Four-dimensional objects: an approach to their visualization

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Abstract

Can we imagine geometric objects in a four-dimensional space? We claim that this is possible: the basic remark is simply to consider one of the four variables as the time. In this way an object in a four-dimensional space can be viewed as a moving object in a three-dimensional space. To prove that this is indeed useful, we have written a short story (*"Universes and Dimensions"*) that is available at the URL address:

`http://www.dsm.univ.trieste.it/~logar/`

in which several animations are given to show that four-dimensional objects could indeed be quite familiar to us.

1 The main construction

Can we imagine geometric objects in a four-dimensional space? This paper, and the `html` file *"Universes and Dimensions,"* available on the Web at the address `http://www.dsm.univ.trieste.it/~logar/` try to give an affirmative answer to this question.

Our point of view is the following: an object in a d-dimensional space (with coordinates $x_1, \ldots, x_d$) can be seen as a moving object in a $(d-1)$-dimensional space (with coordinates $x_1, \ldots, x_{d-1}$), considering the remaining coordinate x_d as the time variable. For example: take a triangle A in the plane with coordinates, say, x and y and take the line $y = t$, with t a constant. Then, in general, this line either meet the triangle A in two points or in none. If t varies with continuity, then on the one-dimensional space $y = t$ we get two moving points (eventually disappearing). In other words, the triangle can be "seen" in the one-dimensional space given by the line $y = t$ as two moving points.

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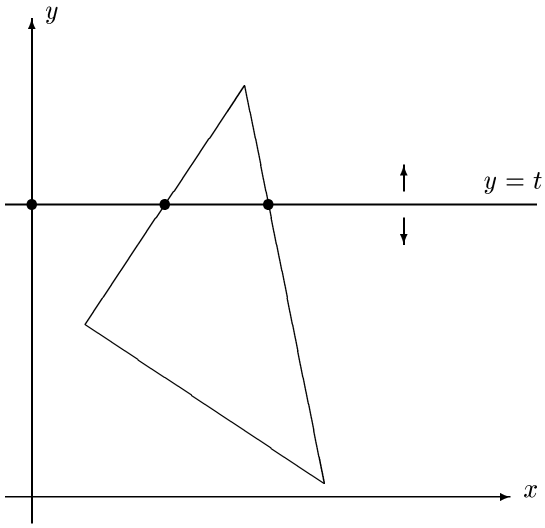

Based on this remark, we have written the small story "*Universes and Dimensions*," partially inspired by the book "Flatland" (see [1] and [2]). It is addressed to students of an undergraduate level and its aim is to give them some feeling about objects in a four-dimensional space. It was presented at the Conference "Matematica 2000", hold at the Mathematics Department of the University of Trieste (Italy) on May 20-21, 1998 (see `http://www.dsm.univ.trieste.it/`). The aim of the meeting was to give a (possibly positive) contact with mathematics to the students of a secondary level.

The story contains several animations that should give more vivacity and clarity to the text. All the animations were produced with the computer algebra system *Maple* (see [3]).

The story begins with the very simple situation of a one-dimensional world: there, very few geometric objects can be seen (basically, only points ...) but, using motion, as said above, an inhabitant of that world can also perceive two-dimensional objects. We show with some animations that triangles, squares, circles ... can be in some sense perceived also in that world and, moreover, we suggest a way in which an imaginary inhabitant of that world can discover some elementary geometric theorems on triangles.

Successively, we go one step further: we try to show that inhabitants of a two-dimensional world can perceive three-dimensional objects. We give in this case a specific example (that will also be useful in the third part of the story), i.e. we consider the following problem: *given two concentric circles, is it possible to unthroead the smaller outside the bigger one without touching them?* Of course, this is not possible to do in a two-dimensional space, but if we can lift one of the two circles (hence using a third dimension), then the solution is trivial. We give an animation to show first how this is done in a three-dimensional space and then how this solution can be perceived by an inhabitant of a two-dimensional world.

