface equation

$$\left(1+u_y^2\right) u_{xx} - 2u_x u_y u_{xy} + \left(1+u_x^2\right) u_{yy} = 0 \quad \text{in} \quad \mathbf{R}^2, \tag{1}$$

then $u(x,y) = ax + by + c$, where $a, b, c \in \mathbf{R}$. This assertion is known as the Bernstein theorem [1]. Its simple proof and generalizations to arbitrary dimensions were given in [2]. Obviously, there is a close relationship between the Bernstein theorem and Liouville's classical theorem; but the function u must be bounded in the latter [3].

In this paper, we try to answer the following natural question: suppose that $u \in C^2\left(\mathbf{R}^2\right)$ is a solution of the differential inequality

$$-\operatorname{div}\left(Du/\sqrt{1+|Du|^2}\right) \geq 0 \quad \text{in} \quad \mathbf{R}^2 \tag{2}$$

(here the left-hand side contains the minimal-surface operator in nonparametric form); what assertion about the function $u(x)$ can be put forth then?

As to radial solutions of problem (2), it is clear that specific results can be obtained only if u is bounded below. Indeed, in this case, if $u(x,y) = u\left(\sqrt{x^2+y^2}\right) \geq C$ for all $(x,y) \in \mathbf{R}^2$, then $u(x,y) \equiv \text{const}$ for all $(x,y) \in \mathbf{R}^2$. This assertion is similar to the fact that an arbitrary superharmonic function bounded below on $\mathbf{R}^2$ is identically equal to a constant. It is well-known that this property follows from Harnack's inequality [3, 4].

In this paper, using an elementary method, we prove assertions like the following: "if $u \in C^1$ is a solution of a differential inequality and is bounded below, then $u = \text{const}$" (see below for rigorous statements). The paper is organized as follows: we formulate and prove the main result in Section 1 and discuss some generalizations in Section 2.

1. MAIN RESULT

Let us state and prove the main result of the reported research. We introduce the class of differential operators to be studied.

Definition 1. A function $\mathscr{A} : \mathbf{R}^N \to \mathbf{R}^N$ given by the formula $A_i(p) = A(|p|)p_i$, $p \in \mathbf{R}^N$, $i = 1, \ldots, N$, where $A \in C([0,\infty); \mathbf{R})$, generates an operator of the type of average curvature if there exists a $C > 0$ such that

$$0 < A(|p|) \leq C \qquad \forall p \in \mathbf{R}^N. \tag{3}$$

If $u \in C^2\left(\mathbf{R}^N; \mathbf{R}\right)$, then the differential operator generated by A is given by the formula

$$\operatorname{div}(A(|Du|)Du).$$

Remark 1. The average-curvature operator occurring in (3) corresponds to the function A given by the formula $\mathscr{A}(p) = A(|p|)p_i$, where $A(|p|) = \left(1+|p|^2\right)^{-1/2}$, $p \in \mathbf{R}^N$, and $|\cdot|$ is a finite-dimensional norm.

The main result of the reported research is given in the following assertion.

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Theorem 1. *Let $u \in C^1(\mathbf{R}^N)$, $N = 1, 2,$ be a weak solution of the problem*

$$-\operatorname{div}(A(|Du|)Du) \geq 0 \quad in \quad \mathbf{R}^N, \tag{4}$$

and let u be bounded below. Then the function u is identically constant everywhere in $\mathbf{R}^N$ $(N = 1, 2)$.

Remark 2. A function $u \in C^1(\mathbf{R}^N)$ is defined as a weak solution of problem (4) if

$$0 \leq \int\limits_{\mathbf{R}^N} A(|Du|)(Du, D\varphi)dx \tag{5}$$

for any nonnegative test function $\varphi \in C_0^1(\mathbf{R}^N)$.

The proof of Theorem 1 is based of the following assertion.

