CLASSIFICATION OF CUBIC SURFACES WITH COMPUTATIONAL METHODS

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Abstract. The aim of the paper is the study the orbits of the action of PGL_4 on the space $\mathbb{P}^{19}$ of the cubic surfaces of $\mathbb{P}^3$, i.e. the classification of cubic surfaces up to projective motions. All the cubic surfaces with finitely many lines are parametrized by a variety $\mathcal{Q} \subset \mathbb{P}^{19}$, explicitely constructed as the union of (22) disjoint irreducible components which are either points or open subsets of linear spaces. More precisely, the orbit of each cubic surface intersects one component X of $\mathcal{Q}$ in a finite number of points and the action of PGL_4 restricted on each component X is equivalent to the action of a finite group G_X, which is explicitely computed. Finally the cubic surfaces of each component of $\mathcal{Q}$ are studied in details by determining their stabilizers, their rational representations and whether they can be expressed as the determinant of a 3×3 matrix of linear forms.

The results are obtained with computational techniques and with the aid of some computer algebra systems like CoCoA, Macaulay and Maple.

Introduction

Algebraic surfaces of degree 3 in the projective space have been studied for many years from several different points of view. The nice configuration of the 27 lines on a smooth cubic surface and its degenerations in the singular cases, as well as the possibility of a classification of these surfaces have been investigated since the last century by Salmon [Sa], Schläfli [Sc1] and Cayley [Ca] and, more recently, by other authors: among them we recall B. Segre [S] and Yu. Manin [M]. Therefore many aspects of cubic surfaces are classical and very well-understood.

In the last two decades, the development of algorithms, their successful implementation in various computer algebra systems and, above all, the improvement of computer technology have provided additional important tools for investigating new and old mathematical problems.

In this paper we propose to use such tools to present a novel approach to the study of cubic surfaces. By doing so, we not only recover well–known results, but are also able to classify cubic surfaces, giving explicit models of each class and studying each model completely.

More precisely, this paper deals with the study of the action of the group PGL_4 of linear transformations of $\mathbb{P}^3$ on the variety of cubic surfaces. This problem was already considered by a number of authors (see [B], [BD], [Se1], [Se2], [DO], ...) but, apparently, a complete answer has not been given so far. What we obtain here is an explicit construction and the complete classification of the orbits of all the cubic surfaces (with finitely many lines).

The first remark we make in this paper is that almost all cubic surfaces of $\mathbb{P}^3$ (and in particular all the smooth ones) contain a specific configuration of five lines (here called an "L-set") which can be fixed up to a projective motion of $\mathbb{P}^3$. Moreover such L-sets characterize the elements of PGL_4, i.e. given two L-sets, there exists exactly one projectivity which maps the first one into the second one. Therefore, in order to classify cubic surfaces up to projectivity, the first step can be to fix a specific L-set and study the linear system of cubic surfaces through it, which turns out to be a four–dimensional projective space. We show that this $\mathbb{P}^4$ parametrizes *all* the smooth cubic surfaces and also some singular ones. Therefore, we determine the equation of the subvariety (hypersurface) Σ of $\mathbb{P}^4$ whose points correspond to singular cubic surfaces. However the space $\mathbb{P}^4 \setminus \Sigma$ is not yet the *minimal* space parametrizing smooth cubic surfaces, since more than one point of it (precisely $25,920$ in the general case) belongs to the same orbit. A deeper analysis of this situation leads to realize that there is a further action of a finite group G (an index two subgroup of the

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Weyl group $\mathbb{E}_6$, hence of order 25,920) on $\mathbb{P}^4 \setminus \Sigma$ such that its quotient by G is precisely the orbit space of smooth cubic surfaces.

The next step is the study of singular cubic surfaces. First we consider those which do not contain an L-set: they give rise (when irreducible and not ruled) to 12 distinct orbits (represented by specific cubic surfaces, say $T_1, \ldots, T_{12}$) and a set of orbits parametrized by $\mathbb{P}^1 \setminus \Delta$ (where Δ is a finite set) and represented by a one–dimensional family $T_{13}(p,q)$, $(p,q) \in \mathbb{P}^1 \setminus \Delta$. The orbits listed here correspond precisely to those cubic surfaces which contain a specific configuration of lines, e.g. the orbit of T_1 is the set of all cubic surfaces containing exactly one line, the orbit of T_2 consists of those cubic surfaces containing two meeting lines, ..., the union of the orbits of the $T_{13}(p,q)$'s is the set of all cubic surfaces containing precisely 7 lines having a peculiar incidence relation (see Figure 1 below for the complete list of the configurations of lines).

Successively, we consider the singular cubic surfaces containing an L-set. They are already parametrized by $\Sigma \subset \mathbb{P}^4$ (at least in a rough way: the orbit of any singular cubic surface of this type intersects Σ only in a finite number of points). We can however be much more precise: we can indeed divide the orbits of these singular cubic surfaces into 8 other disjoint classes (according to the number of singular points and the number of lines): the first class consists of cubic surfaces with one singular point and 21 lines and is parametrized by an open subset of a three–dimensional linear space, the second class consists of cubic surfaces with two singular points and 16 lines and is parametrized by an open subset of a two–dimensional linear space, ..., each of the last two classes consists of a single orbit, which collects, respectively, cubics with four singular points and 9 lines and three singular points and 8 lines (see Theorem 1.9). This is, in short, the contents of Section 1.

In Section 2 we begin the study of the orbits of $T_1, \ldots, T_{12}, T_{13}(p,q)$ and we show that all these cubic surfaces, except those projectively equivalent to T_1, can be expressed as the determinant of 3×3 matrices of linear forms. Since all the remaining cubic surfaces (i.e. those containing an L-set, ruled surfaces, cones and the reducible ones) also have this property, we prove in this way that T_1 is the only cubic surface, up to a projective motion, which is not the determinant of a matrix of linear forms. Moreover, we give a rational representation of $T_1, \ldots, T_{12}, T_{13}(p,q)$, expressing each of them as the blow–up of the plane at six points.

In Section 3 we analyze the groups of projective motions of $T_1 \ldots, T_{12}, T_{13}(p,q)$, describing each of them in terms of generators and relations.

In Section 4 we study the smooth cubic surfaces: we represent each of them as the determinant of a 3×3 matrix and also as a rational variety (i.e. as the blow–up of the plane at six points in general position), we determine the explicit equations of their 27 lines and we describe the action of the Weyl group $\mathbb{E}_6$ in terms of L-sets.

Finally, Section 5 and 6 contain the proofs of the results stated in Section 1, together with a more detailed study of the singular cubic surfaces containing an L-set. In this paper we do not analyze cubic surfaces with infinitely many lines, but their classification (although rather long) is quite easy.

The classification of the cubic surfaces given in this paper can be compared with that given by Schläfli and Cayley (based on the identification of the types of singularity) and presented in a modern language by Bruce and Wall in [BW]. It turns out that the known classification is coarser than that one given here. First of all, many singular cubic surfaces (with the same number of lines and the same kind of singularities) are here parametrized by linear spaces of positive dimension. Even more, we remark that the specific cubic surfaces T_9 and T_{10} (which lye in the same class according to Schläfli–Cayley) are not projectively equivalent.

As already noted, on the space $\mathbb{P}^4 \setminus \Sigma$ acts a finite group G such that $(\mathbb{P}^4 \setminus \Sigma)/G$ is the orbit space of smooth cubic surfaces. An analogous construction can be worked out for each of the other components of the variety $\mathcal{Q}$ (parametrizing the cubic surface with finitely many lines) introduced in Theorem 1.9: one can find suitable finite groups acting on these components in such a way that the corresponding quotients give the orbit spaces (see Section 6). A deeper study of the actions of these groups with the aim of obtaining either the moduli spaces and a complete description of the groups of automorphisms of all the cubic surfaces will be the objective of a further investigation.

One of the aims of this paper is to give a self-contained exposition of the subject presented; the techniques used to prove most theorems are quite elementary, but in many cases they require a careful use of a computer algebra system. For instance, a typical intermediate computation we shall perform many times is the following (see Lemma 2.2): we have the equation of a cubic surface S, whose coefficients depend on several parameters; we intersect S with the generic plane π through a line l (of given equation) contained in S, obtaining a plane cubic curve which splits into the line l and a conic $C \subset \pi$. The usual problem we have to solve is to find out for which planes π the conic C is reducible, i.e. its discriminant vanishes. Of course, this procedure is elementary from a theoretical point of view; nevertheless, if there are many parameters involved (as it indeed happens in our situation) one can carry out such computations only with a 'suitable' use of some symbolic computation package.

In particular, many computations were performed using Maple (mainly for manipulating multivariate polynomials and for their factorization), while in a few instance some checks were done with Gröbner bases techniques and the aid of Macaulay (see [BS]) and CoCoA (see [CNR]).

The computations where done on a DEC Alpha 3000/300 workstation; some of them could be carried out successfully on a smaller computer.

1. Main results

Let K be an algebraically closed field of characteristic 0.
We recall that the space of all cubic surfaces of $\mathbb{P}^3 := \mathbb{P}^3(K)$ can be parametrized by the projective space $\mathbb{P}^{19} := \mathbb{P}^{19}(K)$, i.e. if

$$\begin{aligned} F := a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xyt + a_6xz^2 + a_7xzt + a_8y^2t + a_9yzt + a_{10}yt^2 + \\ a_{11}x^3 + a_{12}x^2t + a_{13}xt^2 + a_{14}y^3 + a_{15}y^2z + a_{16}yz^2 + a_{17}z^3 + a_{18}z^2t + a_{19}zt^2 + a_{20}t^3 \end{aligned} \tag{0}$$

is a cubic form in the variables x, y, z, t defining a cubic surface $S := V(F) \subset \mathbb{P}^3$, then we associate with it the point $(a_1, \ldots, a_{20}) \in \mathbb{P}^{19}$; observe that with this notation we have fixed particular coordinates of $\mathbb{P}^{19}$ and these will be the coordinates of $\mathbb{P}^{19}$ we shall use from now on.
By $\mathrm{PGL}_4 := \mathrm{PGL}_4(K)$ we denote the group of projectivities of $\mathbb{P}^3$, which acts canonically on $\mathbb{P}^{19}$ as follows: given a cubic surface $S := V(F) \subseteq \mathbb{P}^3$ and an element $A \in \mathrm{PGL}_4$, denoting by X the column vector of the variables x, y, z, t, let us define the form $AF \in k[x, y, z, t]$ by

$$(AF)(x, y, z, t) := F(AX).$$

In this way the cubic form $(AF)(x, y, z, t)$ defines a new cubic surface, say $A(S)$, in $\mathbb{P}^3$. We shall denote by $\mathcal{O}_S^{\mathrm{PGL}_4}$ (or simply by $\mathcal{O}_S$) the orbit of S under the above action.

We want to study these orbits in $\mathbb{P}^{19}$, i.e. we want to describe cubic surfaces up to a linear change of coordinates. First of all we partition $\mathbb{P}^{19}$ into two classes: $\mathcal{A}$ and $\mathcal{B} := \mathbb{P}^{19} \setminus \mathcal{A}$, where

$$\mathcal{A} := \{\text{cubic surfaces containing a finite number of lines}\}$$

and

$$\mathcal{B} := \{\text{cubic surfaces containing an infinite number of lines}\}.$$

It is clear that $\mathcal{A}$ and $\mathcal{B}$ are stable under the action of PGL_4. Moreover, if S is a c.s. having a finite number n of lines, then all the cubic surfaces in $\mathcal{O}_S$ have the same number n of lines and, even more, with the same incidence relations.

The following well–known result (see for instance [C]) characterizes the objects of class $\mathcal{B}$:

Theorem 1.1. *The class $\mathcal{B}$ consists of:*
i) the reducible cubic surfaces;
ii) the irreducible cones;
iii) the irreducible cubic surfaces having a double line (i.e. the general ruled cubic surfaces).

Proof. See Section 5.

In the sequel surfaces of type *iii)* above will be simply called "ruled surfaces" (RS); it is understood that cones and reducible cubic surfaces are not in the class RS.

In this paper we are concerned about the c.ss. in the class $\mathcal{A}$. The first tool we shall use is the following:

Definition. Let $(l_1,\ldots,l_5)$ be a 5-tuple of lines of $\mathbb{P}^3$ intersecting according to the following conditions:

l_2 meets l_1,l_3,l_5 in three different points, l_4 meets l_1 and l_3 in two different points
and there are no other intersections

Such a system of lines will be called an *L-set (of lines)*.

Notation. If A and B are two points, then $\langle A+B\rangle$ denotes the line through them; if r and s are two meeting lines, then $\langle r+s\rangle$ denotes the plane containing them.

The following fact holds:

Lemma 1.2. *Let $(l_1,\ldots,l_5)$ and $(l'_1,\ldots,l'_5)$ be two L-sets. Then there exists exactly one projectivity which maps l_i to l'_i, for any $i=1,\ldots,5$.*

Proof. Let A,B,C,D,E be the five intersection points of the first L-set, i.e.

$$A=l_1\cap l_2,\quad B=l_1\cap l_4,\quad C=l_3\cap l_4,\quad D=l_2\cap l_3,\quad E=l_2\cap l_5$$

and let $P:=l_4\cap\langle l_2+l_5\rangle$, $Q:=\langle P+D\rangle\cap l_5$. It is easy to see that A,B,C,E,Q are five points in general position. Denoting by A',B',C',E',Q' the analogous points arising from the L-set $(l'_1,\ldots,l'_5)$, from the fundamental theorem of projectivities, there exists exactly one projectivity mapping A,B,C,E,Q into A',B',C',E',Q', respectively, hence mapping also l_i into l'_i, for $i=1,\ldots,5$. □

We see therefore that L-sets characterize linear changes of coordinates in $\mathbb{P}^3$. The first application of L-sets is the following:

Theorem 1.3. *Any smooth cubic surface of $\mathbb{P}^3$ passes through an L-set; more precisely, it contains exactly 25,920 L-sets.*

Proof. See Section 5.

As far as properties defined up to a linear change of coordinates are concerned, lemma 1.2 suggests that it is sufficient to consider a specific L-set; we fix for instance the following

$$l_1:=(y,z);\quad l_2:=(x,y);\quad l_3:=(x,t);\quad l_4:=(x-t,y-z);\quad l_5:=(x-y,z+t) \tag{1}$$

where the symbol (f,g) represents the line of equations $f=0=g$. From now on, by L^* we shall denote the L-set (l_1,l_2,l_3,l_4,l_5) above.

If we take the generic cubic surface and we require that it contains the L^*-set, we obtain a linear system of cubic surfaces of dimension 4 which can be described for instance by the following equation:

$$a(2x^2y-2xy^2+xz^2-xzt-yt^2+yzt)+b(x-t)(xz+yt)+c(z+t)(yt-xz)+d(y-z)(xz+yt)+g(x-y)(yt-xz)=0 \tag{2}$$

If $P=(a,b,c,d,g)$ is a point of the space $\mathbb{P}^4$ defined above, then by S_P we mean the corresponding c.s. of $\mathbb{P}^3$ given by (2).

We denote by

$$\phi:\mathbb{P}^4\longrightarrow\mathbb{P}^{19}\quad\text{the map defined by}\quad P=(a,b,c,d,g)\mapsto\ \text{coordinates of } S_P$$

i.e., if we use the coordinates of $\mathbb{P}^{19}$ introduced before in (0) and the equation (2), we get:

$$\phi(a,b,c,d,g):=(2a,b-g,-2a,d+g,b+g,a-d-c,-a-b-c,d-g,a+c-d,-a-b+c,0,\ldots,0).$$

Note that ϕ is a linear, one–to–one map. In the sequel we shall identify $\phi(\mathbb{P}^4)$ with $\mathbb{P}^4$, hence (a,b,c,d,g) can represent either a point of $\mathbb{P}^4$ or the corresponding point $\phi(a,b,c,d,g)\in\mathbb{P}^{19}$.

From 1.3 and 1.2 we see that the family (2) describes, in particular, all the smooth c.ss. up to projectivities. Of course, among the surfaces of (2), there are also singular and reducible ones. The following theorems describe the points of $\mathbb{P}^4$ which correspond to them and give some information about the number of lines of the surfaces in the family (2).

Theorem 1.4. *Let $P := (a,b,c,d,g) \in \mathbb{P}^4$, and let S_P be the corresponding cubic surface. Then the following facts are equivalent:*

i) S_P is smooth;

ii) S_P has 27 distinct lines;

iii) $P \notin \Sigma$, where $\Sigma := V(\sigma)$ is the hypersurface of $\mathbb{P}^4$ defined by:

$$\begin{aligned}\sigma := {} & c(a+b-c)(2a+b-d)(a-c-d)(a+c+g)(a+c-g) \\ & (4ac-g^2)(a^2+ac-2ad+ag+d^2-dg)(a^2+2ab+ac-ag+b^2-bg) \\ & (4a^2+3ab-4ac-3ad-bc-2bd+bg+cd+dg) \\ & (4a^3+4a^2b-8a^2c-4a^2d+ab^2-4abc-2abd+2abg+4ac^2+4acd+ad^2+ \\ & \ 2adg+b^2c+b^2g+2bcd-2bcg+cd^2-2cdg-d^2g).\end{aligned}$$

Proof. See Section 5.

Theorem 1.5. *All the irreducible cubic surfaces passing through the L^*-set contain a finite number of lines (i.e. are in $\mathcal{A}$).*

Proof. See Section 5.

Finally we characterize the points of $\mathbb{P}^4$ corresponding to reducible c.ss.:

Theorem 1.6. *Let $P := (a,b,c,d,g) \in \mathbb{P}^4$, then the cubic surface S_P is reducible if and only if $P \in \mathcal{R}$, where*

$$\mathcal{R} := V(b+d, a-c-d) \cup V(a+b-c, 2a+b-g) \cup V(a,c,g) \cup V(2a+b-d, ac+(a+b)(a+b-g))$$

Proof. See Section 5.

Using 1.4 and the previous results, we conclude that all the smooth cubic surfaces are, up to a linear change of coordinates, parametrized by the points of $\mathbb{P}^4 \setminus \Sigma$. Analogously, all the singular and irreducible cubic surfaces containing an L-set are, up to a linear change of coordinates, parametrized by $\Sigma \setminus \mathcal{R}$.

These facts can be summarized by the following:

Theorem 1.7. *With the above notations:*

1) The image of the map

$$\psi_1 : \mathrm{PGL}_4 \times \left(\mathbb{P}^4 \setminus \Sigma\right) \longrightarrow \mathbb{P}^{19} \quad \textit{defined by} \quad (A,P) \mapsto A(S_P)$$

is the open set of $\mathbb{P}^{19}$ consisting of all the smooth cubic surfaces.

2) The image of the map

$$\psi_2 : \mathrm{PGL}_4 \times (\Sigma \setminus \mathcal{R}) \longrightarrow \mathbb{P}^{19}$$

defined as above is the variety of cubic surfaces containing an L-set, singular and irreducible. □

Clearly, the map ψ_1 is finite: it is straightforward to see that, if S is a smooth c.s., then $\sharp(\psi_1^{-1}(S))$ is the number of L-sets contained in S (the same holds for the map ψ_2). Hence, from 1.3, we obtain that 25,920 points of $\mathbb{P}^4 \setminus \Sigma$ belong to the same orbit in $\mathbb{P}^{19}$. Theorem 1.10 below will show that this set of points is itself an orbit with respect to an action of a suitable group on $\mathbb{P}^4 \setminus \Sigma$.

So far we have considered cubic surfaces in the class $\mathcal{A}$ that contain an L-set. Here we want to classify the remaining cubic surfaces in $\mathcal{A}$. Let us summarize the situation: the c.ss. in $\mathcal{A}$ not containing an L-set split into 13 families: 12 of them are orbits, say $\mathcal{O}_{T_1}, \ldots, \mathcal{O}_{T_{12}}$. Obviously, to each of these orbits there corresponds a specific configuration of lines of its cubic surfaces which we shall denote by L_i, $(i = 1, \ldots, 12)$, and conversely, if the lines of a cubic surface are in one of the configurations L_i $(i = 1, \ldots, 12)$, then it turns out that the cubic surface is in $\mathcal{O}_{T_i}$. The last family of cubic surfaces in $\mathcal{A}$ is slightly more complicated to describe: it is given by the set of c.ss. containing precisely 7 lines as in a configuration of type L_{13}

(see Figure 1). This family, say $\mathcal{W}$, can indeed be parametrized by the points of a one–dimensional quasi–projective variety. The configurations of the lines $L_1, \dots, L_{13}$ are described as follows:

Figure 1

In the above figure, the dashed lines have the following meaning: they either join coplanar lines (in L_4) or collinear points (in L_9, L_{10}, L_{12}, L_{13}). Moreover lines which do not meet in the Figure are mutually skew and a small circle indicates a singular point on the corresponding surface.

Here we give a list of representatives $T_1, \dots, T_{12}$ of the above orbits and a family $T_{13}(p,q)$, such that the orbits of the c.ss. in it cover the whole $\mathcal{W}$, as long as (p,q) varies in $\mathbb{P}^1 \setminus \Delta$, where $\Delta = \{(1,0),(0,1),(-2,1)\}$: these three points correspond exactly to those values of (p,q) for which $T_{13}(p,q)$ is reducible.

