Advanced Algebra/Istituzioni di Algebra Superiore
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1 Lecture 1

Introduction to the course. Some definitions: ring, commutative ring, unitary ring, zero divisors of a ring, invertible (or unitary) element of a ring, a ring with no zero divisors (except 0) is called an integral domain. First examples of rings: $\mathbb{Z}$ and the polynomial ring $K[x_1, \ldots, x_n]$.
Ideals in a commutative ring. Quotient of a ring w.r.t. an ideal. Finitely generated and principal ideals. A domain where all the ideals are principal is called a principal ideal domain (PID). The ring of integers and the ring $K[x]$ (where K is a field) are examples of PID.
Homomorphisms of rings. The kernel of an homomorphism. The canonical projection $\pi : R \longrightarrow R/I$, where $I \subseteq R$ is an ideal of R. The theorems of homomorphism:

Teorema 1 *Let $f : R \longrightarrow S$ be a ring homomorphism, and let $\overline{f} : R/\ker(f) \longrightarrow S$ be given by: $\overline{f}([a]) = f(a)$. Then $\overline{f}$ is a well defined homomorphism and $f = \overline{f} \circ \pi$, where π is the canonical projection from R to $R/\ker(f)$.*

Teorema 2 *Let R be a ring and $I \subseteq R$ an ideal. Let*

$$\mathcal{A} = \{J \subseteq R \mid J \text{ is and ideal and } I \subseteq J\},$$

$$\mathcal{B} = \{K \subseteq R/I \mid K \text{ is an ideal of } R/I\}$$

then there is a bijection $F : \mathcal{A} \longrightarrow \mathcal{B}$ given by $F(J) = J/I$ whose inverse is G given by $G(K) = \pi^{-1}(K)$ (where π is the canonical projection from R to R/I.

Teorema 3 *Let $f : R \longrightarrow S$ be a ring homomorphism, then $R/\ker(f)$ is isomorphic to $\mathrm{im}\,(f)$. In particular, if f is surjective, $R/\ker(f)$ is isomorphic to S.*

2 Lecture 2

Modules over a commutative ring. Submodules, quotient modules, homomorphisms of modules. Examples of modules: ideals of a ring, vector sapces over a field. The fundamental theorems of homomorphism for modules (quite similar to the case of rings). Product (finite or infine) of modules. External direct sum of modules. The universal property of the direct sum of modules.