Lemma 1. *Let $u \in C^1(\mathbf{R}^N)$ be a weak solution of problem* (4)*. Suppose that u is bounded below and set $m = \inf_{x \in \mathbf{R}^N} u(x)$. Then there exists a $K > 0$ such that*

$$\int\limits_{B_R} A(|Dv|)|Dv|^2 v^{-2} dx \leq K R^{N-2}, \tag{6}$$

where $v := 1 + u - m$ and $B_R = \{x \in \mathbf{R}^N : \ |x| < R\}$.

Proof. Obviously, v is a weak solution of problem (4), and $v(x) \geq 1$ for all $x \in \mathbf{R}^N$. Let $\psi \in C_0^1(\mathbf{R}^N)$ be a nonnegative function. We set $\varphi(x) := v(x)^{-1}\psi(x)$, $x \in \mathbf{R}^N$. This function can be admitted as a test function. Then, from (5), we obtain

$$0 \leq -\int\limits_{\mathbf{R}^N} A(|Dv|)|Dv|^2 v^{-2}\psi(x)dx + \int\limits_{\mathbf{R}^N} A(|Dv|)(Dv, D\psi)v^{-1}dx. \tag{7}$$

Further, by the Cauchy inequality with a parameter $\varepsilon > 0$, we have

$$\int\limits_{\mathbf{R}^N} A(|Dv|)|Dv|^2 v^{-2}\psi(x)dx \leq \frac{\varepsilon}{2}\int\limits_{\mathbf{R}^N} A(|Dv|)|Dv|^2 v^{-2}\psi(x)dx + \frac{1}{2\varepsilon}\int\limits_{\mathbf{R}^N} A(|Dv|)|D\varphi|^2\psi^{-1}dx.$$

Choosing $\varepsilon < 2$, we find from the latter equation that

$$\left(1 - \frac{\varepsilon}{2}\right)\int\limits_{\mathbf{R}^N} A(|Dv|)|Dv|^2 v^{-2}\psi(x)dx \leq \frac{C}{2\varepsilon}\int\limits_{\mathbf{R}^N} |D\psi|^2\psi^{-1}dx. \tag{8}$$

It remains to show that the right-hand side is bounded for an appropriate choice of the test functions $\psi \in C_0^1(\mathbf{R}^N)$. To this end, we select the test function in the form ψ^γ (with a sufficiently large $\gamma > 0$) (see [5, 6]).

We choose $\psi_0 \in C_0^1(\mathbf{R})$ such that

$$\psi_0(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t \leq 1 \\ 0 & \text{if} \quad t \geq 2, \end{cases}$$

and, setting

$$\psi(x) = \psi_0\left(|x|^2/R^2\right) \tag{9}$$

for $R > 0$, from (8), we derive the desired assertion.

Proof of Theorem 1. First, suppose that $N = 1$. It follows from (6) that

$$\int_{-\infty}^{+\infty} A(v')\,|v'|^2\,v^{-2}dx = 0;$$

consequently, $v' = 0$ and $u \equiv \text{const}$ for all $x \in \mathbf{R}$.

Now consider the case $N = 2$. As in the proof of Lemma 1, we choose ψ and use the notation $\Omega_R = \left\{x \in \mathbf{R}^N :\ R \le |x| \le \sqrt{2}R\right\}$; then from (7), we obtain

$$\int_{B_{\sqrt{2}R}} A(|Dv|)|Dv|^2 v^{-2}\psi\,dx \le \int_{\Omega_R} A(|Dv|)|Dv||D\psi|v^{-1}dx,$$

and consequently,

$$\begin{aligned}\int_{B_R} A(|Dv|)|Dv|^2 v^{-2}dx &\le \left(\int_{\Omega_R} A(|Dv|)|Dv|^2 v^{-2}dx\right)^{1/2} \left(\int_{\Omega_R} A(|Dv|)|D\psi|^2\psi^{-1}dx\right)^{1/2} \\ &\le C^{1/2}\left(\int_{\Omega_R} A(|Dv|)|Dv|^2 v^{-2}dx\right)^{1/2} K,\end{aligned} \tag{10}$$