Similarly to the identification $\phi(\mathbb{P}^4 \setminus \Sigma) = \mathbb{P}^4 \setminus \Sigma$ made before, we identify $\mathbb{P}^1 \setminus \Delta$ with the subvariety of $\mathbb{P}^{19}$ given by

$$\{(p+q, -p, -p, p, -q, p, 0, 2p, -2p, 0, \dots, 0) \in \mathbb{P}^{19} \mid (p,q) \in \mathbb{P}^1,\ pq(p+2q) \neq 0\}$$

obtained by computing the coefficients of $T_{13}(p,q)$.

$$\begin{aligned}
T_1 &:= xy^2 + yt^2 + z^3 \\
T_2 &:= xyt + xz^2 + y^3 \\
T_3 &:= xyt - xzt + y^3 \\
T_4 &:= x^2y - x^2z - xy^2 + xz^2 + y^3 - y^2t + yzt \\
T_5 &:= x^2y + xz^2 + y^2t \\
T_6 &:= xy^2 + xyt + xzt + yt^2 \\
T_7 &:= 2x^2y - x^2z + xy^2 - xyz - xyt - y^2t + yzt \\
T_8 &:= x^2y - x^2z - 2xy^2 + 2xyz - xyt - xz^2 + xzt + y^2t \\
T_9 &:= x^2y - x^2z + xy^2 - xyz + xz^2 - y^2t \\
T_{10} &:= x^2y - x^2z - 2xyz + xz^2 + y^2t \\
T_{11} &:= x^2y + x^2z - xy^2 + xyt - xzt - yt^2 \\
T_{12} &:= x^2y + xyz - 2xyt - xz^2 + xzt - y^2t + yzt \\
T_{13}(p,q) &:= pxy(x-t) + q(y-z)(x^2 - xy - xz + 2yt)
\end{aligned} \tag{3}$$

It is easy to see that each configuration L_i is unique up to projectivity. Hence, we choose the specific configurations of type L_i (say L_i^*) given by the lines lying on the surface T_i, for any $i = 1, \dots, 13$; let us list them:

$$
\begin{array}{rcl}
L_1{}^* & := & [l_1] \\
L_2{}^* & := & [l_1, l_2] \\
L_3{}^* & := & [l_1, l_2, (y,t)] \\
L_4{}^* & := & [l_1, l_2, (y, x-z)] \\
L_5{}^* & := & [l_1, l_2, l_3] \\
L_6{}^* & := & [l_1, l_2, l_3, (y,t)] \\
L_7{}^* & := & [l_1, l_2, l_3, l_4, (x, y-z)] \\
L_8{}^* & := & [l_1, l_2, l_3, l_4, (y, x+z-t)] \\
L_9{}^* & := & [l_1, l_2, l_3, l_4, (y, x-z), (x-t, z-t)] \\
L_{10}{}^* & := & [l_1, l_2, l_3, l_4, (y, x-z), (x-t, y-z+t)] \\
L_{11}{}^* & := & [l_1, l_2, l_3, (x-y, y-t), (y, x-t), (x-y, z+t)] \\
L_{12}{}^* & := & [l_1, l_2, l_3, l_4, (y, z-t), (x, y-z), (x-t, y-z+t)] \\
L_{13}{}^* & := & [l_1, l_2, l_3, l_4, (y, x-z), (x, y-z), (x-t, y-z+t)]
\end{array} \tag{4}
$$

where the lines $l_1, \ldots, l_4$ (already introduced in the definition of the L^*-set) are:

$$l_1 = (y,z), \quad l_2 = (x,y), \quad l_3 = (x,t), \quad l_4 = (x-t, y-z).$$

The situation described before can be stated precisely in the following result:

Theorem 1.8. *Let $\mathcal{P} \subseteq \mathbb{P}^{19}$ be the variety given by:*

$$\mathcal{P} := (\mathbb{P}^4 \setminus \Sigma) \cup (\Sigma \setminus \mathcal{R}) \cup (\mathbb{P}^1 \setminus \Delta) \cup \left(\bigcup_{i=1}^{12} T_i \right).$$

This union is disjoint and the orbits of the points of $\mathcal{P}$ cover $\mathcal{A}$. More precisely, the orbits of the points of $\mathbb{P}^4 \setminus \Sigma$ describe all the smooth cubic surfaces; the orbits of the points of $\Sigma \setminus \mathcal{R}$ describe the singular cubic surfaces with finitely many lines containing an L-set; the orbits of the points $\mathbb{P}^1 \setminus \Delta$ describe all the cubic surfaces with the lines as in configuration L_{13}; the orbits of the points T_i describe all the cubic surfaces with the lines as in configuration L_i, for $i = 1, \ldots, 12$.

Proof. See Section 5.

The above theorem can be improved by classifying also the cubic surfaces in Σ; in Section 6 we will see that, although Σ has several components, to describe all the singular c.ss. containing an L-set up to projectivity it is enough to consider only one of them. In particular we take the component of equation $a + c - g = 0$ in $\mathbb{P}^4$: the surfaces in this component are precisely those having $C = (0,1,1,0)$ as singular point. Substituting $c = g - a$ in (2), we get a linear system of dimension 3 which is denoted by $\mathbb{P}^3_C$; by Σ_1 we denote its *degenerate locus* consisting of the c.ss. either reducible or having a smaller number of lines than the general element of $\mathbb{P}^3_C$. Analogously, in Section 6, we will introduce other suitable linear spaces and their degenerate loci and we obtain the following extension of Theorem 1.8:

Theorem 1.9. *Let $\mathcal{Q} \subseteq \mathbb{P}^{19}$ be the variety given by:*

$$\mathcal{Q} := (\mathbb{P}^4 \setminus \Sigma) \cup (\mathbb{P}^3_C \setminus \Sigma_1) \cup (\mathbb{P}^2_{C,D} \setminus \Sigma_2) \cup (\mathbb{P}^1_{C,D,E} \setminus \Sigma_3) \cup$$
$$\cup (\mathbb{P}^2_C \setminus \Sigma_1') \cup (\mathbb{P}^1_C \setminus \Sigma_1'') \cup (\mathbb{P}^1_{C,D} \setminus \Sigma_2') \cup T_{15} \cup T_{14} \cup (\mathbb{P}^1 \setminus \Delta) \cup \left(\bigcup_{i=1}^{12} T_i \right).$$

Then this union is disjoint and the orbits of the points of $\mathcal{Q}$ cover $\mathcal{A}$. Moreover:
- *the component $\mathbb{P}^3_C \setminus \Sigma_1$ parametrizes the cubic surfaces having one singularity and precisely 21 lines;*
- *$\mathbb{P}^2_{C,D} \setminus \Sigma_2$ parametrizes the cubic surfaces having two singularities and precisely 16 lines;*

- $\mathbb{P}^1_{C,D,E} \setminus \Sigma_3$ *parametrizes the cubic surfaces having three singularities and precisely* 12 *lines;*
- $\mathbb{P}^2_C \setminus \Sigma_1'$ *parametrizes the cubic surfaces having one singularity and precisely* 15 *lines;*
- $\mathbb{P}^1_C \setminus \Sigma_1''$ *parametrizes the cubic surfaces having one singularity and precisely* 10 *lines;*
- $\mathbb{P}^1_{C,D} \setminus \Sigma_2'$ *parametrizes the cubic surfaces having two singularities and precisely* 11 *lines;*
- *the point* T_{15} *parametrizes the cubic surfaces having four singularities and precisely* 9 *lines;*
- *the point* T_{14} *parametrizes the cubic surfaces having three singularities and precisely* 8 *lines.*

Proof. See Section 6.

Let us remark that the above theorem does not give yet the space of the orbits of cubic surfaces in $\mathbb{P}^3$: for instance, the space of the orbits of smooth c.ss. is a quotient of the main component $\mathbb{P}^4 \setminus \Sigma$ of $\mathcal{Q}$.

More precisely, recalling that the group of the admissible permutations of the 27 lines of a smooth cubic surface is a finite group of order $51,840$, denoted by $\mathbb{E}_6$ (see [H], Ch. V, 4.10.1 and Section 4), we will prove the following:

Theorem 1.10. *There exists an action of an index two subgroup G of $\mathbb{E}_6$ on $\mathbb{P}^4 \setminus \Sigma$ in such a way that the points of the variety $(\mathbb{P}^4 \setminus \Sigma)/G$ are in a one–to–one correspondence with the orbits of the smooth cubic surfaces.*

Proof. See Section 4.

For a similar result see also [B] and [BD].

Remark 1.11. Analogous considerations could be repeated for all the components of $\mathcal{Q}$, since several points of these varieties (when of dimension at least one) represent the same orbit and can be identified under actions of suitable finite groups. In Section 6 we explicitly give these groups and their specific actions. A more detailed description of the quotient of each component, also from the point of view of Invariant Theory (see for instance [St]), will be the subject of a further investigation.

Here we describe, in addition to Theorem 1.10, only the situation concerning the component $\mathbb{P}^1 \setminus \Delta \subset \mathcal{Q}$, obtaining the following:

Proposition 1.12. *There is an action of the symmetric group S_3 on $\mathbb{P}^1 \setminus \Delta$ such that there is a one–to–one correspondence between the points of $(\mathbb{P}^1 \setminus \Delta)/S_3$ and the orbits of the cubic surfaces of the family $\mathcal{W}$.*

Proof. See Section 5.

2. Study of $T_1, \ldots, T_{13}$: lines, rational and determinantal representations

We shall give detailed information on the cubic surfaces introduced in the previous section, assuming all the results stated there; more precisely, in this section we are going to find, for each c.s. T_i, its parametric representation as a rational variety and (whether it is possible) a 3×3 matrix of linear forms whose determinant gives the equation of T_i. Finally, in the next section, we will give the list of the groups $\mathrm{Stab}(T_i)$ of the projective motions of $T_i \subset \mathbb{P}^3$, for any i.

First of all we introduce some notation and we describe a procedure which allows, under suitable hypothesis, to compute the lines on a cubic surface and which will be applied many times.

Notation 2.1. Let S be an irreducible cubic surface and let $r = (f, g)$ be a line contained in S (f and g are linear forms in x, y, z, t); we denote by $\pi_r(p, q)$ the pencil of planes of center r, i.e.

$$\pi_r(p, q) : \ pf(x, y, z, t) + qg(x, y, z, t) = 0$$

and by $C_r(p, q)$ the conic residual to r on the plane $\pi_r(p, q)$, i.e. such that

$$S \cap \pi_r(p, q) = r \cup C_r(p, q).$$

The discriminant of $C_r(p, q)$ is a homogeneous polynomial in p and q, say $D_r(p, q)$.

The proof we are giving here of the following result (see also [R1]), although quite elementary, involves polynomials in too many variables to be handled without the use of a computer. On the other hand, in this specific case, as in many other parts of this paper, the computation can almost immediately be carried out by a computer algebra system.

Lemma 2.2. *Let S be a cubic surface containing a line r. If $D_r(p,q) \equiv 0$, then there are infinitely many lines on S meeting r; otherwise $D_r(p,q)$ has degree 5 in p and q and hence there are at most 10 distinct lines meeting r.*

Proof. This fact can be shown by a direct computation: up to a linear change of coordinates, we can assume that r has equation (x,y). Let $f(x,y,z,t)=0$ be the equation of the generic surface S through r (f can be obtained from (0) by imposing $a_{17}=a_{18}=a_{19}=a_{20}=0$) and let $\pi_r(p,q):\ px+qy=0$ the pencil of planes containing r. Assuming $p\neq 0$, the substitution $x=-q/py$ in f gives, up to a non–zero constant, the polynomial $yg(y,z,t)$, where

$$\begin{aligned} g := &\left(a_{14}p^3 - a_3p^2q + a_1pq^2 - a_{11}q^3\right)y^2 + \left(a_{15}p^3 - a_4p^2q + a_2pq^2\right)yz + \left(a_{16}p^3 - a_6p^2q\right)z^2 + \\ &+ \left(a_8p^3 - a_5p^2q + a_{12}pq^2\right)yt + \left(a_9p^3 - a_7p^2q\right)zt + \left(a_{10}p^3 - a_{13}p^2q\right)t^2 \end{aligned}$$

is the equation of the conic $C_r(p,q)$ on the plane $\pi_r(p,q)$. It is clear that $C_r(p,q)$ is reducible if and only if its discriminant $D_r(p,q)$ is zero. The direct computation of the discriminant gives the following polynomial of degree 5 in p,q

$$\begin{aligned} &\left({a_8}^2a_{16} - a_8a_9a_{15} + {a_9}^2a_{14} - 4a_{10}a_{14}a_{16} + a_{10}{a_{15}}^2\right)p^5 + (\ldots)p^4q + (\ldots)p^3q^2 + (\ldots)p^2q^3 + (\ldots)pq^4 + \\ &\qquad + \left(-{a_2}^2a_{13} + a_2a_7a_{12} + 4a_6a_{11}a_{13} - a_6{a_{12}}^2 - {a_7}^2a_{11}\right)q^5 \end{aligned}$$

(for short, we indicate only the first and the last coefficient). Hence, in general, there are at most 10 lines meeting r.
If $D_r(p,q)\equiv 0$ then any conic $C_r(p,q)$ is the union of two lines. Suppose that there are finitely many lines on S meeting r; then almost every plane $\pi_r(p,q)$ meets S in r^3. This implies that every point of r is triple for S, which is then reducible. □

In the case $D_r(p,q)\not\equiv 0$, let us denote by $\pi_1(r),\ldots,\pi_5(r)$ the planes (not necessarily distinct) corresponding to the five roots of $D_r(p,q)$; on each of them, the plane section of S splits into three lines. Conversely, if s is a line on S that meets r, then s lies on one of these five planes.

Definition. Let $S\subseteq\mathbb{P}^3$ be any cubic surface passing through two coplanar lines r_1 and r_2. The plane defined by r_1 and r_2, if not contained in S, intersects S in a further line (not necessarily distinct from r_1 and r_2) that will be called *residual* of r_1 and r_2 and will be denoted by $\mathrm{res}\,(r_1,r_2)$.

We sketch here a procedure that allows us to compute the lines of a cubic surface that passes through two given incident lines r_1 and r_2.

Algorithm 2.3
Input: A cubic surface S passing through two incident lines r_1 and r_2;
Output: Either all the lines on S or 'infinity' if there are infinitely many lines on S;

$r_3 := \mathrm{res}\,(r_1,r_2)$;
lines:= $\{r_1,r_2,r_3\}$;
For l **in** $\{r_1,r_2,r_3\}$ **do**
  Let $\pi_l(p,q)$ be the generic plane of the pencil of planes through l;
  if $D_l(p,q)\equiv 0$ **then**
    RETURN('infinity')
  end if
  $X := \{(p,q)\in\mathbb{P}^1 \mid D_l(p,q)=0\}$;
  For (p,q) **in** X **do**
    call σ_1 and σ_2 the two linear factors in which $C_l(p,q)$ splits;
    lines:= lines $\bigcup$ $\{(\pi_l(p,q),\sigma_1),(\pi_l(p,q),\sigma_2)\}$
  end For
end For
RETURN(lines)
end

The output of the above Algorithm is therefore a set of couple of planes, where each couple defines a line.

The correctness of the Algorithm follows essentially from two facts: on one hand any line contained in S intersects the plane $\langle r_1 + r_2 \rangle$ in a point which is either on r_1 or on r_2 or on $\operatorname{res}(r_1, r_2)$. On the other hand, if S contains two meeting lines l and l', which therefore define a plane $\langle l + l' \rangle = \pi_l(p_0, q_0)$, then necessarily the corresponding conic $C_l(p_0, q_0)$ splits into the product of two linear forms in x, y, z, t.

The applicability of the Algorithm clearly depends on the possibility of finding the solutions X in (p, q) of the equation $D_l(p, q) = 0$ (and on the fact that one has to know in advance two meeting lines of the cubic surface).

The above Algorithm is anyway enough to verify that the lines of T_i ($i = 2, \ldots, 13$) are in the configuration $L_i{}^*$, as soon as one takes as input two meeting lines of $L_i{}^*$. A slight modification of Algorithm 2.3 allows one to check that T_1 contains only the line l_1.
Again using Algorithm 2.3, we shall be able to obtain the equations of the 27 lines on a smooth cubic surface passing through the fixed L^*-set (see Section 4).

It is well–known that the generic cubic surface is a rational variety and that its equation can be expressed as a determinant of a 3×3 matrix of linear forms in $k[x, y, z, t]$ (see for instance [S] or [C]). We sketch here a method which allows us to explicitly give a rational representation of those cubic surfaces, introduced in (3), which have at least three lines as in configuration L_5, and moreover, which expresses the equation of each of them as a determinant of a 3×3 matrix of linear forms.
In the L^*-set and in the configurations L_i^*, for $i = 5, \ldots, 13$, there are always the three lines $l_1 = (y, z)$, $l_2 = (x, y)$, $l_3 = (x, t)$, so we can study them considering cubic forms F such that $V(F)$ contains l_1, l_2, l_3, i.e.:

$$F := a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xyt + a_6xz^2 + a_7xzt + a_8y^2t + a_9yzt + a_{10}yt^2. \tag{5}$$

Let $Q_1 := (w, 0, 0, v)$ be a point of l_1 and $Q_3 := (0, w, v, 0)$ be a point of l_3 ($u, v, w \in K$) and let $l(u, v, w)$ be the line $\langle Q_1 + Q_3 \rangle$ (note that the coordinates are chosen in such a way that $l(0, 0, 1) = l_2$).
Then in general $l(u, v, w)$ intersects $S := V(F)$ in three points: Q_1, Q_3 and $P(u, v, w)$, whose coordinates in $\mathbb{P}^3$ immediately give the parametrization of S:

$$\begin{cases} x = f_1(u,v,w) = \alpha w \\ y = f_2(u,v,w) = -\beta w \\ z = f_3(u,v,w) = -\beta v \\ t = f_4(u,v,w) = \alpha u \end{cases} \quad \text{where} \quad \begin{aligned} \alpha &= a_9\, uv + a_8\, uw + a_6\, v^2 + a_4\, vw + a_3\, w^2 \\ \beta &= a_{10}\, u^2 + a_7\, uv + a_5\, uw + a_2\, vw + a_1\, w^2 \end{aligned} \tag{6}$$

To obtain the parametric representation of the c.s. listed in thm. 1.8 it is enough to specialize (6). This well–known construction is due to Clebsch (see for instance [C], I, Ch.II, 13).

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ be the rational map given by $f(u, v, w) := (f_1(u, v, w), \ldots, f_4(u, v, w))$. Then $\overline{\operatorname{Im}(f)} = V(F)$. If now we call $I := (f_1, \ldots, f_4) \subseteq R := k[u, v, w]$, it is well known that, for almost all choices of the coefficients, $V(I)$ consists of 6 points of $\mathbb{P}^2$ in general position, then a minimal free resolution of R/I has the form

$$0 \longrightarrow R(-4)^3 \xrightarrow{\psi} R(-3)^4 \xrightarrow{\phi} R \longrightarrow R/I \longrightarrow 0$$

where ϕ is the matrix $(\, f_1 \quad f_2 \quad f_3 \quad f_4 \,)$ and ψ is a 4×3 matrix of linear forms in u, v, w (see, for instance, [G]). The matrix ψ can easily be computed starting from $f_1, \ldots, f_4$, using linear algebra in $K(a_1, \ldots, a_{10})$.

We briefly recall now how to use the above resolution to express a cubic form F of equation (5) as a determinant of a 3×3 matrix. If $(u, v, w) \in \mathbb{P}^2$, then the point $(x, y, z, t) = f(u, v, w)$ is on $V(F)$ and therefore $(x, y, z, t) \cdot \psi = 0$. This relation gives three bilinear equations in x, y, z, t and u, v, w, which can also be viewed as linear equations in the variables u, v, w. Calling B the 3×3 associated matrix (of linear forms in x, y, z, t), we have that the system has a non trivial solution if and only if $\det(B) = 0$. Hence we have proved that $\operatorname{Im}(f) \subseteq V(\det(B))$. Since $V(F) = \overline{\operatorname{Im}(f)}$, then $\det(B)$ is the equation of $V(F)$. (For more details, see [G]).

This leads to construct the matrix B of the following proposition:

Proposition 2.4. *Let $S := V(F)$ be any cubic surface through l_1, l_2, l_3, where F is given by (5). Then $F = \det(B)$, where*

$$B := \begin{pmatrix} a_5x + a_8y + a_9z + a_{10}t & a_2x + a_4y + a_6z + a_7t & a_1x + a_3y \\ 0 & -y & z \\ -x & 0 & t \end{pmatrix}. \tag{7}$$

Proof. A straightforward computation of the determinant of B. □

Remark 2.5. Note that the way used to construct the matrix B of Prop. 2.4 requires further hypothesis on F: the c.s. $V(F)$ is initially given as the blow–up of the plane at six points in general position. Nevertheless, $\det(B)$ gives exactly the form F without any other assumption but the requirement that $V(F)$ contains the lines l_1, l_2, l_3.