A student of an undergraduate level can easily imagine the constructions presented above and in this way he/she is also ready to understand, we hope, the analogy that we want to use in order to go further. More precisely, the third part of our story wants to present some geometric objects that could be seen only in a four-dimensional space.
The first example is the following: *given two different concentric spheres, is it possible to unthroead the smaller one from the bigger one (without breaking their border)?* This is, of course, the analog of the previous problem of the two circles and the solution clearly cannot be given in a three-dimensional space. If however we can embed our three-dimensional space in a space of dimension four, then precisely the same kind of construction we have used to unthread a small circle from a big one, can now be used in this situation. Also in this case we present several animations which should lead the reader to understand the construction. In order to see the mathematical background, we add here some explicit details of the involved equations. First of all, let x, y, z, t be the coordinates of our four-dimensional space $\mathbf{R}^4$ and suppose that our three-dimensional space (call it X_0) corresponds to the points of $\mathbf{R}^4$ with the last coordinate $t = 0$. Then the unit sphere S in X_0 has equation: $x^2 + y^2 + z^2 - 1 = 0$. Let us consider the following change of coordinates: $x \leftrightarrow x,\ y \leftrightarrow t,\ z \leftrightarrow z$. In this change of coordinates, the space X_0 is sent to the space Y corresponding to the subspace or $\mathbf{R}^4$ given by $y = 0$ and the sphere $S \subseteq X_0$ is sent to the sphere $T \subseteq Y$ whose equation is $x^2 + z^2 + t^2 - 1 = 0$. The sphere T is not anymore contained in the space X_0: indeed $T \cap X_0$ is only the circle $x^2 + z^2 - 1 = 0,\ y = 0$. Hence an inhabitant of the space X_0 can see only a circle of T. In order to see the "whole" T, we can move X_0 along the direction of the t-axis. In other words, let us consider the space X_a given by the points of $\mathbf{R}^4$ with coordinates (x, y, z, a), where $a \in \mathbf{R}$ is a fixed number. Then the three-dimensional space X_a meets T along the circle $x^2 + z^2 + (a^2 - 1) = 0,\ y = 0$. If we consider a as a parameter varying with continuity, then we can view these circles as a moving circle in the space. This is the way we use to visualize our spheres and to show how to unthroead one from the other.

The last set of examples we consider comes from a mathematical problem. The situation to describe in this case is a little bit more technical. Take the equation $y = P(x)$, where $P(x)$ is a polynomial. If we take x and y in the field of real numbers, it clearly gives a graph in the plane. If we take x and y in the field of complex numbers, the above equation gives a surface S in the four-dimensional space $\mathbf{R}^4$. (Indeed, let $y = u + iv$ and $x = z + it$, then the equation $u + iv = P(z + it)$ gives rise to two equations in the four real variables u, v, z, t). Again, we want to see this surface contained in a four-dimensional space. Also here, we can find the section of the surface with a three-dimensional space X_a given by the points of R^4 of the form (u, v, z, a). The intersection $X_a \cap S$ is a curve C_a and the family of all C_a's gives a moving curve which *is* our surface. What is interesting of this example, is that in this way we can visualize the complex roots of the polynomial $P(x) = 0$. Indeed, these roots are given by the intersection of the surface S with the plane $u = 0,\ v = 0$ of $\mathbf{R}^4$, that is, by the intersections of the curves C_a with the line $u = 0,\ v = 0$ of X_a (note that the line $u = 0,\ v = 0$ is a fixed line of the space X_a). In the file "Universes and Dimensions" you can find some examples of this last problem: the visualizations there given concern polynomials of degree 2, 3, 4 and 5. In each of these animations you can see a moving curve and a fixed line in the space and they meet in the first case twice, in the second case three times etc., as should be expected from the fundamental theorem of algebra.

Finally, we remark that many other four-dimensional geometric objects could be visualized with the technique above presented. In particular, one interesting example to consider is the following: let $P(x, y)$ be a cubic polynomial. It is then know that the affine variety $P(x, y) = 0$ is a curve of genus 1 (i.e. topologically, a torus in $\mathbf{R}^4$). It would be interesting to make a suitable animation to show that this is precisely the case.

References

[1] Edwin Abbott, Flatland: a romance of many dimensions, Verplanck, N.Y.: Emerson 1982.

[2] Arte e matematica: Flatlandia, a film by Michele Emmer.

[3] Maple V Release 5, a computer algebra system. Waterloo Maple, Canada.