where C and K are the constants occurring in (3) and (8), respectively. Since, by (6),

$$A(|Dv|)|D\ln v|^2 \in L^1\left(\mathbf{R}^2\right),$$

it follows that there exists a sequence $\{R_k\}$ with $R_k \to \infty$ such that

$$\lim_{k\to\infty} \int_{\Omega_{R_k}} A(|Dv|)|D\ln v|^2 dx = 0,$$

which, combined with (10), implies the relation $\int_{\mathbf{R}^2} A(|Dv|)|D\ln v|^2dx = 0$, which completes the proof of Theorem 1.

Corollary 1. Let $u \in C^2(\mathbf{R}^2)$ be a solution of the differential inequality

$$-\operatorname{div}\left(Du/\sqrt{1+|Du|^2}\right) \ge 0 \quad \text{in} \quad \mathbf{R}^2.$$

If u is bounded below, then $u(x, y) = \text{const}$ for all $(x, y) \in \mathbf{R}^2$.

Remark 3. In the general case, an analog of the theorem fails in higher dimensions. Indeed, the function $u(x) = \varepsilon\left(1+|x|^2\right)^{1/(1-q)}$, $x \in \mathbf{R}^N$, $N > 2$, where $\varepsilon > 0$ is sufficiently small and $q > N/(N-2)$, is a global solution of the differential inequality

$$-\operatorname{div}\left(Du/\sqrt{1+|Du|^2}\right) \ge u^q \quad \text{in} \quad \mathbf{R}^N.$$

Remark 4. The one-sided Bernstein (Liouville) theorem, which is similar to Theorem 1, fails for operators in nondivergent form. This fact was noticed by Bernstein (see [7] and discussion therein). Indeed, the function $u(x, y) = e^{x-y^2}$ is a positive solution of the equation

$$2\left(1+2y^2\right)u_{xx} + 4yu_{xy} + u_{yy} = 0 \quad \text{in} \quad \mathbf{R}^2.$$

2. GENERALIZATIONS

This section outlines some generalizations of Theorem 1 to other types of differential inequalities. Let us start from the introduction of the class of operators to be discussed.

Definition 2. Let $A : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ be a continuous function. We say that A generates an operator of the type of the m-Laplacian if there exist $a, b > 0$ such that

$$(A(x,t,p),p) \geq a|p|^m \geq b|A(x,t,p)|^{m'} \tag{11}$$

for all $(x,t,p) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N$, where $1/m + 1/m' = 1$ and $m > 1$.

If $u \in C^2(\mathbf{R}^N)$, then the operator of the type of the m-Laplacian generated by A is given by the formula $\operatorname{div}(A(x,u,Du))$, $x \in \mathbf{R}^N$.

Remark 5. The class of operators satisfying condition (11) was used in Serrin's classical work [8, 9]. Relation (11) is known to imply Harnack's weak inequality.

Remark 6. We can readily show that if $A : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ has the form $A(x,t,p) = \mathscr{A}(x,t,|p|)p$, $p \in \mathbf{R}^N$, i.e., $A_i(x,t,p) = \mathscr{A}(x,t,|p|)p_i$, $i = 1,\ldots,N$, where $\mathscr{A} : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is continuous, then A satisfies condition (11) if and only if $L^{-1}s^{m-2} \leq \mathscr{A}(x,t,s) \leq Ls^{m-2}$, $s \in \mathbf{R}_+$, where $m > 1$.

In particular, the m-Laplacian given by the formula $\operatorname{div}(|D\cdot|^{m-2}D\cdot)$ belongs to this class.

Definition 3. Let A be an operator of the type of the m-Laplacian. We say that $u \in C^1(\mathbf{R}^N)$ is a weak solution of the differential inequality

$$-\operatorname{div}(A(x,u,Du)) \geq 0 \quad \text{in} \quad \mathbf{R}^N, \tag{12}$$

if

$$0 \leq \int\limits_{\mathbf{R}^N} (A(x,u,Du), D\varphi)dx \tag{13}$$

for all nonnegative $\varphi \in C_0^1(\mathbf{R}^N)$.