Specializing (6) and (7), we can explicitly describe those cubic surfaces considered in thm. 1.8 having at least an L_5 configuration of lines: we can write each of them either as a rational surface (giving only, for simplicity, the polynomials $\alpha(u, v, w)$ and $\beta(u, v, w)$ defined in (6)) and as a determinant of a 3×3 matrix, possibly modifying the specialization of (7) by elementary rows and columns operations.

To express the cubic forms $T_1, \ldots, T_4$ as rational surfaces, we can use either the fact that each of them has a double point or the fact that each of their equations is linear in (at least) one variable. Moreover we can find, by direct (or *ad hoc*) computations, three matrices which give, respectively, T_2, T_3, T_4.

Hence we obtain the list in Table 1 (at the end of the paper), in which we have also added the coordinates of the singular points of T_i, for $i = 1, \ldots, 13$ (see also Figure 1) and the coordinates of the 6 points in $\mathbb{P}^2$, center of the blow–up which gives T_i.

As far as the c.s. $T_1 : xy^2 + yt^2 + z^3 = 0$ is concerned, we prove the following fact:

Proposition 2.6. *The polynomial*

$$F(x, y, z, t) = xy^2 + yt^2 + z^3$$

cannot be expressed as determinant of a 3×3 matrix whose entries are linear forms in x, y, z, t.

Proof. One way to follow could be to try the direct computation, i.e. to consider the 36 coefficients appearing in a general 3×3 matrix A of linear forms and to study the ideal I generated by the 20 equations (in the above 36 variables) obtained by imposing $\det(A) = F$. One could compute a Gröbner basis G of I and verify that $G = \{1\}$. Unfortunately this way seems to be unfeasible, since the computations involved are probably too large for any computer algebra system available. Hence we have to use a slightly different strategy; by "e.t." we mean elementary transformations on the rows and columns of a matrix.

Since the monomial xy^2 occurs in F, we can assume (up to e.t.) that the variable x appears with coefficient one in $A(1,1)$ and, clearly, we can "delete" it (by e.t.'s) from all the other elements in the first row and the first column of A; so that we start with the following matrix:

$$A := \begin{pmatrix} x + b_{11}y + c_{11}z + d_{11}t & b_{12}y + c_{12}z + d_{12}t & b_{13}y + c_{13}z + d_{13}t \\ b_{21}y + c_{21}z + d_{21}t & a_{22}x + b_{22}y + c_{22}z + d_{22}t & a_{23}x + b_{23}y + c_{23}z + d_{23}t \\ b_{31}y + c_{31}z + d_{31}t & a_{32}x + b_{32}y + c_{32}z + d_{32}t & a_{33}x + b_{33}y + c_{33}z + d_{33}t \end{pmatrix}.$$

Two possibilities occur: either $a_{22} = a_{23} = a_{32} = a_{33} = 0$ and so we get the matrix

$$A_1 := \begin{pmatrix} x + b_{11}y + c_{11}z + d_{11}t & b_{12}y + c_{12}z + d_{12}t & b_{13}y + c_{13}z + d_{13}t \\ b_{21}y + c_{21}z + d_{21}t & b_{22}y + c_{22}z + d_{22}t & b_{23}y + c_{23}z + d_{23}t \\ b_{31}y + c_{31}z + d_{31}t & b_{32}y + c_{32}z + d_{32}t & b_{33}y + c_{33}z + d_{33}t \end{pmatrix}$$

or at least one of these coefficients is non zero. In this second case (up to suitable e.t.) we assume that $a_{22} = 1$ and $a_{23} = a_{32} = 0$; if $A(3,3) \neq 0$, then a monomial containing x^2 arises in $\det(A)$, while x appears in F only at degree one. Thus we get the matrix:

$$A_2 := \begin{pmatrix} x + b_{11}y + c_{11}z + d_{11}t & b_{12}y + c_{12}z + d_{12}t & b_{13}y + c_{13}z + d_{13}t \\ b_{21}y + c_{21}z + d_{21}t & x + b_{22}y + c_{22}z + d_{22}t & b_{23}y + c_{23}z + d_{23}t \\ b_{31}y + c_{31}z + d_{31}t & b_{32}y + c_{32}z + d_{32}t & 0 \end{pmatrix}.$$

Let us first consider the matrix A_1: the monomial xy^2 must arise from the x at the place $(1,1)$, so at least two cross coefficients among $b_{22}, b_{23}, b_{32}, b_{33}$ are non–zero. As usual (up to e.t.) we can fix $b_{22} = b_{33} = 1$; moreover, using the entries $A_1(2,2)$ and $A_1(3,3)$ and a suitable sequence of e.t.'s, we can delete the y's in all the remaining places, except $(1,1)$; so we get a matrix of the form:

$$A_1' := \begin{pmatrix} x + b_{11}y + c_{11}z + d_{11}t & c_{12}z + d_{12}t & c_{13}z + d_{13}t \\ c_{21}z + d_{21}t & y + c_{22}z + d_{22}t & c_{23}z + d_{23}t \\ c_{31}z + d_{31}t & c_{32}z + d_{32}t & y + c_{33}z + d_{33}t \end{pmatrix}.$$

Finally note that the only monomial of F containing y^2 is xy^2 and this implies $b_{11} = c_{11} = d_{11} = 0$ in A_1' and so we obtain the matrix

$$A_1'' := \begin{pmatrix} x & c_{12}z + d_{12}t & c_{13}z + d_{13}t \\ c_{21}z + d_{21}t & y + c_{22}z + d_{22}t & c_{23}z + d_{23}t \\ c_{31}z + d_{31}t & c_{32}z + d_{32}t & y + c_{33}z + d_{33}t \end{pmatrix}.$$

Clearly, we have to require that

$$\det \begin{pmatrix} y + c_{22}z + d_{22}t & c_{23}z + d_{23}t \\ c_{32}z + d_{32}t & y + c_{33}z + d_{33}t \end{pmatrix} = y^2$$

and the solutions of this equation (easy to compute), once substituted in A_1'', give rise to matrices whose determinant cannot be $F(x, y, z, t)$.

Consider now the matrix A_2; the monomial xy^2 can arise from both the entries $(1,1)$ and $(2,2)$; obviously, using e.t.'s, we can assume that b_{23} and b_{32} are non–zero and, from this, that $b_{12} = b_{22} = b_{21} = 0$. In this way we get the matrix:

$$A_2' := \begin{pmatrix} x + b_{11}y + c_{11}z + d_{11}t & c_{12}z + d_{12}t & b_{13}y + c_{13}z + d_{13}t \\ c_{21}z + d_{21}t & x + c_{22}z + d_{22}t & b_{23}y + c_{23}z + d_{23}t \\ b_{31}y + c_{31}z + d_{31}t & b_{32}y + c_{32}z + d_{32}t & 0 \end{pmatrix}.$$

Note also that $b_{11} = 0$ since the monomial y^3 does not appear in F. Using $b_{23} \neq 0$ we can delete b_{13} by a rows transformation; in this way we introduce a monomial in x at $A_2'(1,2)$, but we can delete it by $A_2'(1,1)$. Let us remark that, at the end of this sequence of e.t.'s, the coefficient of y at the place $(3,2)$ must be still non–zero since the monomial xy^2 appears in F. In the same way, from $b_{32} \neq 0$ we can delete b_{31} and the coefficient of y at the place $(2,3)$ must be still non–zero. Now, up to scaling the third row and the third column, we get a matrix of the following type:

$$A_2'' := \begin{pmatrix} x + c_{11}z + d_{11}t & c_{12}z + d_{12}t & c_{13}z + d_{13}t \\ c_{21}z + d_{21}t & x + c_{22}z + d_{22}t & y + c_{23}z + d_{23}t \\ c_{31}z + d_{31}t & y + c_{32}z + d_{32}t & 0 \end{pmatrix}.$$

Observing that in F the only monomial containing y^2 is xy^2, we immediately obtain $c_{11} = d_{11} = 0$ in A_2''; again, an easy computation shows that no values of the coefficients of A_2'' give $\det(A_2'') = F$. $\square$

Remark 2.7. Since it is easy to see that any cubic surface in class $\mathcal{B}$ can be expressed as the determinant of a suitable matrix, from the previous results it follows that:
T_1 is the only cubic form (up to projectivities) of $K[x, y, z, t]$ that cannot be expressed as determinant of a 3×3 matrix of linear forms.

3. Study of $T_1, \dots, T_{13}$: stabilizers and orbits

So far we have seen a detailed study of the representatives of the orbits $\mathcal{O}_{T_1}, \dots, \mathcal{O}_{T_{13}}$ introduced in theorem 1.8. We want now to study the orbits themselves. If S is a cubic surface, the dimension of the orbit

$\mathcal{O}_S$ depends on the dimension of the *group of projective motions* of $S \subset \mathbb{P}^3$ (which is indeed the subgroup of PGL_4 stabilizing S):

$$\mathrm{Stab}(S) := \{g \in \mathrm{PGL}_4 \mid g(S) = S\}.$$

where $g(S)$ denotes as usual the cubic surface S transformed by g. More precisely

$$\dim(\mathcal{O}_S) = 15 - \dim(\mathrm{Stab}(S)).$$

Thus, in order to determine the dimensions of the orbits of $T_1, \ldots, T_{13}$, we have to study the groups of projective motions of each T_i.

Example. The cubic surface T_7 contains only the five lines $l_1 = (y, z)$, $l_2 = (x, y)$, $l_3 = (x, t)$, $l_4 = (x - t, y - z)$, $m = (x, y - z)$ of L_7^* and if $A \in \mathrm{Stab}(T_7)$, then clearly A is in the stabilizer of L_7^* and also in the stabilizer of the set of the three singular points $P_1 := (0, 0, 0, 1), P_2 := (0, 0, 1, 0), P_3 := (0, 1, 1, 0)$ of T_7 (see Figure 1 and Table 1).
We denote by $\mathrm{Stab}(L_7^*, \mathrm{sing})$ the subgroup of PGL_4 consisting of the matrices which map L_7^* and the singular locus $\{P_1, P_2, P_3\}$ of T_7 into themselves. It is easy to see that

$$\mathrm{Stab}(L_7^*, \mathrm{sing}) = H^{(7)} \cup g_2^{(7)} H^{(7)}$$

where

$$H^{(7)} := \left\{ \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & c-b & b & 0 \\ a & 0 & 0 & d-a \end{pmatrix} \text{ where } a, b, c, d \in K,\ (a-d)bcd \neq 0 \right\}$$

is the group fixing each line of L_7^* (and therefore each P_i) and $g_2^{(7)}$ is a matrix which stabilizes m, exchanges l_1 with l_4 and l_2 with l_3 (hence it exchanges the points P_1 and P_3, while mapping P_2 into itself), e.g.:

$$g_2^{(7)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

As we already noticed before, $\mathrm{Stab}(T_7)$ consists of the elements of $H^{(7)} \cup g_2^{(7)} H^{(7)}$ fixing the cubic surface T_7. Thus we look for the matrices A of $\mathrm{Stab}(L_7^*, \mathrm{sing})$ satisfying

$$T_7 \left(A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \right) = \lambda T_7 \ , \quad \text{for a suitable } \lambda \in K \setminus \{0\}.$$

Let us indicate the type of computation one has to perform: if $A \in H^{(7)}$, the above equality requires to solve the system

$$\begin{cases} bd^2 - \lambda & = 0 \\ abc - bcd + \lambda & = 0 \\ acd - cd^2 + \lambda & = 0 \end{cases}.$$

The case $A \in g_2^{(7)} H_1^{(7)}$ is analogous. A direct computation gives that $\mathrm{Stab}(T_7)$ is $H_1^{(7)} \cup g_2^{(7)} H_1^{(7)}$, where $H_1^{(7)}$ is the part of $\mathrm{Stab}(T_7)$ entirely contained in $H^{(7)}$, and precisely:

$$H_1^{(7)} = \left\{ \begin{pmatrix} ab & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & b(b-a) & ab & 0 \\ a(b-a) & 0 & 0 & a^2 \end{pmatrix} \mid a, b \in K, ab \neq 0 \right\}.$$

In particular, we see that it is a 1-dimensional group, therefore $\mathcal{O}_{T_7}$ is a subvariety of $\mathbb{P}^{19}$ of dimension 14.

As follows from the previous example, in order to compute the group $\mathrm{Stab}(T_i)$, it is useful to compute first $\mathrm{Stab}(L_i^*, \mathrm{sing})$, defined as the group of matrices of PGL_4 which map L_i^* to itself and the singular locus $\mathrm{Sing}(T_i)$ to itself. Let us describe how to proceed in general.
We shall denote by $H^{(i)}$ the subgroup of PGL_4 which stabilizes each line of L_i^* and each singular point of T_i, i.e.

$$H^{(i)} := \bigcap_{l \in L_i^*} \mathrm{Stab}(l) \quad \cap \quad \bigcap_{P \in \mathrm{Sing}(T_i)} \mathrm{Stab}(P).$$

Note that (as one can see in Figure 1) for every T_i but T_1 and T_2 each point of $\mathrm{Sing}(T_i)$ is the intersection of (at least) two lines of L_i^*, hence $H^{(i)} := \bigcap_{l \in L_i^*} \mathrm{Stab}(l)$, for every $i = 3, \dots, 13$.

Now consider the finite group $K^{(i)}$ of permutations π of the lines of L_i^* which preserve their incidence relations and stabilize the singular locus of T_i. The peculiar configurations $L_1^*, \dots, L_{13}^*$ allow us to realize any permutation π as a matrix of PGL_4, hence $K^{(i)}$ itself can be realized as a subgroup of PGL_4. Therefore $\mathrm{Stab}(L_i^*, \mathrm{sing}) = \langle H^{(i)}, K^{(i)} \rangle \subseteq \mathrm{PGL}_4$. Moreover, one can infer from Figure 1 that the following properties hold:

a) $hk = kh$, for any $h \in H^{(i)}$ and $k \in K^{(i)}$;
b) $H^{(i)} \cap K^{(i)} = \{1_{\mathrm{PGL}_4}\}$,

hence it is straightforward to see that

$$\mathrm{Stab}(L_i^*, \mathrm{sing}) \cong H^{(i)} \times K^{(i)}.$$

Moreover, as one can see from Figure 1, the groups $K^{(i)}$ are isomorphic either to S_2 or to S_3 or to a product of them. To underline the generators of the (factors of) the $K^{(i)} \subseteq \mathrm{PGL}_4$ we use the following notation: $S_2(g_2^{(i)})$ will mean the order two subgroup of PGL_4 generated by the matrix $g_2^{(i)}$, while $S_3(g_2^{(i)}, g_3^{(i)})$ is the order six subgroup of PGL_4 generated by those two elements, such that $\left(g_2^{(i)}\right)^2 = 1$ and $\left(g_3^{(i)}\right)^3 = 1$.
Finally set

$$H_1^{(i)} := \{g \in H^{(i)} \ : \ g(T_i) = T_i\}.$$

We present here the list of $\mathrm{Stab}(L_i^*, \mathrm{sing})$ and $\mathrm{Stab}(T_i)$ obtained.
(If a matrix contains some parameters, it is understood that it indicates the group of all matrices having that form, where the parameters vary in the field K in such a way that the matrix is non singular).

List of stabilizers

Surface T_1:

$$\mathrm{Stab}(L_1^*, \mathrm{sing}) = H^{(1)}, \quad \mathrm{Stab}(T_1) = H_1^{(1)}$$

where

$$H^{(1)} := \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & m_{44} \end{pmatrix} \quad H_1^{(1)} := \begin{pmatrix} a^6 & -a^4b^2 & 0 & -2a^5b \\ 0 & c^6 & 0 & 0 \\ 0 & 0 & c^4a^2 & 0 \\ 0 & c^3a^2b & 0 & c^3a^3 \end{pmatrix}$$

Surface T_2:

$$\mathrm{Stab}(L_2^*, \mathrm{sing}) \cong H^{(2)}, \quad \mathrm{Stab}(T_2) = H_1^{(2)}$$

where

$$H^{(2)} := \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & m_{44} \end{pmatrix} \quad H_1^{(2)} := \begin{pmatrix} a^4 & 0 & 0 & 0 \\ 0 & b^2a^2 & 0 & 0 \\ 0 & b^2ac & b^3a & 0 \\ 0 & -b^2c^2 & -2b^3c & b^4 \end{pmatrix}$$

Surface T_3:
$$\mathrm{Stab}(L_3^*,\mathrm{sing}) \cong H^{(3)} \times S_3(g_2^{(3)}, g_3^{(3)}), \quad \mathrm{Stab}(T_3) \cong H_1^{(3)} \times S_3(g_2^{(3)}, g_3^{(3)})$$
where:
$$H^{(3)} := \begin{pmatrix} g & a & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & c & f & 0 \\ 0 & d & 0 & e \end{pmatrix} \; g_2^{(3)} := \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; g_3^{(3)} := \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
and
$$H_1^{(3)} := \begin{pmatrix} ab^2 & 0 & 0 & 0 \\ 0 & ab\,(c+a) & 0 & 0 \\ 0 & abc & a^2b & 0 \\ 0 & 0 & 0 & (c+a)^3 \end{pmatrix}$$

Surface T_4:
$$\mathrm{Stab}(L_4^*,\mathrm{sing}) \cong H^{(4)} \times S_3(g_2^{(4)}, g_3^{(4)}), \quad \mathrm{Stab}(T_4) \cong H_1^{(4)} \times S_2(g_2^{(4)})$$
where
$$H^{(4)} := \begin{pmatrix} m_{11} & m_{12} & 0 & 0 \\ 0 & m_{22} & 0 & 0 \\ 0 & m_{32} & m_{11} & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \quad g_2^{(4)} := \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad g_3^{(4)} := \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
and $H_1^{(4)} = H_{1,1}^{(4)} \times S_2(f_2^{(4)})$, where
$$H_{1,1}^{(4)} := \begin{pmatrix} a^2 & -ab & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ -2ab & b(a+b) & ab & a^2 \end{pmatrix} \quad f_2^{(4)} := \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 1+i\sqrt{3} & 0 & 0 \\ 0 & 3+i\sqrt{3} & -2 & 0 \\ 6-2i\sqrt{3} & 0 & 0 & 1+\sqrt{3} \end{pmatrix}$$

Surface T_5:
$$\mathrm{Stab}(L_5^*,\mathrm{sing}) \cong H^{(5)}, \quad \mathrm{Stab}(T_5) = H_1^{(5)}$$
where
$$H^{(5)} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & e & c & 0 \\ f & 0 & 0 & d \end{pmatrix} \qquad H_1^{(5)} = \begin{pmatrix} a^2b^2 & 0 & 0 & 0 \\ 0 & b^4 & 0 & 0 \\ 0 & 0 & ab^3 & 0 \\ 0 & 0 & 0 & a^4 \end{pmatrix}$$

Surface T_6:
$$\mathrm{Stab}(L_6^*,\mathrm{sing}) \cong H^{(6)}, \quad \mathrm{Stab}(T_6) = H_1^{(6)}$$
where
$$H^{(6)} := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix} \qquad H_1^{(6)} := \begin{pmatrix} a^3 & 0 & 0 & 0 \\ 0 & ab^2 & 0 & 0 \\ 0 & b^2(b-a) & b^3 & 0 \\ 0 & 0 & 0 & a^2b \end{pmatrix}$$

Surface T_7:
$$\mathrm{Stab}(L_7^*,\mathrm{sing}) \cong H^{(7)} \times S_2(g_2^{(7)}), \quad \mathrm{Stab}(T_7) \cong H_1^{(7)} \times S_2(g_2^{(7)})$$
where
$$H^{(7)} := \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & c-b & b & 0 \\ a & 0 & 0 & d-a \end{pmatrix} \quad H_1^{(7)} := \begin{pmatrix} ab & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & b(b-a) & ab & 0 \\ a(b-a) & 0 & 0 & a^2 \end{pmatrix} \quad g_2^{(7)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Surface T_8:

$$\mathrm{Stab}(L_8^*, \mathrm{sing}) \cong H^{(8)} \times S_2(g_2^{(8)}), \quad \mathrm{Stab}(T_8) \cong H_1^{(8)} \times S_2(g_2^{(8)})$$

where

$$H^{(8)} := \begin{pmatrix} b+c & 0 & 0 & 0 \\ 0 & a+c & 0 & 0 \\ 0 & a & c & 0 \\ b & 0 & 0 & c \end{pmatrix} \quad g_2^{(8)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad H_1^{(8)} := \begin{pmatrix} (a+b)^2 & 0 & 0 & 0 \\ 0 & b(a+b) & 0 & 0 \\ 0 & ab & b^2 & 0 \\ a(a+2b) & 0 & 0 & b^2 \end{pmatrix}$$

Surface T_9:

$$\mathrm{Stab}(L_9^*, \mathrm{sing}) \cong H^{(9)} \times S_3(g_2^{(9)}, g_3^{(9)}), \quad \mathrm{Stab}(T_9) = S_3(g_2^{(9)}, g_3^{(9)})$$

where

$$H^{(9)} := \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ -a+b & 0 & 0 & a \end{pmatrix} \qquad g_2^{(9)} := \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix} \qquad g_3^{(9)} := \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

Surface T_{10}:

$$\mathrm{Stab}(L_{10}^*, \mathrm{sing}) \cong H^{(10)} \times S_3(g_2^{(10)}, g_3^{(10)}), \quad \mathrm{Stab}(T_{10}) \cong H_1^{(10)} \times S_3(g_2^{(10)}, g_3^{(10)})$$

where

$$H^{(10)} := \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & c-b & b & 0 \\ b-a & 0 & 0 & a \end{pmatrix} \qquad H_1^{(10)} := \begin{pmatrix} a^2b & 0 & 0 & 0 \\ 0 & a^3 & 0 & 0 \\ 0 & a^2(a-b) & a^2b & 0 \\ b(a^2-b^2) & 0 & 0 & b^3 \end{pmatrix}$$

$$g_2^{(10)} := \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & 1 & 1 \end{pmatrix} \qquad g_3^{(10)} := \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Surface T_{11}:

$$\mathrm{Stab}(L_{11}^*, \mathrm{sing}) \cong H^{(11)} \times S_2(g_2^{(11)}), \quad \mathrm{Stab}(T_{11}) = \{Id, h_1^{(11)}, h_2^{(11)}, h_3^{(11)}\} = \left\langle h_2^{(11)} \right\rangle$$

where

$$H^{(11)} := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & b-a & b & 0 \\ a-b & 0 & 0 & b \end{pmatrix} \qquad g_2^{(11)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$h_1^{(11)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix} \quad h_2^{(11)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1+\mathrm{i} & 1-\mathrm{i} & -\mathrm{i} & -\mathrm{i} \\ 1-\mathrm{i} & 0 & 0 & \mathrm{i} \end{pmatrix} \quad h_3^{(11)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1-\mathrm{i} & 1+\mathrm{i} & \mathrm{i} & \mathrm{i} \\ 1+\mathrm{i} & 0 & 0 & -\mathrm{i} \end{pmatrix}$$

Surface T_{12}:

$$\mathrm{Stab}(L_{12}^*, \mathrm{sing}) \cong H^{(12)} \times S_2(g_2^{(12)}), \quad \mathrm{Stab}(T_{12}) = S_2(g_2^{(12)})$$

where

$$H^{(12)} := \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & -b+a & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \qquad g_2^{(12)} := \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

Surface $T_{13}(p,q)$:

$$\mathrm{Stab}(L_{13}{}^*, \mathrm{sing}) \cong H^{(13)} \times S_3(g_2^{(13)}, g_3^{(13)}) \times S_2(f_2^{(13)})$$

where the above matrices are defined as follows: let

$$\begin{array}{rl} L_{13}{}^* = & [l_1 = (y,z),\ l_2 = (x,y),\ l_3 = (x,t),\ l_4 = (x-t, y-z), \\ & m_5 = (y, x-z),\ m_6 = (x, y-z),\ m_7 = (x-t, y-z+t)] \end{array}.$$

Then $g_2^{(13)}$ permutes l_4 with m_7 and l_1 with m_5, $g_3^{(13)}$ induces the cycles $l_4 \to m_7 \to l_3 \to l_4$ and $l_1 \to m_5 \to l_2 \to l_1$ and $f_2^{(13)}$ permutes the lines as follows: $l_1 \leftrightarrow l_4$, $m_5 \leftrightarrow m_7$, $l_2 \leftrightarrow l_3$; note that each of this matrices either fix or permutes the two singular points of T_{13}. With the computations described at the beginning of this section, we obtain:

$$H^{(13)} := \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & a-b & b & 0 \\ b-c & 0 & 0 & c \end{pmatrix}$$

$$g_2^{(13)} := \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}, \quad g_3^{(13)} := \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \quad f_2^{(13)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

The computation of $\mathrm{Stab}(T_{13}(p,q))$ is more complicated (indeed it depends on the parameters p and q) and it will be described at the end of Section 5. We anticipate that, for every p and q, it is a one–dimensional subgroup of PGL_4.

Therefore we have proved the following

Theorem 3.1. *The orbits of the c.ss. $T_1, \dots, T_{13}$ have the following dimensions:*

$$\dim(\mathcal{O}_{T_1}) = \dim(\mathcal{O}_{T_2}) = \dim(\mathcal{O}_{T_3}) = 13$$
$$\dim(\mathcal{O}_{T_9}) = \dim(\mathcal{O}_{T_{11}}) = \dim(\mathcal{O}_{T_{12}}) = 15$$

and the remaining ones have dimension 14. □

4. Smooth cubic surfaces: computation of the 27 lines and action of $\mathbb{E}_6$

Let us now consider the generic cubic surface S through the fixed L^*-set, i.e. belonging to $\mathbb{P}^4$:

$$a(2x^2y-2xy^2+xz^2-xzt-yt^2+yzt)+b(x-t)(xz+yt)+c(z+t)(yt-xz)+d(y-z)(xz+yt)+g(x-y)(yt-xz)=0. \tag{2}$$

Specializing (6) we obtain a parametric equation of S:

$$\begin{cases} x = \left((a+c-d)uv + (d-g)uw + (d+g)vw + (a-c-d)v^2 - 2aw^2\right) w \\ y = \left((a+b-c)u^2 + (a+b+c)uv - (b+g)uw - (b-g)vw - 2aw^2\right) w \\ z = \left((a+b-c)u^2 + (a+b+c)uv - (b+g)uw - (b-g)vw - 2aw^2\right) v \\ t = \left((a+c-d)uv + (d-g)uw + (d+g)vw + (a-c-d)v^2 - 2aw^2\right) u \end{cases} \tag{8}$$

and from 2.4 (and suitable elementary transformations on rows and columns) we obtain its cartesian equation (2) as the determinant of the matrix

$$\begin{pmatrix} a(z-t)+b(x-t)+d(y-z) & g(y-x)-c(z+t) & a(y-x) \\ y & y & z \\ x & -x & t \end{pmatrix}. \tag{9}$$

In this section we shall study smooth cubic surfaces which, as we noted in Section 1, can be parametrized by the open set $\mathbb{P}^4 \setminus \Sigma = \phi(\mathbb{P}^4 \setminus \Sigma)$. In the sequel we denote $\mathbb{P}^4 \setminus \Sigma$ by M_{L^*}.

We want to determine the 27 lines on $S \in M_{L^*}$. Let us try to apply Algorithm 2.3 starting from the lines l_1 and l_2 of the L^*-set.
Consider the planes passing through l_1. To find the five reducible residual conics it is useful to completely factorize the degree 5 polynomial $D_{l_1}(p,q)$. The pencil of planes containing l_1 is:

$$\pi_{l_1}(p,q): \quad py+qz=0.$$

The discriminant of the residual conic $C_{l_1}(p,q)$ is:

$$D_{l_1}(p,q) := q(p+q)(cp^2+gpq+aq^2)(\lambda p+\mu q)$$

where

$$\lambda := (2a+b-d)(a-c-d); \qquad \mu := -2a(a+b-c)-(b+d)(g-d)$$

The crucial remark is that the factor of degree 2 of D_{l_1} does not factorize with the initial choice of parameters; nevertheless as soon as we introduce the relations

$$g = e+f \qquad \text{and} \qquad ac = ef$$

we obtain:

$$c(cp^2+gpq+aq^2) = (cp-fq)(cp-eq).$$

In this way, up to a factor c which can be assumed non zero (since if $c=0$ then S is singular, as follows from the equation of Σ in 1.4), we get

$$D_{l_1}(p,q) = q(p+q)(cp-fq)(cp-eq)(\lambda p+\mu q)$$

and in addition the discriminants corresponding to l_2 and $s := \mathrm{res}\,(l_1,l_2)$ split into linear factors. This enables us to completely apply Algorithm 2.3 and compute the 27 lines of S, whose grassmannian coordinates in $G(1,3) \subset \mathbb{P}^5$ can be found in Table 2, at the end of the paper.

Remark 4.1. The above change of parameters corresponds to the following map: consider the projective space $\mathbb{P}^4$, with coordinates (a,b,c,d,g), parametrizing the family (2), the projective space $\mathbb{P}^5$, with coordinates (a,b,c,d,e,f) and the quadric $V \subset \mathbb{P}^5$ of equation $ac=ef$. The map

$$\psi: \quad \mathbb{P}^5 \supset V \longrightarrow \mathbb{P}^4 \quad \text{defined by} \quad (a,b,c,d,e,f) \mapsto (a,b,c,d,e+f)$$

is a two–to–one covering of $\mathbb{P}^4$ (in fact it is simply the projection of V from the point $(0,0,0,0,1,-1) \in \mathbb{P}^5 \setminus V$ onto the hyperplane $f=0$). Obviously, two distinct points $P=(a,b,c,d,e,f)$ and $P'=(a',b',c',d',e',f')$ of V represent the same cubic surface (i.e. have the same image on $\mathbb{P}^4$, via ψ) if and only if their coordinates fulfill $a=a', b=b', c=c', d=d', e=f', f=e'$ (up to a non zero factor).

In this way, the quadric V introduced above, still parametrizes the c.ss. through L^*; their equation (immediately obtained from (2)) is then:

$$\begin{aligned} &a(2x^2y-2xy^2+xz^2-xzt-yt^2+yzt)+b(x-t)(xz+yt)+ \\ &\quad +c(z+t)(yt-xz)+d(y-z)(xz+yt)+(e+f)(x-y)(yt-xz)=0 \end{aligned} \tag{2'}$$

where $(a,b,c,d,e,f) \in \mathbb{P}^5$ and fulfills the relation: $ac = ef$.

By $V_{\rm scs}$ we shall denote the open subset of the quadric V parametrizing the *smooth cubic surfaces* containing L^*. So $V_{\rm scs} = \psi^{-1}(\mathbb{P}^4 \setminus \Sigma) = \psi^{-1}(M_{L^*})$; in particular, any c.s. of M_{L^*} is represented by two points of $V_{\rm scs}$.
Clearly, we can lift in $\mathbb{P}^5$ the equation $\sigma = 0$, which defines Σ (see 1.4) and obtain the equation of $V \setminus V_{\rm scs}$.

Finally note that the singular locus of V, given by $a = c = e = f = 0$, corresponds to a component of $\mathcal{R}$ (see 1.6). Hence, in particular, $V_{\rm scs}$ is smooth.

An immediate consequence of the above construction is the possibility of obtaining all the (smooth) cubic surfaces having all the lines with coefficients in the rational field $\mathbb{Q} \subset K$. If $(a,b,c,d,e,f) \in \mathbb{P}^5(\mathbb{Q})$ is a point of $V_{\rm scs}$, the corresponding cubic surface S given by $(2')$ has all the lines with rational coefficients, as clearly follows from the explicit computation of the lines given in Table 2. Moving each of the cubic surfaces obtained in this way with the matrices with rational entries, we describe all the cubic surfaces whose lines are rational.

Conversely, given an explicit cubic surface S, it is possible to detect if its lines are all rational: one way is to compute the sub-variety $H \subseteq \mathcal{G}(1,3)$ (where $\mathcal{G}(1,3)$ denotes the Grassmannian of the lines of $\mathbb{P}^3$) given by the lines contained in S. The computations of the primary decomposition of the ideal defining H with one of the known techniques (see for instance [GTZ], [EHV] and the references given there), allows us to determine the points of H. A line of S is rational iff it is given by a point of H that can be expressed with rational coordinates.

The computation of the variety H is straightforward: if $F(x,y,z,t) = 0$ is the equation of S and if $(p_1,\ldots,p_6)$ denotes a point of $\mathcal{G}(1,3)$ (hence satisfying the condition $p_1p_6 - p_2p_5 + p_3p_4 = 0$), then four points in $\mathbb{P}^3$ of the line $(p_1,\ldots,p_6)$ are: $P_1 := (0,p_1,p_2,p_3)$, $P_2 := (-p_1,0,p_4,p_5)$, $P_3 := (-p_2,-p_4,0,p_6)$, $P_4 := (-p_3,-p_5,-p_6,0)$, so an ideal defining H is: $(F(P_1),\ldots,F(P_4),\ p_1p_6 - p_2p_5 + p_3p_4)$.

Remark 4.2. It is well–known that the generic smooth cubic surface has trivial stabilizer (see for instance [S], Ch. XIV). This fact can also be obtained from the computation sketched at the beginning of this section; indeed it is enough to show that a specific smooth c.s. has this property. One way to see this might be the following: take a specific smooth c.s. S having only rational lines; compute its lines and all the L-sets L_i $(i = 1,\ldots,25,920)$ of S. Then move S with all the matrices A_i which map L^* to L_i and verify that $A_i(S) \neq S$, for all $i = 1,\ldots,25,920$ (this computation can be done in a reasonable amount of time with the help of a computer algebra system).

The results obtained so far, allow us to conclude this section with the proof of Theorem 1.10.

Given a point $(a,b,c,d,e,f) \in V_{\rm scs}$, we denote by $\mathcal{L}$ the set $\{E_1,\ldots,E_6,G_1,\ldots,G_6,F_{12},\ldots,F_{56}\}$ of the 27 lines of the corresponding cubic surface $S := S(a,b,c,d,e,f) \in V_{\rm scs}$. The label attached to the lines follows the notation introduced, for instance, in [H] (Ch. V, 4), i.e. their incidence relations are

$$\begin{array}{ll} E_i \cap E_j = \emptyset \quad \text{for } i \neq j; & E_i \cap F_{kl} = \emptyset \ \Leftrightarrow \ i \neq k,l \\ G_i \cap G_j = \emptyset \quad \text{for } i \neq j; & G_i \cap F_{kl} = \emptyset \ \Leftrightarrow \ i \neq k,l \\ E_i \cap G_j = \emptyset \ \Leftrightarrow \ i = j; & F_{ij} \cap F_{kl} = \emptyset \ \Leftrightarrow \ \{i,j\} \cap \{k,l\} \neq \emptyset \end{array}$$

and, in particular, the incidence relations are independent of the smooth cubic surface $S := S(a,b,c,d,e,f)$. From the list of the lines in Table 2, we see that the L-set $L^* = (l_1,l_2,l_3,l_4,l_5)$ defined in (1) becomes (E_1,G_4,E_2,G_3,E_3).

Let $\mathbb{E}_6$ be the group of the permutations on the 27 lines on a smooth c.s. preserving the incidence relations; in particular, $\mathbb{E}_6$ is a subgroup of the symmetric group S_{27}. An easy computation shows that $|\mathbb{E}_6| = 51,840$ (for further details see, for instance, [H], Ch. V, 4.10.1). Note that the group $\mathbb{E}_6$ can be either regarded as an abstract group of permutations or as the group of permutations of the lines of any specific cubic surface $S := S(a,b,c,d,e,f) \in V_{\rm scs}$.

Notation. If $g \in \mathbb{E}_6$ and $l \in \mathcal{L}$, we denote by $g(l)$ the corresponding line of $\mathcal{L}$ obtained via the permutation g.

Note that any $g \in \mathbb{E}_6$ preserves the incidence relations of the lines of a cubic surface, in particular, if L is an L-set, then also $g(L)$ is an L-set.

Definition. Let $S := S(a,b,c,d,e,f) \in V_{\mathrm{scs}}$ be a fixed c.s.; for any $g \in \mathbb{E}_6$ let $A_g \in \mathrm{PGL}_4$ be the matrix (unique by lemma 1.2) such that:

$$A_g^{-1}(g(l)) = l, \quad \text{for } l \in \{E_1, G_4, E_2, G_3, E_3\}.$$

Let us remark that the matrix A_g depends not only on $g \in \mathbb{E}_6$, but also on the coordinates (a,b,c,d,e,f) of the surface S.
Since A_g is a projectivity, it clearly follows that:

$$A_g^{-1}(\mathrm{res}\,(l,m)) = \mathrm{res}\,(A_g^{-1}(l), A_g^{-1}(m)) \quad \text{for any } l,m \in \mathcal{L}.$$

From these relations and the definition of A_g it is easy to see that

$$A_g^{-1}(g(l)) = l, \quad \text{for all } l \in \mathcal{L} \text{ having the indices in } \{1,2,3,4\} \text{ or } l = F_{56}.$$

but the equality does not hold for all the elements of $\mathcal{L}$. This is essentially due to the fact that the indices 5 and 6 cannot be distinguished starting from L^*: e.g. E_5 and E_6 have the same incidence relations with all the lines of L^*. The same is true for the permutations belonging to $\mathbb{E}_6$: if $g, g' \in \mathbb{E}_6$ coincide on the lines of L^*, then they coincide only up to the exchange of the indices 5 and 6. Hence we denote by $\sigma \in \mathbb{E}_6$ the permutation which exchanges only the indices 5 and 6, i.e.

$$\begin{gathered}\sigma(l) = l, \quad \text{for } l \in \mathcal{L} \text{ having the indices in } \{1,2,3,4\} \text{ or } l = F_{56} \\ \sigma(E_5) = E_6,\ \sigma(E_6) = E_5,\ \sigma(G_5) = G_6,\ \sigma(G_6) = G_5, \\ \sigma(F_{i5}) = F_{i6},\ \sigma(F_{i6}) = F_{i5}, \quad \text{for any } i.\end{gathered}$$

The above observations immediately imply:

Proposition 4.3. *Let $S \in V_{\mathrm{scs}}$ and $g \in \mathbb{E}_6$; we have:*
i) if $g' \in \mathbb{E}_6$ coincides with g on the lines of L^*, then either $g' = g$ or $g' = \sigma g$;
ii) $A_g^{-1} = A_{\sigma g}^{-1}$. Moreover, either $A_g^{-1}(g(l)) = l$ for every $l \in \mathcal{L}$, or $A_g^{-1}(\sigma g(l)) = l$ for every $l \in \mathcal{L}$. □

Corollary 4.4. *The map $\mathbb{E}_6 \longrightarrow \{L\text{-sets of } S\}$ defined by $g \mapsto g(L^*)$ is surjective.*

Proof. It is enough to observe that if $g(L^*) = h(L^*)$ then (from 4.3) either $h = g$ or $h = \sigma g$ and conclude the proof by recalling (see 1.3) that $\sharp\{L\text{-sets of } S\} = 25{,}920 = |\mathbb{E}_6|/2$. □

Finally, from the list of the grassmannian coordinates of the 27 lines in Table 2, one can easily see that the exchange $e \leftrightarrow f$ in the equation $(2')$ gives exactly the permutation σ of the lines.
From this and taking into account 4.1, we obtain the following:

Proposition 4.5. *Let G_0 be the order two subgroup of $\mathbb{E}_6$ generated by σ. Consider the map*

$$\beta : G_0 \times V_{\mathrm{scs}} \longrightarrow V_{\mathrm{scs}}$$

defined by $(\mathbf{1}, (a,b,c,d,e,f)) \mapsto (a,b,c,d,e,f)$ and $(\sigma, (a,b,c,d,e,f)) \mapsto (a,b,c,d,f,e)$. Then β is an action of G_0 on V_{scs} and $V_{\mathrm{scs}}/G_0 \cong M_{L^*}$. □

Now let $G := \{g \in \mathbb{E}_6 \mid A_g^{-1}(g(l)) = l \text{ for all } l \in \mathcal{L}\}$. The following result holds:

Proposition 4.6. *For every $g, h \in G$ we have:*

$$A_{gh}^{-1} = A_h^{-1} \circ A_g^{-1}. \tag{10}$$

Moreover G is a normal subgroup of $\mathbb{E}_6$ of index two.

Proof. To see the above equality, it is enough to show (by lemma 1.2) that $A_h^{-1} \circ A_g^{-1}$ and A_{gh}^{-1} coincide on an L-set, e.g. on the specific one $(gh(E_1), gh(G_4)), gh(E_2), gh(G_3), gh(E_3))$. This easily follows, since for

every $l \in L^*$: $A_{gh}^{-1}(gh(l)) = l$ and $A_h^{-1}\left(A_g^{-1}(gh(l))\right) = A_h^{-1}(h(l)) = l$, by definition of the involved matrices and since $g \in G$.
From (10), we infer that if $g, h \in G$ then also $gh \in G$; in fact, for every line $l \in \mathcal{L}$:

$$A_{gh}^{-1}(gh(l)) = A_h^{-1}\left(A_g^{-1}(gh(l))\right) = l.$$

Moreover if $g \in G$, then $l = A_{g^{-1}}^{-1}\left(A_g^{-1}(l)\right) = A_{g^{-1}}^{-1}\left(A_g^{-1}(g(g^{-1}(l)))\right) = A_{g^{-1}}^{-1}\left(g^{-1}(l)\right)$ hence $g^{-1} \in G$, so G is a group. Finally, from 4.3 it follows that G is a subgroup of index two in $\mathbb{E}_6$, hence it is normal. □

Remark 4.7. Let $S \in M_{L^*}$ be a c.s.; then, as we have seen in 4.1, it is represented by two points of V_{scs} of coordinates (a,b,c,d,e,f) and (a,b,c,d,f,e), respectively. From 4.3 and 4.5 it follows immediately that the map $f_S : G \longrightarrow \mathrm{PGL}_4$ defined by $g \mapsto A_g^{-1}$ does not depend on the coordinates of S in V_{scs} (hence it is well–defined). Moreover, from (10), we have that f_S is an anti–monomorphism.