The following assertion is valid.

Theorem 2. *Let $N \geq 1$, and A be an operator of the type of the m-Laplacian. Let $u \in C^1(\mathbf{R}^N)$ be a weak solution of problem* (12). *If $u \geq 0$ in $\mathbf{R}^N$ and $N \leq m$, then $u(x) \equiv \mathrm{const}$ for all $x \in \mathbf{R}^N$.*

Proof. The proof of Theorem 2 is similar to that of Theorem 1. Indeed, suppose that the assertion fails and u is not identically zero in $\mathbf{R}^N$. Since the strong maximum principle is valid for the operator A, we can assume that $u > 0$ in $\mathbf{R}^N$.

Note the theorem can be proved without using the strong maximum principle; to this end, it suffices to set $u_\varepsilon = (u+\varepsilon)^\alpha$. Multiplying by $\varphi_\varepsilon = u_\varepsilon \psi(x)$, $\varepsilon > 0$, and letting $\varepsilon \to 0$, we can perform all required calculations without referring to the strong maximum principle.

Let $\psi \in C_0^1(\mathbf{R}^N)$ be a nonnegative function. We set $\varphi(x) = u^\alpha(x)\psi(x)$, $x \in \mathbf{R}^N$, where $\alpha < 0$ is to be specified below. Substituting φ into (13), we obtain

$$|\alpha| \int\limits_{\mathbf{R}^N} \left(\sum_{i=1}^n A_i(x,u,Du) \frac{\partial u}{\partial x_i} \right) u^{\alpha-1}\psi dx \leq \int\limits_{\mathbf{R}^N} \left(\sum_{i=1}^n A_i(x,u,Du) \frac{\partial \psi}{\partial x_i} \right) u^\alpha dx, \tag{14}$$

which, combined with (11) and the Young inequality with parameter $\varepsilon > 0$, implies the inequality

$$\begin{aligned} |\alpha| b \int\limits_{\mathbf{R}^N} |A(x,u,Du)|^{m'} u^{\alpha-1}\psi\, dx \leq \frac{\varepsilon^{m'}}{m'} \int\limits_{\mathbf{R}^N} |A(x,u,Du)|^{m'} u^{\alpha-1}\psi\, dx \\ + \frac{\varepsilon^{-m}}{m} \int\limits_{\mathbf{R}^N} u^{((1-\alpha)+\alpha m')/m'} |D\psi|^m \psi^{1-m} dx. \end{aligned} \tag{15}$$

By setting $\alpha = 1 - m$ in (15) and choosing a sufficiently small $\varepsilon > 0$, from (15), we obtain

$$\int_{\mathbf{R}^N} |A(x,u,Du)|^{m'} u^{-m}\psi\, dx \le K \int_{\mathbf{R}^N} |D\psi|^m \psi^{1-m} dx, \tag{16}$$

where $K = K(\varepsilon, m, b) > 0$. Obviously, using (11) again, from (14) and (16), we obtain

$$\begin{aligned} |\alpha| a \int_{\mathbf{R}^N} |Du|^m u^{-m}\psi\, dx &\le \left(\int_{\mathbf{R}^N} |A(x,u,Du)|^{m'} u^{-m}\psi\, dx \right)^{1/m'} \left(\int_{\mathbf{R}^N} |D\psi|^m \psi^{1-m} dx \right)^{1/m} \\ &\le K^{1/m'} \int_{\mathbf{R}^N} |D\psi|^m \psi^{1-m} dx. \end{aligned} \tag{17}$$