Note that, if S contains L^* and $g \in \mathbb{E}_6$, then the c.s. $A_g^{-1}(S)$ still contains L^*; in fact S also contains the L-set $g(L^*)$. Hence we have:

Theorem 4.8. *The map*

$$\alpha : G \times M_{L^*} \longrightarrow M_{L^*} \quad \textit{defined by} \quad (g, S) \mapsto A_g^{-1}(S)$$

where $A_g^{-1} := f_S(g)$, is an action of G on M_{L^*}; more precisely it is the restriction of the action of PGL_4 to M_{L^*}. In particular, every smooth c.s. of $\mathbb{P}^3$ can be represented, up to a projectivity, by a unique element of the quotient $\mathcal{M} := M_{L^*}/G$.

Proof. From (10), it immediately follows that $\alpha(gh, S) = \alpha(h, \alpha(g, S))$, so α is a group action. It remains to be shown that, if $S, T \in M_{L^*}$ are two projectively equivalent c.ss., then there exists $g \in G$ such that $T = A_g^{-1}(S)$. Since there exists $B \in \mathrm{PGL}_4$ such that $T = BS$, then T contains the L-set $B(L^*)$; by 4.4 $B(L^*) = A_g^{-1}(L^*) = A_{\sigma g}^{-1}(L^*)$, for a suitable $g \in \mathbb{E}_6$, so the projectivities B and A_g^{-1} coincide. But either g or σg are in G and this concludes the argument.
Finally, from the first part of the proof, it is clear that $\mathcal{M}$ parametrizes the smooth c.ss. up to projectivities and it has the structure of quasi–projective algebraic variety since it is a quotient of a quasi–projective algebraic variety by a finite group (see [Ha], Ch. 10). □

This concludes the **Proof of Theorem 1.10**.

Remark 4.9. As an immediate consequence of 4.8, we have that for any $S \in M_{L^*}$:

$$\mathcal{O}_S^G = \mathcal{O}_S^{\mathrm{PGL}_4} \cap M_{L^*}.$$

In particular, G acts transitively on $\mathcal{O}_S^{\mathrm{PGL}_4} \cap M_{L^*}$. If $S \in M_{L^*}$ is a general smooth cubic surface, then $\mathrm{Stab}(S) = \{1\}$ (see Remark 4.2); so also $\mathrm{Stab}_G(S) = \{1\}$, then $\sharp(\mathcal{O}_S^{\mathrm{PGL}_4} \cap M_{L^*}) = |G| = 25,920$.

5. Proofs of the theorems (I)

In this final section we collect the proofs of all the results stated in Section 1, but Theorem 1.9.

Proof of theorem 1.1. It suffices to show the following:

5.1. *Let S be an irreducible cubic surface having infinitely many lines and assume that S is not a cone. Then S has a double line.*

From 2.2 there exists a plane σ meeting S in a plane cubic curve Γ which is union of three lines (not necessarily distinct). Since any line of S meets σ, there exists a line, say r, among these three meeting infinitely many lines of S. Note that, since S is irreducible and not a cone, there exist infinitely many mutually skew lines meeting r; let m_1, m_2, m_3 be three of them. We may assume that, up to a projectivity,

$$r = (x, y); \quad m_1 = (y, z); \quad m_2 = (x, t); \quad m_3 = (x - y, z + t).$$

Moreover, again from 2.2 we conclude that the discriminant $D_r(p,q)$ of the residual conic $C_r(p,q)$ on the plane $\pi_r(p,q) : px + qy = 0$ must be zero for all p's and q's. From the explicit computation (of the type explained below) of $D_r(p,q)$, the obtained conditions force S to be either reducible or a c.s. with a double line. $\square$

Now we are going to prove theorems 1.8 and 1.3, 1.4, 1.5, 1.6.
We want to describe the equations (up to a linear change of coordinates) of the cubic surfaces that do not contain an L-set. To this end, we first prove that any c.s. contains at least one line and then we proceed in this way: we consider the following 5 configurations of lines

Figure 2

Note that each of these configurations is contained in the next one, that M_5 is an L-set and each of them can be fixed, if necessary, up to a projectivity by 1.2. We find the c.ss. that contain a configuration of lines M_i but do not contain a configuration M_{i+1} (for $i = 1, \ldots, 4$). Since, as we will see, any c.s. contains at least a configuration of type M_1, the union of all the cubic surfaces obtained in this way is the totality of c.ss. not passing through an L-set.

In order to determine these surfaces, we use the following strategy:

Step 1 Let us call $\mathcal{S}_i$ the linear system of all the c.ss. through a fixed set of lines M_i (and satisfying, if necessary, some other conditions that can still be fixed up to a projectivity).

Step 2 We analyze some necessary conditions on the parameters of $\mathcal{S}_i$, imposed by the fact that there are no other lines on $\mathcal{S}_i$ extending the fixed M_i to a configuration of type M_{i+1}. In this way we obtain sets of equations on the parameters of $\mathcal{S}_i$.

Step 3 We determine all the solutions of the equations obtained in Step 2. Call X_i the set of these solutions.

Step 4 We substitute each solution of X_i into the parameters of $\mathcal{S}_i$ and we study the families of surfaces obtained in this way. We usually find that for some elements of X_i the corresponding c.ss. are either reducible or cones or ruled surfaces; hence by 1.1 the remaining families of c.ss. (if they exist) contain a finite number of lines. We study their configurations of lines and collect those families of c.ss. which do not contain *any* M_{i+1}; we denote them by $\mathcal{S}_i^{(1)}, \mathcal{S}_i^{(2)}, \ldots$ etc.

Step 5 We analyze the families $\mathcal{S}_i^{(j)}$ obtained in Step 4 and we select specific c.ss. such that all their orbits cover the above families.

The tools needed in the above steps are essentially as follows:

- an algorithm for factorizing multivariate polynomials (over the field of rational numbers);
- Algorithm 2.3 to study the configuration of lines on the irreducible cubic surfaces obtained;
- a method for finding the solutions of Step 3 above, which is based on the fact that, every time we have to perform this step, at least one equation either factorizes or is linear in one variable. In this second case, the substitution of such a variable in the remaining equations gives rise to polynomials which factorize into polynomials, linear in some other variables. This procedure can be repeated until the complete resolution of the initial equations.

The above steps are repeated for $i = 1, \ldots, 4$. Here we summarize the results obtained after Step 4:

i) The cubic surfaces containing a system of lines M_1 but not M_2 are given by: $\mathcal{S}_1^{(1)}, \mathcal{S}_1^{(2)}$;

ii) The cubic surfaces containing a system of lines M_2 but not M_3 are given by: $\mathcal{S}_2^{(1)}, \mathcal{S}_2^{(2)}, \mathcal{S}_2^{(3)}$;

iii) The cubic surfaces containing a system of lines M_3 but not M_4 are given by: $\mathcal{S}_3^{(1)}, \ldots, \mathcal{S}_3^{(6)}$;

iv) The cubic surfaces containing a system of lines M_4 but not M_5 are given by: $\mathcal{S}_4^{(1)}, \dots, \mathcal{S}_4^{(11)}$.

(The explicit equations of the families $\mathcal{S}_j^{(i)}$'s are written below).

Then we give some examples and the outline of the procedure followed through Step 5, to obtain the list of the c.ss. $T_1, \dots, T_{13}$.

Finally, we study c.ss. containing an L-set, distinguishing the singular and the reducible ones.

Notation. If S is an irreducible c.s., then there exists a non-singular point $P \in S$; let $T_P(S)$ denote the tangent plane to S at the point P and $C_P = T_P(S) \cap S$ be the corresponding plane cubic curve, necessarily singular at P.

Before going on with the steps listed above, we need the following result:

A. *Every cubic surface contains a line.*

Here we give a self–contained proof of this well–known fact, using a method which is a suitable modification of [R1], Ch. 7, and that also gives an algorithm for an easy computation of the lines on a c.s. with numerical approximation.

5.2. *If a cubic surface S does not contain an infinite number of lines, then it contains either a plane nodal curve or a plane cuspidal curve.*

Clearly, the surface S is irreducible, then there exists a non-singular point $P \in S$. Three possibilities occur: either C_P is reducible (hence $C_P \subset S$ contains a line) or C_P is irreducible and has a node at P or C_P is irreducible and has a cusp at P. Hence, if S does not contain any plane nodal or cuspidal curve, then for any non-singular point $P \in S$, there exists a line l_P such that $P \in l_P \subset S$, i.e. S is a ruled surface; in particular, it contains infinitely many lines. $\square$

5.3. *All plane nodal curves are projectively equivalent.*

(see for instance [Ha], Ch.10).

5.4. *If an irreducible cubic surface S contains a plane nodal curve then, up to a projectivity, it has equation:*

$$a(x^3 - x^2z + y^2z) + bx^2t + cxyt + dy^2t + exzt + fyzt + gxt^2 + hyt^2 - izt(z+t) = 0. \tag{11}$$

By 5.3, we may assume that S passes through the fixed plane cubic curve $\Gamma = V(t, y^2z - x^2(z-x))$ and (as in the proof of 5.2) that S is smooth at $(0,0,1,0)$ (the node of Γ). Take now a line l through $(0,0,1,0)$ and not contained in the plane $\pi : t = 0$; hence l intersects S in two other distinct points A and B; it is easy to see that there exists a unique projectivity $F \in \mathrm{PGL}_4$ extending the identity on π and such that $F(A) = (0,0,0,1)$, $F(B) = (0,0,1,1)$. So we may assume that, up to a linear change of coordinates, S contains the plane cubic curve Γ and passes through the points $(0,0,0,1)$ and $(0,0,1,1)$; imposing these conditions on S we can see, by a direct computation, that S has equation (11).

5.5. *Let S be an irreducible c.s. of the above family; then S contains at least one line.*

Note that in equation (11) we can assume $a \neq 0$ (otherwise the corresponding polynomial has t as a factor). Consider the following parametrization of the curve Γ in 5.4: $\phi : \mathbb{P}^1 \longrightarrow \Gamma \subseteq S$ defined by:

$$\phi(u,v) := (4uv(v-u), 4uv(v+u), (u-v)^3, 0)$$

Note that $\phi(0,1) = \phi(1,0) = (0,0,1,0)$ is the node of Γ.
To characterize the points $X = (u,v) \in \mathbb{P}^1$ such that through $P := \phi(X) \in S$ passes a line of S, consider the plane cubic curve $C_P = S \cap T_P(S)$. Using (11) we can explicitly compute $T_P(S)$, where $P = P(u,v)$, obtaining the coefficients of the equation:

$$T_P(S) : \quad \alpha(u,v)x + \beta(u,v)y + \gamma(u,v)z + \delta(u,v)t = 0$$

where $\gamma(u,v) = au^3v^3 \neq 0$; in particular, this implies that S is smooth at each point $P \in \Gamma$. Thus a straightforward computation yields the equation of C_P.
After the change of coordinates in the plane $T_P(S)$:

$$\begin{cases} x = X + (v-u)Z \\ y = -X + (v+u)Z \\ t = 2vY \end{cases}$$

the point P becomes $(0,0,1)$. Let us call again (x,y,z) the new coordinates in $T_P(S)$ instead of (X,Y,Z); hence C_P has an equation of the form:

$$C_P: \quad f_3(x,y) + f_2(x,y)z = 0$$

where

$$f_3 = e_{11}y^3 + uve_{12}xy^2 + u^2v^2e_{13}x^2y + u^3v^3e_{14}x^3, \quad f_2 = g_{11}y^2 + uvg_{12}xy + u^2v^2g_{13}x^2.$$

and $e_{11}, e_{12}, e_{13}, g_{11}, g_{12}$ are suitable polynomials in u and v not divisible either by u and by v, while e_{14} and g_{13} are easier to write and are precisely $e_{14} = u^3$ and $g_{13} = v^3 - u^3$.
Clearly, the plane cubic C_P splits into a line and a conic if and only if f_3 and f_2 have a common factor, i.e. if and only if $\mathrm{Res}(f_2, f_3) = 0$, where $\mathrm{Res}(f_2, f_3)$ (the resultant of f_2 and f_3) is a polynomial in $a, \ldots, i, u, v$, homogeneous in u and v.
The explicit computation of it gives:

$$\mathrm{Res}(f_2, f_3) = u^9v^9H(u,v) \quad \text{where} \quad H(u,v) = i^6v^{27} + \cdots - i^6u^{27}.$$

Note that here we write only the coefficients of u^{27} and v^{27} (the entire polynomial is quite big and almost impossible to be obtained without the aid of a computer) but they are enough to show that $H(u,v)$ has a root different from $(1,0), (0,1), (0,0)$ and has degree at most 27. Namely $i \neq 0$, since the computation of the jacobian of the cubic surfaces of (11) with $i = 0$ gives that the point $(0,0,1,0)$ is singular, against our assumption. $\square$

This shows **A.** in the case that the surfaces contain a plane nodal curve. In the case of a cuspidal plane curve, we can proceed essentially in the same way.

Remark 5.6. 1) A slight modification of the above proof is enough to show that almost all cubic surfaces contain 27 lines: one has essentially to show that if through a point of Γ pass two lines, then $H(u,v)$ has a double root.
2) The above proof suggests an algorithm for the numerical computation of the lines on a given cubic surface S; one can follow this procedure: take first a point on S, compute the tangent plane to S through that point and finally intersect this plane with S. In general we get a plane nodal curve Γ, which can be parametrized by a map $\phi : \mathbb{P}^1 \longrightarrow \Gamma$. Following the argument of the above proof, we see that the lines on S correspond to the zero of a homogeneous polynomial in the two variables u, v, which are quite easy to find with numerical methods.

Now we can start with the first four Steps described at the beginning of this section.

B. *Study of the cubic surfaces containing a configuration M_1 but not a configuration M_2.*

We follow the steps and the notations introduced at the beginning of the section. Fix $M_1 := \{l_1\}$, where $l_1 = (y,z)$. Then the linear space $\mathcal{S}_1$ of all c.ss. passing through M_1 is given by (0) with the conditions

$$a_{11} = a_{12} = a_{13} = a_{20} = 0.$$

We are looking for the conditions on the coefficients a_i's that avoid the extension of M_1 to a configuration of type M_2. From lemma 2.2 (using the notation introduced in 2.1, i.e. $\pi_{l_1}(p,q)$ denotes the pencil of planes $py + qz = 0$) we know that the discriminant $D_{l_1}(p,q)$ cannot be identically zero and therefore it is a homogeneous polynomial in p, q of degree 5, so are defined at least one and at most five planes in $\mathbb{P}^3$. Up to a projectivity, we assume that the plane $y = 0$ is one of them, i.e. we assume that $(p, q) = (1, 0)$ is a solution of D_{l_1}. Moreover, the intersection of the plane $y = 0$ with any surface of $\mathcal{S}_1$ must be l_1^3, since we require that there are no lines meeting l_1. Hence the coefficients of $\mathcal{S}_1$ have to satisfy:

$$a_2 = a_6 = a_7 = a_{18} = a_{19} = 0.$$

Again we call $\mathcal{S}_1$ the subfamily of (0) fulfilling all these conditions:

$$\mathcal{S}_1 : \ a_1\, x^2y + a_3\, xy^2 + a_4\, xyz + a_5\, xyt + a_8\, y^2t + a_9\, yzt + a_{10}\, yt^2 + a_{14}\, y^3 + a_{15}\, y^2z + a_{16}\, yz^2 + a_{17}\, z^3 = 0.$$

Suppose now that there is another solution of D_{l_1} different from $(1, 0)$. Up to projectivity, we can assume that it is given by $(p, q) = (0, 1)$, i.e. by the plane $z = 0$. Again it must happen that $\{z = 0\} \cap S = l_1^3$ for every $S \in \mathcal{S}_1$. Hence the parameters of $\mathcal{S}_1$ have to satisfy:

$$a_1 = a_3 = a_5 = a_8 = a_{10} = 0.$$

However, using Algorithm 2.3, we infer that such S' have infinitely many lines. Therefore we are forced to consider the other possibility: all the 5 planes defined by D_{l_1} coincide with $y = 0$. The computation of the discriminants $D_{l_1}(p, q)$ for $\mathcal{S}_1$ give:

$$D_{l_1}(p, q) = q^2 \left(\alpha_3 p^3 + \alpha_2 p^2 q + \alpha_1 p q^2 + \alpha_0 q^3\right)$$

where

$$\begin{aligned}
\alpha_0 = &\ {a_5}^2 a_{14} - a_3 a_5 a_8 + a_1 {a_8}^2 + {a_3}^2 a_{10} - 4 a_1 a_{14} a_{10} \\
\alpha_1 = &\ a_3 a_5 a_9 - 2 a_1 a_8 a_9 + 4 a_1 a_{15} a_{10} - 2 a_4 a_3 a_{10} + a_4 a_5 a_8 - {a_5}^2 a_{15} \\
\alpha_2 = &\ a_1 {a_9}^2 - a_4 a_5 a_9 - 4 a_1 a_{16} a_{10} + {a_4}^2 a_{10} + {a_5}^2 a_{16} \\
\alpha_3 = &\ a_{17} \left(4 a_{10} a_1 - {a_5}^2\right)
\end{aligned}$$

and this polynomial has the only solution $(p, q) = (1, 0)$ if and only if $\alpha_3 = \alpha_2 = \alpha_1 = 0$. In order to simplify the resolution to these equations, we first note that $a_{17} = 0$ implies the reducibility of the surfaces of $\mathcal{S}_1$ (indeed $a_{17} = 0$ implies that the plane $y = 0$ is a component of any element of $\mathcal{S}_1$). Then, from $\alpha_3 = 0$ we get that either $a_1 = a_5 = 0$ or $a_{10} = a_5^2/(4a_1)$. From the first condition, it is straightforward to find the solutions. The second one, substituted in the first three equations, gives a system that can easily be solved as soon as one factorizes over $\mathbb{Q}$ the involved polynomials. All the solutions obtained are then:

$$X_1 := \left\{ [a_1 = 0, a_5 = 0, a_{10} = 0],\ [a_1 = 0, a_4 = 0, a_5 = 0],\ \left[a_{10} = \frac{a_5^2}{4\, a_1}, a_9 = \frac{a_4\, a_5}{2\, a_1} \right] \right\}.$$

(In this last one, a_1 is assumed $\neq 0$).
Now, as exposed in Step 4, we substitute each element of X_1 into the parameters of $\mathcal{S}_1$ and we analyze the surfaces obtained:

$$\begin{aligned}
& a_3\, xy^2 + a_4\, xyz + a_{14}\, y^3 + a_{15}\, y^2z + a_8\, y^2t + a_{16}\, yz^2 + a_9\, yzt + a_{17}\, z^3 = 0 \\
\mathcal{S}_1^{(1)} : \quad & a_3\, xy^2 + a_{14}\, y^3 + a_{15}\, y^2z + a_8\, y^2t + a_{16}\, yz^2 + a_9\, yzt + a_{10}\, yt^2 + a_{17}\, z^3 = 0 \\
\mathcal{S}_1^{(2)} : \quad & 4{a_1}^2 x^2 y + 4 a_3 a_1 x y^2 + 4 a_4 a_1 xyz + 4 a_5 a_1 xyt + 4 a_{14} a_1 y^3 + 4 a_{15} a_1 y^2 z + \\
& \qquad + 4 a_8 a_1 y^2 t + 4 a_{16} a_1 y z^2 + 2 a_4 a_5 yzt + {a_5}^2 y t^2 + 4 a_{17} a_1 z^3 = 0
\end{aligned}$$

Using Algorithm 2.3 we conclude that the first family consists of ruled (or reducible) surfaces, while the other ones are cubic surfaces containing the line l_1 and no other lines (when not reducible).

Therefore we have proved that, if a cubic surface does not contain a configuration of lines of type M_2, then, up to a linear change of coordinates, it is in one of the two families $\mathcal{S}_1^{(1)}$ or $\mathcal{S}_1^{(2)}$, which will be studied in Step 5, at the end of this section.

C. *Study of the cubic surfaces containing a configuration M_2 but not M_3.*

We can fix $M_2 := \{l_1 = (y,z), l_2 = (x,y)\}$; the c.ss. of (0) containing M_2 have to satisfy the conditions

$$a_{11} = a_{12} = a_{13} = a_{17} = a_{18} = a_{19} = a_{20} = 0$$

giving a family $\mathcal{S}_2$, say. We distinguish two cases: a) the line $\mathrm{res}\,(l_1, l_2)$ coincides with one the two lines of M_2 (say l_1, up to a projectivity); b) the lines $l_1, l_2, m := \mathrm{res}\,(l_1, l_2)$ are distinct.
a) The conditions in this case are:

$$a_2 = a_7 = 0$$

and they give a family $\mathcal{S}_{2a}$. If the configuration M_2 on a surface $S \in \mathcal{S}_{2a}$ cannot be extended to M_3, then one of the two possibilities occurs:
a_1) no other lines intersect l_1 or l_2, hence S contains only l_1 and l_2;
a_2) all the lines of S pass through the point $(0,0,0,1)$.

In case a_1), we first take the discriminant $D_{l_1}(p,q)$ of the conics lying on $\pi_{l_1}(p,q) : py + qz = 0$. This polynomial must have the only solution $(p,q) = (1,0)$. In fact, if not, we can assume that, up to a projectivity, the plane $y = z$ intersects S in three lines that must be all equal to l_1; hence the coefficients have to satisfy

$$a_1 = a_5 = a_{10} = a_8 + a_9 = a_3 + a_4 + a_6 = 0$$

and, from Algorithm 2.3, we easily see that in this case S contains infinitely many lines.
Then we take the discriminant $D_{l_2}(p,q)$ of the conics lying on $\pi_{l_2}(p,q) : py + qx = 0$. If $D_{l_2}(p,q)$ has the only solution $(p,q) = (1,0)$, we get a set of equations on the coefficients of S that, together with the above conditions given by $D_{l_1}(p,q)$, force all the corresponding c.ss. to be either reducible or RS.
Therefore we are left to study the case in which $D_{l_2}(p,q)$ has at least another solution different from $(1,0)$; up to projectivity, we can assume that it is $(0,1)$, corresponding to the plane $x = 0$. The assumption a_1) implies that $S \cap \{x = 0\} = l_2^3$. Hence we impose on $\mathcal{S}_{2a}$ this last condition which becomes

$$a_8 = a_9 = a_{10} = a_{15} = a_{16} = 0.$$

It turns out that these c.ss. already satisfy the initial requirement on $D_{l_1}(p,q)$ (i.e. to have $(p,q) = (1,0)$ as only solution); so we obtain the family:

$$\mathcal{S}_2^{(1)} : \quad a_1\, x^2y + a_3\, xy^2 + a_4\, xyz + a_5\, xyt + a_6\, xz^2 + a_{14}\, y^3 = 0.$$

Using 2.3 we see that these c.ss. (when irreducible) contain l_1, l_2 and no other lines.

In case a_2), we may assume that $S \in \mathcal{S}_{2a}$ contains also the line $m := (x,z)$, i.e. that $a_8 = a_{10} = a_{14} = 0$. By a_2), the lines $\mathrm{res}\,(l_1, m)$ and $\mathrm{res}\,(l_2, m)$ have to contain the point $(0,0,0,1)$; therefore, we obtain the further conditions: $a_5 = a_9 = 0$ and this forces S to be a cone.
b) We have two cases:
b_1) m does not pass through $(0,0,0,1)$ (e.g. $m = (y,t)$);
b_2) m passes through $(0,0,0,1)$ (e.g. $m = (y, x-z)$).

In case b_1), if M_2 does not extend to an M_3 configuration, then necessarily the c.s. contains only l_1, l_2, m; with the usual techniques involving the discriminants D_{l_1}, D_{l_2}, D_m we obtain the family

$$\mathcal{S}_2^{(2)} : \quad a_4a_5xy^2 - a_4a_5xyz - a_5{}^2xyt + a_5{}^2xzt - a_5a_{14}y^3 - a_4a_8y^2z - a_5a_8y^2t + a_5a_8yzt = 0.$$

In case b_2), if M_2 does not extend to an M_3 configuration, then either S has other lines (again through $(0,0,0,1)$), then S turns out to be a cone; or S contains only the above three lines. In the last situation we obtain the following family:

$$\mathcal{S}_2^{(3)} : \quad xy\,(y-z)\,a_4 - x\,(y-z)\,(x-y-z)\,a_6 + yt\,(y-z)\,a_9 - a_{14}y^3 - a_{15}y^2z - a_{16}yz^2 = 0.$$

D. *Study of the cubic surfaces containing a configuration M_3 but not M_4.*

Up to a linear change of coordinates, we may assume that the three lines of M_3 are

$$l_1 = (y, z); \quad l_2 = (x, y); \quad l_3 = (x, t)$$

so that the equation of the generic cubic surface S through them is:

$$\mathcal{S}_3 : \; ax^2y + bx^2z + cxy^2 + dxyz + exyt + fxz^2 + gxzt + hy^2t + iyzt + jyt^2 = 0.$$

5.7. *Let $S \in \mathcal{S}_3$; if M_3 cannot be completed to a configuration of type M_4, then necessarily the discriminant $D_{l_1}(p, q)$ has the only solution $(p, q) = (1, 0)$.*

If M_3 cannot be completed to an M_4, then
$i)$ either it does not exist a further line meeting l_1 and l_3;
$ii)$ or there exists a line $m \neq l_2$ meeting l_1 and l_3, but l_1, l_2, l_3, m is not an M_4 configuration.

$i)$ In this case the conic $C_{l_1}(p, q)$, residual to l_1 on the plane $\pi_{l_1}(p, q) : \; py + qz = 0$, is degenerate only on the plane $y = 0$, i.e. $D_{l_1}(p, q)$ has the only solution $(p, q) = (1, 0)$.
$ii)$ In this case, m must contain one of the two points $A := l_1 \cap l_2$ or $B := l_2 \cap l_3$. Clearly, up to a projectivity, we can assume that m passes through B and intersects l_1 in a point, say C, different from A. Suppose there exists a plane $\sigma \neq \{y = 0\}$ on which the residual conic to l_1 splits in two lines, one of them (say n) passes necessarily through $D := l_3 \cap \sigma$. Clearly, the set $\{l_1, l_2, l_3, m, n\}$ contains an M_4 configuration. Therefore the only plane on which $C_{l_1}(p, q)$ splits must be, also in this case, $y = 0$. □

The computation of the discriminant involved in 5.7 gives:

$$\begin{aligned}
D_{l_1}(p, q) &:= q\left(\alpha_4 p^4 + \alpha_3 p^3 q + \alpha_2 p^2 q^2 + \alpha_1 p q^3 + \alpha_0 q^4\right) \quad \text{where} \\
\alpha_0 =& \quad ah^2 + c^2j - ceh \\
\alpha_1 =& \quad 2ahi + bh^2 + 2cdj - cei - cgh - deh \\
\alpha_2 =& \quad ai^2 + 2bhi + 2cfj - cgi + d^2j - dei - dgh - efh \\
\alpha_3 =& \quad bi^2 + 2dfj - dgi - efi - fgh \\
\alpha_4 =& \quad f(fj - gi).
\end{aligned}$$

Clearly, the only solution of $D_{l_1}(p, q)$ is $(1, 0)$ if and only if the coefficients of S satisfy the system:

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.$$

Taking all the solutions of it (for the sake of shortness, we omit them), we obtain several families of reducible or ruled surfaces and six families, say $\mathcal{S}_3^{(1)}, \ldots, \mathcal{S}_3^{(6)}$, of irreducible cubic surfaces:

$$\begin{aligned}
\mathcal{S}_3^{(1)} :& \quad ax^2y + bx^2z + cxy^2 + exyt + gxzt + jyt^2 = 0 \\
\mathcal{S}_3^{(2)} :& \quad ahx^2y + gcx^2z + chxy^2 + ehxyt + ghxzt + h^2y^2t + jhyt^2 = 0 \\
\mathcal{S}_3^{(3)} :& \quad adhx^2y - d(cg - de)x^2z + cdhxy^2 + d^2hxyz + edhxyt + gdhxzt + h^2dy^2t + gh^2yt^2 = 0 \\
\mathcal{S}_3^{(4)} :& \quad (cgi + d^2j)x^2y + gdix^2z + ci^2xy^2 + di^2xyz + i(gh + 2dj)xyt + i^2gxzt + hi^2y^2t + i^3yzt + ji^2yt^2 = 0 \\
\mathcal{S}_3^{(5)} :& \quad ax^2y + cxy^2 + dxyz + fxz^2 + hy^2t = 0 \\
\mathcal{S}_3^{(6)} :& \quad 4afg^2x^2y + 4bfg^2x^2z + (2gdbi + g^2d^2 + b^2i^2 - 4aifg)xy^2 + 4dfg^2xyz + \\
& \quad + 2g^2(dg + 3bi)xyt + 4f^2g^2xz^2 + 4g^3fxzt - 2ig(-dg + bi)y^2t + 4ifg^2yzt + 4ig^3yt^2 = 0
\end{aligned}$$

It turns out that these ones have no M_4 configuration on them (when not RS).

E. *Study of the cubic surfaces containing a configuration M_4 but not a configuration M_5.*

We can assume, up to a projectivity, that the cubic surface S passes through the lines

$$l_1 = (y, z); \quad l_2 = (x, y); \quad l_3 = (x, t); \quad l_4 = (x - t, y - z)$$

and let $A = l_1 \cap l_2 = (0, 0, 0, 1)$, $B = l_1 \cap l_4 = (1, 0, 0, 1)$, $C = l_3 \cap l_4 = (0, 1, 1, 0)$, $D = l_3 \cap l_2 = (0, 0, 1, 0)$. So $\mathcal{S}_4$ has equation:

$$ax^2(y-z)+bx(y-z)(y+z)+cy^2(x-t)+dy(x-t)(x+t)+ey(yx-zt)+fxy(y-z)+gxy(x-t)+hx(yx-zt)=0$$

If $S \in \mathcal{S}_4$ does not contain an L-set, then necessarily one of the following two cases occurs:

1. the only degenerate conics on S residual to l_2 are on the two planes $x = 0$ and $y = 0$ and the only degenerate conics on S residual to l_1 are on the two planes $y = 0$ and $y = z$;
2. there exists a third plane π passing through l_1 (resp. l_2) with a degenerate residual conic on it.

We are going to study these cases separately: case 1 in 5.8 and 5.9, case 2 in 5.10, respectively.

5.8. *Let S be an irreducible cubic surface of the family $\mathcal{S}_4$; assume that the conic $C_{l_2}(p, q)$ on the plane $\pi_{l_2}(p, q)$: $py + qx = 0$ is degenerate only if $(p, q) \in \{(1, 0), (0, 1)\}$. Then S contains a square of lines such that one of the vertices is a singular point of S; in particular, up to a projectivity, S has equation*

$$ax^2(y - z) + bx(y - z)(y + z) + cy^2(x - t) + dy(xy - zt) + exy(y - z) + fxy(x - t) + gx(xy - zt) = 0 \quad (12)$$

and $(0, 0, 0, 1)$ is a singular point.

The discriminant of $C_{l_2}(p, q)$ is:

$$\begin{aligned}
D_{l_2}(p, q) := pq\left(\alpha_3 p^3 + \alpha_2 p^2 q + \alpha_1 pq^2 + \alpha_0 q^3\right), &\quad \text{where} \\
\alpha_0 = \quad & (c + e)\left(bc - be - e^2 - ef\right) \\
\alpha_1 = \quad & ace + ae^2 - 4b^2d - 4bcd - 2bcg - 4bde - 4bdf + 2beh + 2ceh+ \\
& + cfh + de^2 - df^2 + e^2g + 3e^2h + efg + 2efh \\
\alpha_2 = \quad & 4abd - ach + 2adf - aeg - 2aeh + 4bd^2 + 4bdg + 4bdh + bg^2- \\
& - bh^2 - ch^2 - 2deh - 2egh - 3eh^2 - fgh - fh^2 \\
\alpha_3 = \quad & (a + h)\left(ad - dh - gh - h^2\right).
\end{aligned}$$

From the assumption, all the roots of $D_{l_2}(p, q)$ must belong to the set $\{(1, 0), (0, 1)\}$, i.e. the coefficients α_i's must satisfy one of the following four systems of equations:

$$\alpha_3 = \alpha_2 = \alpha_1 = 0; \quad \alpha_3 = \alpha_2 = \alpha_0 = 0; \qquad \alpha_1 = \alpha_0 = \alpha_3 = 0; \quad \alpha_1 = \alpha_0 = \alpha_2 = 0.$$

In particular, the parameters must satisfy either $\alpha_3 = \alpha_2 = 0$ or $\alpha_1 = \alpha_0 = 0$.
Note that if we exchange a with c, b with d, e with h and f with g in α_3 and α_2, we get, respectively, α_0 and α_1. This gives that it is enough to study only the case $\alpha_3 = \alpha_2 = 0$. All the solutions in $a, \ldots, h$ to this system give rise to reducible cubic surfaces except the following five ones (where the denominators can be assumed not zero):

$$[a = -h, g = -h - 2d]; \quad \left[a = -h, g = \frac{hf + he + hb - 2\,bd}{b}\right]; \quad [d = 0, h = 0, g = 0]; \quad \left[d = 0, h = 0, b = \frac{ae}{g}\right]$$
$$\left[a = \frac{h^2 + gh + dh}{d}, \quad c = \frac{-2\,h^2e + 3\,hbd + hdf - geh - deh + 2\,d^2b + gbd}{h^2}\right]$$

which give rise, respectively, to five families of cubic surfaces (say $V_1, \ldots, V_5$). It is easy to see that V_1 and V_2 are singular at the point $(1, 0, 0, 1)$, V_3 and V_4 are singular at the point $(0, 0, 0, 1)$, while V_5 is singular at the point $P := (-d, 0, 0, h + g + d)$.

As far as $V_1, \dots, V_4$ are concerned, we can conclude that one of the vertices of the square l_1, l_2, l_3, l_4 is singular.
As far as V_5 is concerned, note that the singular point is not one of the four vertices A, B, C, D of the given square; nevertheless, the following lines $l_1, l_2, \mathrm{res}\,(l_2, l_3), \mathrm{res}\,(l_1, l_4)$ give rise to an M_4 configuration (in fact the condition that two vertices coincide implies $d = 0$ or $h = 0$, which in this case is not possible) and moreover the singular point P is one of the vertices ($P = \mathrm{res}\,(l_2, l_3) \cap \mathrm{res}\,(l_1, l_4)$).
So we prove the claim and, up to a linear change of coordinates, we can assume that the surface S passes through $l_1, \dots, l_4$ as above and is singular at the point $A = (0, 0, 0, 1)$. In this way we get the equation (12).
□

5.9. *Let S be an irreducible cubic surface of the family* (12) *and let $\pi_{l_1}(p, q)$: $p(y - z) + qy = 0$ and $\pi_{l_2}(p, q)$: $py + qx = 0$ be the pencil of planes of center l_1 and l_2, respectively. Assume that the three non trivial roots of $D_{l_i}(p, q)$, for $i = 1, 2$, belong to the set $\{(1, 0), (0, 1)\}$. Then S belongs to one of the following families $\mathcal{S}_4^{(1)}, \dots, \mathcal{S}_4^{(8)}$ of irreducible singular cubic surfaces:*

$$
\begin{aligned}
\mathcal{S}_4^{(1)}: &\quad abx(xy + xz - 2yt) + aexy(x - t) - b^2(y - z)(xy + xz - 2yt) - \\
&\quad - bey(x - t)(y - z) + bgx(2xy - 3yt + zt) + egxy(x - t) = 0 \\
\mathcal{S}_4^{(2)}: &\quad bfx^2(y - z) + c^2yt(y - z) - cexy(y - z) - cfxy(x - t) - bcx(y - z)(y + z) = 0 \\
\mathcal{S}_4^{(3)}: &\quad ax^2(y - z) + ey(x - t)(y - z) + fxy(x - t) = 0 \\
\mathcal{S}_4^{(4)}: &\quad ax^2(y - z) + b(y - z)(xy + xz - 2yt) + ey(x - t)(y - z) + fxy(x - t) = 0 \\
\mathcal{S}_4^{(5)}: &\quad ax^2(y - z) + bx(y - z)(y + z) + cy^2(x - t) + exy(y - z) = 0 \\
\mathcal{S}_4^{(6)}: &\quad bx(y - z)^2 - cy^2(x - t) - fx(x - t)(y - z) = 0 \\
\mathcal{S}_4^{(7)}: &\quad ax^2(y - z) + cy^2(x - t) - fxt(y - z) = 0 \\
\mathcal{S}_4^{(8)}: &\quad cy^2(x - t) + exy(y - z) + fx(x - t)(y - z) = 0
\end{aligned}
$$

As usual we compute $D_{l_1}(p, q)$ and $D_{l_2}(p, q)$ and impose the suitable conditions on the coefficients of these polynomials. In this way we obtain (out of the reducible ones) the above families of cubic surfaces arising, respectively, from the following sets of conditions on the coefficients of (12):

$$
\left[d = -c, c = 2b + e, f = -\frac{2ab + ae + 3bg + eg}{b}\right], \ [d = -c, a = -bf/c, g = 0], \ [d = -c, b = 0, c = e, g = 0]
$$
$$
[d = -c, c = 2b + e, g = 0], \ [g = 0, d = 0, f = 0], \ [d = 0, g = -f, a = f, e = -2b]
$$
$$
[b = 0, d = 0, e = 0, g = -f], \ [b = 0, d = 0, g = -f, a = f].
$$

□

5.10. *Let $S \in \mathcal{S}_4$ and assume that there exists a plane, say π, through l_1, out of $y = 0$ and $y - z = 0$, on which the residual conic to l_1 is degenerate. Then there exists a line (on π), say m_5, meeting both l_1 and l_3. Moreover, one of the following two cases occurs:*

i) if m_5 contains one of the vertices of M_4, then either S contains an L-set or it belongs to one of the following three families $\mathcal{S}_4^{(9)}, \mathcal{S}_4^{(10)}, \mathcal{S}_4^{(11)}$ of irreducible singular cubic surfaces:

$$
\begin{aligned}
\mathcal{S}_4^{(9)}: &\quad ax(y - z)(x - t) - bx(y - z)^2 + cy(y + z)(x - t) = 0 \\
\mathcal{S}_4^{(10)}: &\quad ax^2(y - z) + cy(2xy - yt - zt) + dxy(y - z) = 0 \\
\mathcal{S}_4^{(11)}: &\quad ax^2(y - z) + bx(y - z)(y + z) + cy(y + z)(x - t) = 0;
\end{aligned}
$$

ii) if m_5 does not contain any vertex, then either S contains an M_5 configuration or it is in the class $\mathcal{B}$.

Call w and w' the two lines such that $\pi \cap S = l_1 \cup w \cup w'$; then one (say m_5) among w and w' meets l_3.
i) Assume first that m_5 contains a vertex of the square. Then, up to a projectivity, we may assume that m_5

contains the vertex A (and not C, since $\pi \neq \{y-z=0\}$, by assumption); so we can choose $m_5 = (x, y+z)$. Note that in this case the point A is singular. The equation of the generic cubic surface passing through l_1, l_2, l_3, l_4, m_5 is then:

$$x^2(y-z)a + x(y-z)(y+z)b + y(2xy-yt-zt)c + xy(y-z)d + xy(x-t)e + x(xy-zt)f = 0.$$

Let us remark that l_1, l_2, l_3, l_4 and l_1, m_5, l_3, l_4 are two configurations of type M_4; if S does not contain an L-set then, in particular, they cannot be completed into an L-set.
Let $m_6 := \mathrm{res}\,(l_1, m_5)$; in the general case it meets only l_1 among the lines of the first square, giving an L-set. To avoid this situation, one of the following conditions must hold:

$$m_6 = l_1; \quad m_6 = m_5; \quad m_6 \cap l_2 \neq \emptyset; \quad m_6 \cap l_3 \neq \emptyset; \quad m_6 \cap l_4 \neq \emptyset.$$

It is clear that the first two conditions are contained in the other ones, which give, respectively: $e = f,\ d = -c,\ f = -a$.
Let $m_7 := \mathrm{res}\,(l_1, l_2)$; in the general case it meets only l_1 among the lines of the second square, giving an L-set. As in the previous case, the conditions $m_7 = l_1$; $m_7 = l_2$; $m_7 \cap l_3 \neq \emptyset$; $m_7 \cap l_4 \neq \emptyset$; $m_7 \cap m_5 \neq \emptyset$ can be reduced to $b = 0,\ f = -a,\ f = 0$. Hence we can conclude that, if S does not contain an L-set, then one of the following systems of equations must be satisfied:

$$f = -a, \quad \begin{cases} b = 0 \\ e = f \end{cases} \quad \begin{cases} b = 0 \\ d = -c \end{cases} \quad \begin{cases} e = 0 \\ f = 0 \end{cases} \quad \begin{cases} f = 0 \\ d = -c \end{cases}$$

which give, respectively, the following families of surfaces:

$$\begin{aligned}
Y_1:&\quad xz(x-t)a - x(y-z)(y+z)b - y(2xy-yt-zt)c - xy(y-z)d - xy(x-t)e = 0\\
Y_2:&\quad x^2(y-z)a + y(2xy-yt-zt)c + xy(y-z)d + x(2xy-yt-zt)e = 0\\
Y_3:&\quad x^2(y-z)a + y(y+z)(x-t)c + xy(x-t)e + x(xy-zt)f = 0\\
Y_4:&\quad x^2(y-z)a + x(y-z)(y+z)b + y(2xy-yt-zt)c + xy(y-z)d = 0\\
Y_5:&\quad x^2(y-z)a + x(y-z)(y+z)b + y(x-t)(y+z)c + xy(x-t)e = 0.
\end{aligned}$$

With the Algorithm 2.3 we can compute all the lines of Y_1; studying their incidence relations, it is possible to see that, for generic coefficients Y_1 contains an L-set, unless

$$e = -\frac{a(b+c+d)}{b}, \quad d = -2b - c\,;$$

these conditions give rise to the family $\mathcal{S}_4^{(9)}$. Analogous computations show that Y_2 gives rise (with $f = 0$) to the subfamily $\mathcal{S}_4^{(10)}$ and, finally, Y_4 gives rise (with the condition $d = -c$) to the subfamily $\mathcal{S}_4^{(11)}$.
ii) Assume now that m_5 does not contain any vertex of the square. In this case m_5 is skew w.r.t. l_2 and l_4; up to a projectivity we may assume that $m_5:\ t = 0 = z$. The equation of the generic cubic surface passing through l_1, l_2, l_3, l_4, m_5 is

$$(xz^2 - y^2t)a + xz(x-t)f + (x^2z - yt^2)b + x(xz-yt)e - xz(y-z)d + z(xz-yt)c = 0.$$

With the same kind of computations performed in the previous case, one can show that the surfaces of this family either contain an L-set on are in the class $\mathcal{B}$. □

In particular, since all the c.ss. obtained above are singular, we have proved that

5.11. *Every smooth c.s. contains an L-set.* □

(This is part of the forthcoming proof of 1.3).