Finally, after choosing the same $\psi \in C_0^1(\mathbf{R}^N)$ as in (9), from (17), we obtain

$$\int_{B_R} |Du|^m u^{-m} dx \le \text{const} \times R^{N-m}. \tag{18}$$

If $N < m$, then the statement of the theorem is a straightforward consequence of relation (18); but if $N = m$, then we proceed as follows. Setting $\alpha = 1 - m$ and choosing ψ by analogy with (9), from (14), we obtain

$$\begin{aligned} &|\alpha| \int_{B_R} \left(\sum_{i=1}^n A_i(x,u,Du) \frac{\partial u}{\partial x_i} \right) u^{-m}\psi\, dx \\ &\qquad \le \left(\int_{\Omega_R} |A(x,u,Du)|^{m'} u^{-m}\psi\, dx \right)^{1/m'} \left(\int_{\Omega_R} |D\psi|^m \psi^{1-m} dx \right)^{1/m} \\ &\qquad \le \text{const} \times \left(\int_{\Omega_R} |A(x,u,Du)|^{m'} u^{-m}\psi\, dx \right)^{1/m'}. \end{aligned} \tag{19}$$

Consequently, relations (11) and (19) imply that

$$\begin{aligned} a|\alpha| \int_{B_R} |Du|^m u^{-m} dx &\le \text{const} \times \left(\int_{\Omega_R} |A(x,u,Du)|^{m'} u^{-m}\psi\, dx \right)^{1/m'} \\ &\le \text{const} \times \left(\frac{a}{b}\right)^{1/m'} \left(\int_{\Omega_R} |Du|^m u^{-m} dx \right)^{1/m'}. \end{aligned} \tag{20}$$

Taking account of the fact that relation (18) with $m = N$ implies that

$$|Du|^m u^{-m} = |D \ln u|^m \in L^1\left(\mathbf{R}^N\right),$$

we find that there exist $R_k \to \infty$ such that $\lim_{k\to\infty} \int_{\Omega_{R_k}} |D\ln u|^m dx = 0$. Then, from (20), we have

$$\lim_{k\to\infty} \int_{B_{R_k}} |D\ln u|^m dx = \int_{\mathbf{R}^N} |D \ln u|^m dx = 0.$$

Hence $u(x) \equiv \text{const}$ for all $x \in \mathbf{R}^N$. The proof of Theorem 2 is complete.

This theorem results in the following assertion.

Corollary 2. Let $N \geq 1$ and $m > 1$. Suppose that $A : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$, and there exist $a, b > 0$ such that $(A(x,p),p) \geq a|p|^m \geq b|A(x,p)|^{m'}$ for all $(x,p) \in \mathbf{R}^N \times \mathbf{R}^N$.

Let $u \in C^1(\mathbf{R}^N)$ be a weak solution of the problem

$$-\operatorname{div}(A(x,Du)) \geq 0 \quad \text{in} \quad \mathbf{R}^N.$$

If u is bounded below and $N \leq m$, then $u(x) \equiv \text{const}$ for all $x \in \mathbf{R}^N$.

Proof. We apply Theorem 2 to problem (21), where v and m are defined in Lemma 1.

Corollary 3. Let $\sigma \geq 0$ and $m > 1$. Suppose that u is a weak solution of the inequality

$$-\operatorname{div}\left(|x|^\sigma |Du|^{m-2} Du\right) \geq 0$$

in $\mathbf{R}^N$. Let u be bounded below, and let $N + \sigma \leq m$. Then $u(x) = \text{const}$ for all $x \in \mathbf{R}^N$.

The proof reproduces that of Theorem 2.

Various generalizations can be proved, for example, for a right-hand side of (11) depending on x. The details can be reconstructed by the interested reader.

Remark 7. In the general case, an analog of Theorem 2 fails for higher dimensions. Indeed, the function

$$u(x) = \varepsilon\left(1 + |x|^{m/(m-1)}\right)^{(1-m)/(q-m+1)},$$

where $\varepsilon > 0$ is sufficiently small, and $q > N(m-1)/(N-m)$, is a global solution of the inequality $-\operatorname{div}(|Du|^{m-2}Du) \geq u^q$ in $\mathbf{R}^N$, $N > m$ (see [5]).

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