F. *Study of irreducible cubic surfaces not containing an L-set.*

So far we have collected several families of cubic surfaces $\mathcal{S}_i^{(j)}$ which correspond to those not containing certain configurations of lines. Now we want to determine suitable representatives (and their orbits) of these families (according to Step 5 at the beginning of this section). Here we summarize the method used to obtain the list (3).

Let $\mathcal{U}$ be one of the above families of cubic surfaces.

Step 5_1 We determine, using Algorithm 2.3, the lines of the cubic surfaces of $\mathcal{U}$, we study their configuration in the generic case and we collect the degenerate cases which can occur (adding them to the collection of the above families $\mathcal{S}_i^{(j)}$).

Step 5_2 We check if the configuration of lines of the generic cubic surface of $\mathcal{U}$ can be fixed up to projectivity. If this is possible (and it will turn out that this is indeed possible for all the cases to be considered), we choose specific equations for them, obtaining a configuration of lines, say $L^*_{\mathcal{U}}$.
Then we construct a family $\mathcal{U}'$: take the linear system of all the cubic surfaces passing through $L^*_{\mathcal{U}}$ and impose the requirement that the sections with the planes containing two incident lines of $L^*_{\mathcal{U}}$ be of the same type of the corresponding plane sections of $\mathcal{U}$. In particular, from this construction it follows that $\mathcal{U}$ is contained in $\mathcal{U}'$ up to projectivity.
In general the family $\mathcal{U}'$ is simpler to describe than the family $\mathcal{U}$, since the lines of its surfaces are fixed and do not depend on the parameters. (We anticipate that the configuration of lines obtained in this step are those listed in (4)).

Step 5_3 We compute the group $G := \bigcap_{l\in L^*_{\mathcal{U}}} \mathrm{Stab}(l)$.

Step 5_4 We fix a specific cubic surface T in $\mathcal{U}'$ and we check whether moving it with the elements of G we obtain all the elements of $\mathcal{U}'$. If this is the case, $\mathcal{O}_T$ (the orbit of T under the action of PGL_4) contains the whole family $\mathcal{U}$ and we are done. It will turn out that there is only one case in which this does not occur: it is the case of the family T_{13}, studied below.

We observe that Step 5_3 is almost done in Section 3, where we list the groups $H^{(i)}$; indeed, for $i = 3, \ldots, 13$, $H^{(i)} = \bigcap_{l\in L^*_i} \mathrm{Stab}(l)$.
Here we give some examples of the application of the above procedure and then we list all the results.

First example. Consider the family

$$\mathcal{U} := \mathcal{S}_3^{(5)} : \quad ax^2y + cxy^2 + dxyz + fxz^2 + hy^2t = 0.$$

Step 5_1: these cubic surfaces contain only the lines (x, y), (y, z), (x, t) by construction. The intersection of any cubic surface S of $\mathcal{S}_3^{(5)}$ with the plane $y = 0$ is given by fxz^2. If $f = 0$, then we have a degenerate case to consider (which is trivial here, since it corresponds to reducible cubic surfaces). Otherwise we use Algorithm 2.3 applied to the lines (x, y) and (y, z), to compute all the lines of S. If f, h and a are not zero, then S contains exactly the three lines above. Moreover the case $h = 0$ again gives reducible surfaces. Therefore we collect only the family obtained from the condition $a = 0$ (this is again a trivial case, since Algorithm 2.3 immediately shows that these surfaces are ruled surfaces).
Step 5_2 in the present case is unnecessary, since the lines are already fixed. A family $\mathcal{U}'$ meeting the requirement of Step 5_2 is then the family $\mathcal{S}_3^{(5)}$ itself.
Step 5_3: the group G is here $H^{(5)}$ (see Section 3), i.e.

$$H^{(5)} := \left\{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & \epsilon & \gamma & 0 \\ \phi & 0 & 0 & \delta \end{pmatrix} \;\Big|\; \alpha, \ldots, \phi \in K,\ \alpha\beta\gamma\delta \neq 0 \right\}$$

Step 5_4: take a specific cubic surface in the family $\mathcal{U}' = \mathcal{S}_3^{(5)}$, given e.g. by $a = f = h = 1, c = d = 0$ and call it T_5. If we move it with the general matrix A of $H^{(5)}$ we obtain:

$$A(T_5): \quad \alpha^2\beta x^2y + \left(\beta^2\phi + \alpha\epsilon^2 + \alpha\beta^2 + \alpha\beta\epsilon\right) xy^2 + \alpha\gamma\,(2\,\epsilon + \beta)\, xyz + \alpha\gamma^2 xz^2 + \beta^2\delta y^2 t = 0. \tag{13}$$

It is easy to see (equating the coefficients) that for those values of the parameters a, c, d, f, h which give rise to a cubic surface S with exactly the three lines fixed above, there exists a matrix $A \in H^{(5)}$ such that $A(T_5)$ is S. Hence in this way we have shown that the orbit of the cubic surface T_5 consists of all the cubic surfaces containing a configuration of lines of type L_5.

Second example. We now take the family

$$\mathcal{U} := \mathcal{S}_3^{(1)} : \quad ax^2y + bx^2z + cxy^2 + exyt + gxzt + jyt^2 = 0.$$

Step 5_1: from the construction that led to $\mathcal{S}_3^{(1)}$ we know that every $S \in \mathcal{S}_3^{(1)}$ passes through the three lines $(x, y), (y, z)$, and (x, t). The intersection of S with the plane $y = 0$ is: $xz(bx + gt)$. Now we apply Algorithm 2.3 to S and we obtain the list of its lines (for simplicity we call them $m_1, \dots, m_6$ although m_1, m_2, m_3 are indeed l_1, l_2, l_3):

$$m_1 := (x, y),\ m_2 := (y, z),\ m_3 := (x, t),\ m_4 := (y, bx + gt),$$
$$m_5 := \left(\alpha x + cg^2y, bcgy - \alpha t\right),\ m_6 := \left(\alpha x + cg^2y, cg(jb - eg)y + j\alpha t - cg^3z\right)$$

where we have denoted by α the coefficient $ag^2 + b^2j - beg$. The incidence relations of these lines are the following:

$$\begin{array}{c}
\\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5
\end{array}
\begin{bmatrix}
m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\
* & (0,0,0,1) & (0,0,1,0) & (0,0,1,0) & (0,0,1,0) & (0,0,j\alpha, cg^3) \\
 & * & \emptyset & (g,0,0,-b) & \emptyset & \emptyset \\
 & & * & (0,0,1,0) & (0,0,1,0) & \emptyset \\
 & & & * & (0,0,1,0) & \emptyset \\
 & & & & * & (-cg^4, \alpha g^2, (2bj - eg)\alpha, bcg^3)
\end{bmatrix}$$

From the intersections written above, we obtain the following degenerate cases:

-) $m_2 \cap m_4 = (0,0,0,1)$: then $g = 0$. It turns out that each element of the corresponding family is projectively equivalent to T_5.

-) $m_1 \cap m_6 = (0,0,1,0)$: this happens either if $g = 0$ or if $c = 0$. The case $g = 0$ has already been considered; the case $c = 0$ gives a family of ruled surfaces, as one can verify intersecting a surface of this family with any plane passing through the line (y, z): the section always splits into three lines.

-) $m_1 \cap m_6 = (0,0,0,1)$: then $j\alpha = 0$, thus $\alpha = 0$ (since $j = 0$ implies S reducible). If $b = 0$, then we get $a = 0$, since S is irreducible; the corresponding c.ss. are represented by T_6. Otherwise, $j = (beg - ag^2)/b^2$; also in this case, the corresponding surfaces are projectively equivalent to T_6.

It is clear that special positions of the point $m_5 \cap m_6$ lead to conditions already considered.

Step 5_2: in the generic case, the six lines $m_1, \dots, m_6$ are disposed as in configuration L_{11} and they can be fixed up to projectivity: hence we assume they are the lines of the configuration L_{11}^*. Now we take the generic cubic surface passing through L_{11}^* and we impose on it all the conditions that are satisfied by the surfaces of the family $\mathcal{S}_3^{(1)}$, i.e. $\operatorname{res}(m_1, m_3) = m_3$. In this way we obtain the family:

$$\mathcal{U}' : \quad ax^2y + bx^2z - axy^2 + bxyt - bxzt - byt^2 = 0. \tag{14}$$

According to Step 5_2, this family is described by fewer parameters and hence it is simpler to analyze.

Step 5_3: the group G is here $H^{(11)}$ (see Section 3) i.e.

$$H^{(11)} := \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & \beta - \alpha & \beta & 0 \\ \alpha - \beta & 0 & 0 & \beta \end{pmatrix}$$

and we move, as in Step 5_4, a specific surface of $\mathcal{U}'$ (e.g. the surface T_{11}, obtained in case $a = 1, b = 1$) by a matrix $A \in H^{(11)}$:

$$A(T_{11}) : \quad \alpha^3x^2y + \beta^2\alpha\, x^2z - \alpha^3xy^2 + \beta^2\alpha\, xyt - \beta^2\alpha\, xzt - \beta^2\alpha\, yt^2 = 0.$$

A comparison of the coefficients of this family of surfaces with the family (14) shows that, up to projectivity, T_{11} represents any surface of $\mathcal{U}'$. Hence we have proved that $\mathcal{O}_{T_{11}}$ describes all the cubic surfaces with a configuration of lines of type L_{11} and, in particular, the generic element of the family $\mathcal{S}_3^{(1)}$.

Third example. Let us consider the family

$$\mathcal{U} := \mathcal{S}_4^{(4)}: \quad ax^2(y-z)+b(y-z)(xy+xz-2yt)+ey(x-t)(y-z)+fxy(x-t)=0.$$

Step 5_1 and 5_2: the configuration of the lines of the generic element of $\mathcal{S}_4^{(4)}$ is of type L_{13} and can be fixed up to projectivity: we refer to L_{13}^* as in (4). The family $\mathcal{U}'$, consisting of the c.ss. having, in general, exactly L_{13}^* as configuration of lines, is:

$$\mathcal{U}': \quad X\,(y-z)\left(x^2-xy-xz+2\,yt\right)+Y y\,(x-t)\,(y-z)+Zxy\,(x-t)=0.$$

In particular, we can identify $\mathcal{U}'$ with a two dimensional space $\mathbb{P}^2$ (with coordinates X, Y, Z).
Step 5_3: contrary to what we have done in the previous cases, we now take for G the whole group

$$G := \mathrm{Stab}({L_{13}}^*) \cong H^{(13)} \times S_3(g_2^{(13)}, g_3^{(13)}) \times S_2(f_2^{(13)})$$

of matrices which map L_{13}^* to itself.
Step 5_4: consider the action of G on the irreducible cubic surfaces of $\mathcal{U}'$. Since the reducible c.ss. of $\mathcal{U}'$ have infinitely many lines, then necessarily at least one of the three discriminants D_{l_1}, D_{l_2}, D_m (where $m := (y, x-z) = \mathrm{res}\,(l_1, l_2)$) is zero. It turns out that this occurs if and only if (X, Y, Z) belongs to the plane quartic curve

$$Q: \quad XZ(Y-2X)(2X-Y+Z)=0.$$

Conversely, if $(X, Y, Z) \in Q$ then the corresponding c.s. of $\mathcal{U}'$ is reducible, as follows from the factorization of the involved polynomials.

Therefore we have to consider the action of G on $\mathcal{U}' \setminus Q$. Moreover, it is easy to see that $f_2^{(13)}(S) = S$, for any $S \in \mathcal{U}'$; so the orbit $\mathcal{O}_S^G$ of S under the action of G coincides with that obtained under the action of $S_3(g_2^{(13)}, g_3^{(13)})$.

If $S = (a, b, c)$ is an irreducible cubic surface of $\mathcal{U}'$, it is easy to compute its orbit $\mathcal{O}_S^G$ which turns out to be the union of the following six lines in $\mathbb{P}^2$:

$$2cX-cY-(2a-b)Z, \quad 2cX-cY+(2a-b+c)Z, \quad 2(2a-b)X-(2a-b)Y+(2a-b+c)Z$$

$$2(2a-b+c)X-(2a-b+c)Y+(2a-b)Z, \quad 2(2a-b)X-(2a-b)Y-cZ, \quad 2(2a-b+c)X-(2a-b+c)Y+cZ.$$

From this we see that (although G has been taken as large as possible) no 0-dimensional subvariety of $\mathcal{U}' \setminus Q$ can intersect all the orbits, hence, in order to parametrize the orbits, we are forced to consider a subvariety of $\mathcal{U}'$ of dimension at least 1. Take, for instance, the line of equation $Y = 0$. If we set $X = q$ and $Z = p$, we obtain exactly the one–dimensional subfamily of cubic surfaces of $\mathcal{U}'$:

$$T_{13}(p, q): \quad pxy(x-t)+q(y-z)(x^2-xy-xz+2yt)=0$$

which is irreducible iff $pq(2q+p) \neq 0$ (as follows by intersecting Q with the line $Y = 0$). For simplicity let us call W this open set of the line $Y = 0$ in $\mathcal{U}' \cong \mathbb{P}^2_{X,Y,Z}$; in this way we introduce the variety $\mathbb{P}^1 \setminus \Delta$ (where $\Delta := \{(1,0), (0,1), (-2,1)\}$) and we identify it with $W \subset \mathcal{U}'$. From the above argument, it is clear that the irreducible surfaces of the initial family $\mathcal{U}$ are represented by $\mathcal{U}' \setminus Q$ and that this space is parametrized by $\mathbb{P}^1 \setminus \Delta$. In fact, if $S \in \mathcal{U}' \setminus Q$ is an irreducible c.s., then

$$\mathcal{O}_S \cap (\mathcal{U}' \setminus Q) = \mathcal{O}_S^G \quad \text{hence} \quad \mathcal{O}_S \cap W = \mathcal{O}_S^G \cap W.$$

In particular, any orbit $\mathcal{O}_S$ intersects $W \cong \mathbb{P}^1 \setminus \Delta$ (this gives the part of the **Proof of Theorem 1.8** concerning $\mathbb{P}^1 \setminus \Delta$). More precisely, $\mathcal{O}_S \cap W$ consists (in general) of 6 points, hence 6 points of W belong to the same orbit. We can explicitly compute their coordinates: if $S = (a, c)$ is a generic point of W, then $\mathcal{O}_S \cap W$ consists of:

$$(a, c), \quad (2a + c, -2c), \quad (2a + c, -4a), \quad (-a, 2a + c), \quad (c, 4a), \quad (-c, 4a + 2c).$$

(This gives the **Proof of Theorem 1.12**, since S_3 acts on $\mathbb{P}^1 \setminus \Delta$ and the orbits of this action are the above sets of 6 points).
This concludes the study of the family $\mathcal{S}_4^{(4)}$.

The above description allows us to compute $\mathrm{Stab}(S)$, where $S := T_{13}(p, q)$ and $pq(2q + p) \neq 0$. As in Section 3, we compute $H_1^{(13)} := \mathrm{Stab}(S) \cap H^{(13)}$, obtaining:

$$H_1^{(13)} := \left\{ \begin{pmatrix} rs & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & r\,(r-s) & rs & 0 \\ s\,(r-s) & 0 & 0 & s^2 \end{pmatrix} \;\middle|\; r, s \in K, \; rs \neq 0 \right\}$$

We have two cases to distinguish: if $4q+p \neq 0$ then it is easy to verify that $\mathrm{Stab}(S) \cap \left(H^{(13)} \times S_3(g_2^{(13)}, g_3^{(13)})\right)$ is again $H_1^{(13)}$, so taking into account that $f_2^{(13)}(S) = S$ we obtain that

$$\mathrm{Stab}(S) \cong H_1^{(13)} \times S_2(f_2^{(13)}).$$

If $4q + p = 0$, then the equation of S is:

$$3x^2y + x^2z + xy^2 - 4xyt - xz^2 - 2y^2t + 2yzt = 0.$$

For this specific cubic surface, $\mathrm{Stab}(S) \cap g_2^{(13)} H^{(13)} = g_2^{(13)} H_1^{(13)}$; more precisely it turns out that

$$\mathrm{Stab}(S) \cong H_1^{(13)} \times S_2(g_2^{(13)}) \times S_2(f_2^{(13)}).$$

5.12. *With the techniques used in the above examples we obtain that:*

- *each element of* $\mathcal{S}_1^{(1)}$ *and* $\mathcal{S}_1^{(2)}$ *is projectively equivalent to* T_1*;*
- *each element of* $\mathcal{S}_2^{(1)}$ *is p.e. to* T_2*;*
- *each element of* $\mathcal{S}_2^{(2)}$ *is p.e. to* T_3*;*
- *each element of* $\mathcal{S}_2^{(3)}$ *is p.e. to* T_4*;*
- *the generic elements of* $\mathcal{S}_3^{(1)}, \mathcal{S}_3^{(2)}, \mathcal{S}_3^{(3)}, \mathcal{S}_3^{(4)}, \mathcal{S}_3^{(6)}$ *are p.e. to the c.s.* T_{11}*;*
- *two special subfamilies of* $\mathcal{S}_3^{(1)}, \mathcal{S}_3^{(2)}$ *are represented by* T_6*;*
- *the generic elements of* $\mathcal{S}_3^{(5)}$ *and of a subfamily of* $\mathcal{S}_3^{(1)}$ *are p.e. to* T_5*;*
- *each element of* $\mathcal{S}_4^{(1)}$ *and* $\mathcal{S}_4^{(2)}$ *is p.e. to* T_{12}*;*
- *each element of* $\mathcal{S}_4^{(3)}$ *and* $\mathcal{S}_4^{(8)}$ *is p.e. to* T_7*;*
- *the generic element of* $\mathcal{S}_4^{(4)}$ *is p.e. to* $T_{13}(p, q)$*, while a subfamily of* $\mathcal{S}_4^{(4)}$ *is represented by* T_7*;*
- *the generic element of* $\mathcal{S}_4^{(5)}$ *is p.e. to* T_9*, while a subfamily of* $\mathcal{S}_4^{(5)}$ *is represented by* T_{10}*;*
- *each element of* $\mathcal{S}_4^{(6)}$ *and* $\mathcal{S}_4^{(7)}$ *is p.e. to* T_8*;*
- *the generic element of* $\mathcal{S}_4^{(9)}, \mathcal{S}_4^{(10)}, \mathcal{S}_4^{(11)}$ *is p.e. to* T_{12} *again;*
- *two subfamilies of* $\mathcal{S}_4^{(10)}$ *and* $\mathcal{S}_4^{(11)}$*, respectively, are represented by* T_8
  (here "each" means "each, when not reducible"). $\square$

The above study concludes the **Proof of Theorem 1.8**.

G. *Study of cubic surfaces containing an L-set.*

5.13. *Let S be an irreducible cubic surface and let $r \subset S$ be a line such that $D_r(p,q) \not\equiv 0$ and the planes $\pi_1(r), \dots, \pi_5(r)$ are distinct. Then S is smooth out of r.*

This can easily be checked as follows: suppose that S is singular at a point $P \notin r$; up to a projectivity we can assume $r = (x,y)$ and $P = (1,0,0,0)$. We impose these conditions on (0) and we compute $D_r(p,q)$. By factorizing it, we get that two planes among $\pi_1(r), \dots, \pi_5(r)$ coincide. $\square$

5.14. *Let S be an irreducible cubic surface, $r \subset S$ be a line and let π be a plane through r such that the cubic curve $\pi \cap S$ splits into three lines. If these three lines are not distinct, then S is singular.*

Let us first assume that the three lines all coincide with $r = (x,y)$. This gives 5 linear conditions on (0). The computation of the Jacobian shows that, in this case, the surface must be singular.
Assume now that there are exactly two distinct lines, $r = (x,y)$ and $s = (z,y)$, on the plane section $\pi \cap S$ and assume also that $\operatorname{res}(r,s) = r$. Imposing these conditions on (0), we can see again that the obtained surfaces are singular. $\square$

5.15. *Let S be an irreducible surface containing an L-set. The following conditions are equivalent:*
i) S is smooth;
ii) for any line r contained in S, $D_r(p,q) \not\equiv 0$, the planes $\pi_1(r), \dots, \pi_5(r)$ are distinct and on any plane $\pi_i(r)$ there are two distinct lines out of r, for $i = 1, \dots, 5$;
iii) if s_1 and s_2 are two distinct meeting lines on S, then $D_{s_i}(p,q) \not\equiv 0$, the planes $\pi_1(s_i), \dots, \pi_5(s_i)$ are distinct for $i = 1, 2$ and S is smooth at the point $s_1 \cap s_2$;
iv) the number of lines contained in S is 27.

i) $\Rightarrow$ *ii)* Let us assume first that there exists a line r such that two planes among $\pi_1(r), \dots, \pi_5(r)$ coincide. Clearly, up to a projectivity we may assume $r = (x,y)$ and that the double root of $D_r(p,q)$ is $(1,0)$ (corresponding to the plane $y = 0$). If we impose these conditions on the generic cubic surface, we see at once that S has at least a singular point. Using 5.14 we conclude the proof.
ii) $\Rightarrow$ *i)* Since S contains an L-set of lines, say $L := (l_1, \dots, l_5)$, then it cannot be a cone and l_1, l_2, l_3 form an M_3-type configuration. By assumption the planes $\pi_1(l_i), \dots, \pi_5(l_i)$ $(i = 1, 3)$ are distinct, so by 5.13, S is smooth out of the line l_1 and out of the line l_3; hence S is smooth everywhere.
ii) $\Rightarrow$ *iii)* Obvious, since we have already proved the equivalence *i)* $\Leftrightarrow$ *ii)*.
iii) $\Rightarrow$ *i)* Immediate, from 5.13.
i – ii – iii) $\Rightarrow$ *iv)* Since S contains an L-set, then S contains two incident lines, say s_1 and s_2. From the assumption *ii)*, $s_3 := \operatorname{res}(s_1, s_2)$ is distinct from s_1 and s_2. Let us apply the argument of Algorithm 2.3 to this plane section $s_1 \cup s_2 \cup s_3$: again from *ii)*, through each s_i pass 4 distinct planes (out of the plane containing s_1, s_2, s_3) on which lie two distinct lines of S (out of s_i). Hence there are 8 distinct lines (out of the three starting lines), say $s_i^{(1)}, \dots, s_i^{(8)}$, meeting s_i.
Finally let us notice that $s_i^{(k)}$ does not meet s_j, if $i \neq j$, for any $k = 1, \dots, 8$, otherwise either $s_1, s_2, s_3, s_i^{(k)}$ are four coplanar lines of S (while S is irreducible) or $s_1, s_2, s_3, s_i^{(k)}$ must meet at the same point; but this is impossible since S is smooth, by the assumption *i)*.
Therefore there are exactly 24 distinct lines, out of s_1, s_2, s_3, on the surface S.
iv) $\Rightarrow$ *i)*. Assume that S is singular at a point P; since S contains an L-set, then there are two meeting lines, say s_1 and s_2, on S such that $P \notin s_1$. Let us apply the argument of Algorithm 2.3 to find all the lines on S, starting from the plane section $s_1 \cup s_2 \cup s_3$ ($s_3 := \operatorname{res}(s_1, s_2)$ not necessarily distinct from s_1 and s_2). From 5.13, the planes $\pi_1(s_1), \dots, \pi_5(s_1)$ are not distinct, hence the number of lines of S cannot exceed 25, against the assumption. $\square$

Now we want to find explicit expressions for the conditions in 5.15. To this end, let us assume that S contains $L^* = (l_1, l_2, l_3, l_4, l_5)$ (defined in (1)), so S is in (2). Let $A = l_1 \cap l_2 = (0,0,0,1)$. From 5.15 we immediately obtain :

5.16. *Let S be a cubic surface of the family (2). Then S is smooth if and only if the discriminants $D_{l_1}(p,q), D_{l_2}(p,q)$ have no multiple roots and S is smooth at $A := l_1 \cap l_2$. In particular, S is smooth if and only if $\sigma \neq 0$, where σ is defined in thm. 1.4.*

Computing the two discriminants we get:

$$\begin{aligned}
&D_{l_1}(p,q) := \\
&\quad p(p-q)(ap^2-gpq+cq^2)\big[(2a^2+2ab-2ac-bd+bg-d^2+dg)p+(a-c-d)(2a+b-d)q\big] \\
&D_{l_2}(p,q) :=\ pq(p-q) \\
&\quad \big[(a+c+g)(a^2+ac+2ba-ag+b^2-bg)p^2+ \\
&\quad +(2a^2(a+b-6c-d)-ab(6c+2d-g)+2ac(c+3d)+ag(d+2g)+2bcd-cg(b+d)+g^2(b-d))pq+ \\
&\quad +(a+c-g)(a^2+ac-2ad+ag+d^2-dg)q^2\big]
\end{aligned}$$

Moreover $D_{l_1}(p,q)$ and $D_{l_2}(p,q)$ have multiple roots if and only if their discriminants (w.r.t. p and q) vanish, i.e.:

$$\mathrm{Dis}(D_{l_1}(p,q),p) = 0 = \mathrm{Dis}(D_{l_1}(p,q),q) \qquad (a)$$

$$\mathrm{Dis}(D_{l_2}(p,q),p) = 0 = \mathrm{Dis}(D_{l_2}(p,q),q) \qquad (b)$$

The polynomial σ is given by the square–free l.c.m. of the polynomials appearing in (a) and (b), together with the condition $a+b-c=0$ (equivalent to A singular). □

Proof of Theorem 1.3. We already noticed that, if S is a smooth c.s., then it contains an L-set (from 5.11). To count the number of its L-sets, we can use 5.15 as follows:
- the line l_1 can be chosen in 27 different ways;
- the line l_2 (meeting l_1) can be chosen in 10 different ways;
- the line l_3 (meeting l_2 and skew with l_1) can be chosen in 8 different ways;
- the line l_4 (meeting l_1, l_3 and skew with l_2) can be chosen in 4 different ways;
- the line l_5 (meeting l_2 and skew with l_1, l_3, l_4) can be chosen in 3 different ways.

So $\sharp\{L\text{-sets}\} = 27\cdot 10\cdot 8\cdot 4\cdot 3 = 25,920$. □

Proof of Theorem 1.4. It follows from the equivalence $i) \Leftrightarrow iv)$ of 5.15 and from 5.16.

Proof of Theorem 1.5. It is enough to prove that cones and RS' do not contain an L-set. Clearly, this is true for cones. Now let S be a RS; as in 5.1 we can assume that S contains the lines r, m_1, m_2, m_3 and r is a double line. If we intersect S with any plane $\pi_r(p,q):\ px+qy=0$ containing r, then the computation shows that $\pi_r(p,q)\cap S = r^2 \cup m(p,q)$, where $m(p,q)$ is the corresponding residual line. In particular, any line of S meets either r or m_1. Hence, in order to obtain all the lines of S, it is enough to compute the lines meeting m_1. With the usual procedure, it turns out that two possibilities arise: either there is a further line (skew with r) and meeting all the lines $m(p,q)$'s (general case) or there are no further lines on S (these c.ss. are classically known as *Cayley ruled cubic surfaces*). It is clear that, in both cases, S does not contain an L-set. □

Proof of Theorem 1.6. We want to find the conditions characterizing the reducible surfaces in the family (2). From 1.5 and 1.1, it follows that a surface S passing through an L-set contains infinitely many lines if and only if it is reducible. Hence it is enough to find out for what values of the parameters $a,\ldots,g$ the corresponding c.s. of (2) contains infinitely many lines. Note first that $\mathrm{res}\,(l_2,l_5)$ is distinct from both l_2 and l_5 (in fact l_4 intersects the plane $\langle l_2+l_5\rangle$ in a point not belonging to $l_2\cup l_5$). Therefore, if S contains infinitely many lines, then either l_2 or l_5 or $\mathrm{res}\,(l_2,l_5)$ must meet infinitely many lines, i.e.

$$\text{either}\quad D_{l_2}(p,q)\equiv 0 \quad\text{or}\quad D_{l_5}(p,q)\equiv 0 \quad\text{or}\quad D_{\mathrm{res}\,(l_2,l_5)}(p,q)\equiv 0.$$

Studying these equations, we obtain the following:

5.17. *Let S be a cubic surface of the family (2). Then S is reducible (hence it contains a plane) if and only if one of the following groups of conditions is satisfied:*

$$\begin{cases} b=-d \\ a=c+d \end{cases},\quad \begin{cases} c=a+b \\ g=2a+b \end{cases},\quad \begin{cases} a=0 \\ c=0 \\ g=0 \end{cases},\quad \begin{cases} d=2a+b \\ c=-\dfrac{(a+b)(a+b-g)}{a} \end{cases},$$

which give rise, respectively, to the following subfamilies of (2)*:*

$$
\begin{gathered}
(y-x)\,(2c(xy-zt)+d(2xy-xz-yt)-g(xz-yt))=0\\
(y-z)\,\big(2a(x^2+xt-yt)+b(xz+2xt-yt)+d(xz+yt)\big)=0\\
(b(x-t)+d(y-z))\,(xz+yt)=0\\
(a(x-y+z+t)+b(z+t))\,(2ay(x-t)+(b-g)(xz-yt))=0.
\end{gathered}
$$

□

6. Proofs of the theorems (II): singular cubic surfaces containing an L-set

In this last section we want to classify singular irreducible cubic surfaces containing an L-set; we have already parametrized them by the subvariety $\Sigma \setminus \mathcal{R}$ of $\mathbb{P}^4$; moreover, from the result of Section 2 we know that each of them can be expressed as the determinant of a matrix of linear forms (see Prop. 2.4) and we can get its rational equation.

Now we use an approach quite similar to that considered in Section 5, which gave the parameter space $\mathbb{P}^4$ and the exceptional cases $T_1, \dots, T_{13}$. Most of the required computations has been already done in Section 5, so we only sketch them.

For example, to classify c.ss. with one singular point we consider the configurations $M_1, \dots, M_5$ in Figure 2 and we assume that one intersection point of two lines of M_i, for $i = 2, 3, 4, 5$ (resp. one point of M_1) is singular. Following Step $1, \dots, 5$ of Section 5, one can rather easily prove the analogous result 5.11 in the case of one singular point.

Similarly, we can proceed for c.ss. containing two or three singular points.
In the case of four singularities, it is easier to choose 4 points in general position and to study c.ss. having them as singular points; using Algorithm 2.3, one can describe the configuration of their lines and, in particular, their L-sets.

In this way we can prove the following result, where the names of the lines are as in (1) and the intersection points are:

$$
\begin{gathered}
A := l_1 \cap l_2 = (0,0,0,1); \quad B := l_1 \cap l_4 = (1,0,0,1);\\
C := l_3 \cap l_4 = (0,1,1,0); \quad D := l_2 \cap l_3 = (0,0,1,0); \quad E := l_2 \cap l_5 = (0,0,1,-1).
\end{gathered}
$$

Proposition 6.1. *Let S be an irreducible c.s. with finitely many lines and not projectively equivalent to any T_i, for $i = 1, \dots, 13$. We have the following facts:*

i) if S has (at least) one singular point, then either it is projectively equivalent to a c.s. containing L^* and singular at C or it has more than one singular point;
ii) if S has (at least) two singular points, then either it is projectively equivalent to a c.s. containing L^* and singular at C and D or it has more than two singular points;
iii) if S has (at least) three singular points, then either it is projectively equivalent to a c.s. containing L^* and singular at C, D and E or it has four singular points;
iv) if S has four singular points, then it is projectively equivalent to the (unique) c.s. containing L^* and singular at B, C, $P_1 := l_1 \cap \operatorname{res}(l_1, l_4) = (1,0,0,-1)$, $P_2 := l_3 \cap \operatorname{res}(l_3, l_4) = (0,1,-1,0)$.

Remark 6.2. The computation of the equations of the families arising in the above proposition is straightforward; namely it is enough to impose to the generic cubic surface through L^* (having equation (2)) to be singular at C (resp. at C, D, at C, D, E, at B, C, P_1, P_2). In each case we get a linear system of dimension $4-r$, where r is the number of the singular points; such linear systems will be denoted by $\mathbb{P}^3_C$, $\mathbb{P}^2_{C,D}$, $\mathbb{P}^1_{C,D,E}$, $\mathbb{P}^0_{B,C,P_1,P_2}$ respectively. The equations of their general elements can be obtained from (2) and the linear conditions coming from the imposed singularities; we list here, for each of the above families, the set eq_i of the corresponding equations and the name of its general element:

$$\begin{array}{lll}
\mathbb{P}^3_C : & eq_1 := \{a+c-g=0, & T_{21}(a,b,d,g) \\
\mathbb{P}^2_{C,D} : & eq_2 := \{eq_1, a-c-d=0, & T_{20}(a,b,d) \\
\mathbb{P}^1_{C,D,E} : & eq_3 := \{eq_2, 2a+b-d=0, & T_{18}(a,b) \\
\mathbb{P}^0_{B,C,P_1,P_2} : & eq_4 := \begin{cases} a+c+g=0 \\ eq_1 \\ a+2b+c-g=0 \\ a+c+2d+g=0 \end{cases}, & T_{15}: \quad x^2y-xy^2+xz^2-yt^2=0.
\end{array}$$

The usual application of Algorithm 2.3 allows us to compute the lines of each of the above linear systems, obtaining that the general elements of them have, respectively, $21, 16, 12, 9$ lines.

Remark 6.3. The condition $c = g - a$ defining $\mathbb{P}^3_C$ inside $\mathbb{P}^4$ leads to conclude that $\mathbb{P}^3_C$ is a component of Σ; moreover Proposition 6.1 *i)* implies that every element of $\Sigma \setminus \mathcal{R}$ is projectively equivalent to an element of $\mathbb{P}^3_C$. Finally it is clear that the linear systems defined in 6.2 are in the following chain:

$$\mathbb{P}^4 \supset \Sigma \supset \mathbb{P}^3_C \supset \mathbb{P}^2_{C,D} \supset \mathbb{P}^1_{C,D,E}.$$

Definition. Let $\mathbb{P}^s$ be a linear space parametrizing a family of c.ss. and suppose that the general surface has a constant number m of lines; we call *degenerate locus* of $\mathbb{P}^s$ the subset $\Sigma(\mathbb{P}^s)$ consisting of c.ss. which either are reducible or have less than m lines.

Observe that the subset Σ of $\mathbb{P}^4$ defined in Section 1 is exactly its degenerate locus.
Since we can compute the lines of the families introduced in 6.2 (by Algorithm 2.3), with techniques similar to those used to find out $\Sigma \subset \mathbb{P}^4$, we are able to compute the degenerate locus of $\mathbb{P}^s$ (where $\mathbb{P}^s$ is one of the above families).
Moreover, we study the locus $\Sigma(\mathbb{P}^s)$ and the corresponding subfamilies arising from it; some of these consist of c.ss. having more than the required number of singularities (hence included in another family), some others consist of reducible cubic surfaces. Therefore we keep only the subfamilies of $\mathbb{P}^s$ of irreducible c.ss. having the same number of singular points than the general element of $\mathbb{P}^s$ and a smaller number of lines. Again we compute and study the degenerate locus of each of these collected subfamilies.
Here we list the obtained result:

Proposition 6.4. *The following facts hold:*

i) the degenerate locus of $\mathbb{P}^3_C$ is

$$\Sigma_1: \quad bg(a-g)(2a+b-d)(2a+b-g)(2a-d)(2a-g)(2a-d-g)(d-g)(4a^2-2ad-2ag+bg+dg)=0.$$

The components of Σ_1 which consist of c.ss. with less than 21 lines and only one singular point correspond to the factors $2a-g$, $2a-d$, $d-g$, respectively;

ii) the degenerate locus of $\mathbb{P}^2_{C,D}$ is

$$\Sigma_2: \quad bd(a-d)(2a-d)(2a-d+b)(b+d)=0.$$

Its components consisting of c.ss. with less than 16 lines and only two singular points correspond to the factors d and $a-d$, respectively;

iii) the degenerate locus of $\mathbb{P}^1_{C,D,E}$ is

$$\Sigma_3: \quad b(a+b)(2a+b)=0.$$

Its component consisting of c.ss. with less than 12 lines and only three singular points corresponds to the factor $2a+b$. □

A detailed study of the two families arising in 6.4 *i)* from the conditions $g = 2a$ and $d = 2a$ on $\mathbb{P}^3_C$, which are:

$$w_1: \quad 2a(x-y)(x+t)(y-z)+b(x-t)(xz+yt)+d(y-z)(xz+yt)=0$$

and

$$w_2: \quad 2ay(t-x)(x-y+z+t)+b(t-x)(xz+yt)+g(x-y+z+t)(xz-yt)=0$$

respectively, shows that they are projectively equivalent, i.e. for almost each $S \in w_1$ there exists a c.s. $T \in w_2$ which is p.e. to S. Indeed, if we call a_1, b_1, g_1 the parameters of w_2 instead of a, b, g (to distinguish them from the parameters of w_1), we substitute

$$a = \frac{1}{4}g_1(2a_1 - g_1 + b_1), \quad b = b_1(a_1 - g_1), \quad d = \frac{1}{2}(2a_1 - g_1)(2a_1 - g_1 + b_1)$$

in w_1 and we move the resulting c.s. by the change of coordinates:

$$A_0 := \begin{pmatrix} 0 & (a_1-g_1)(2a_1-g_1) & (g_1-a_1)(2a_1-g_1) & 0 \\ 0 & (2a_1-g_1)^2 & 0 & 0 \\ 4b_1(a_1-g_1) & (2a_1-g_1)^2 & 0 & 0 \\ 2(g_1-a_1)(2a_1-g_1+b_1) & (a_1-g_1)(2a_1-g_1) & (g_1-a_1)(2a_1-g_1) & 2(g_1-a_1)(2a_1-g_1+b_1) \end{pmatrix}$$

we obtain the generic element of w_2. Let us sketch the techniques used to obtain this result. First we need the following notion (which will also be used in the sequel):

Definition. Suppose that S is a cubic surface passing through L^* and that some vertices of L^* are singular points for S. An L-set $L' := (m_1, \dots, m_5) \subset S$ is called a *singular L-set* if the same vertices of L' are singular points for S; i.e. if $l_i \cap l_j$ is singular, then also $m_i \cap m_j$ is singular.

In order to show that w_1 and w_2 are projectively equivalent, we compute all the lines of w_2 (their number is 15) and collect all the singular L-sets of the generic element of this family (their number is 144). Successively we compute the matrix A which maps $L^*(\subseteq w_1)$ into a fixed singular L-set ($\subseteq w_2$), we move w_1 by the change of coordinates A (obtaining, say $A(w_1)$) and we check if there exists a solution in a, b, d, a_1, b_1, g_1 to the equation $A(w_1) = w_2$. In this way, choosing a suitable singular L-set of lines, we find the above transformation and the above matrix A_0. (Note that the determinant of A_0 is contained in the degenerate locus of w_2, hence it is easy to see that the projectivity A_0 is defined exactly outside the degenerate loci of w_1 and w_2).

In the sequel the two dimensional linear space given by w_2 will be denoted by $\mathbb{P}^2_C$ and its generic element by $T_{19}(a, b, g)$.

Concerning the condition $d = g$ (arising in 6.4 *i*)) imposed on the cubic surfaces of $\mathbb{P}^3_C$, it gives in general a family of cubic surfaces with two singular points, (hence considered in *ii*)), unless $g = 2a$. In this case the two singular points are coincident and we get the following family of cubic surfaces, singular only at C and having 10 lines:

$$T_{16}(a,b): \quad 2ax(y-z)(x-y+z+t)+b(t-x)(yt+xz)=0$$

which will be denoted by $\mathbb{P}^1_C$. Its degenerate locus is defined by $ab = 0$ and consists only of two reducible c.ss.

Concerning the case *ii*), the two conditions $a = d$ and $d = 0$ give rise to two one–dimensional families of cubic surfaces which are projectively equivalent (one can see this with the same techniques used to show that w_1 and w_2 are p.e.). Hence we define $\mathbb{P}^1_{C,D}$ to be one of the two families (e.g. the second one) whose equation is:

$$T_{17}(a,b): \quad 2a(x-y)(x+t)(y-z)+b(x-t)(yt+xz)=0.$$

Case *iii*) of Proposition 6.4: the condition $b = -2a$ gives precisely one cubic surface with 3 singular points (the points C, D, E) which is:

$$T_{14}: \quad x^2y - 2x^2z - xy^2 + xyz - y^2t + yzt + yt^2 = 0.$$

In this way, the above arguments, together with 6.1 and 6.4, give the **Proof of Theorem 1.9**.

Now we summarize the properties of the obtained families of type $\mathbb{P}^s$, by listing their degenerate loci $\Sigma(\mathbb{P}^s)$, the number of singularities and lines contained in any surface of $\mathbb{P}^s \setminus \Sigma(\mathbb{P}^s)$ and the number of singular L-sets of the general element of each family.

Finally we list also the stabilizer groups $\mathrm{Stab}(S)$ of the *general* surface S of each family $\mathbb{P}^s$. In order to compute these groups we proceed as usual: clearly $\mathrm{Stab}(S)$ is contained in the set of all matrices, say $g_{L'}$, which map L^* into another singular L-set, say L', of S. Thus it is enough to compute all the singular L-sets: $\{L_1, \dots, L_i\}$ and the corresponding matrices $g_{L_1}, \dots, g_{L_i}$; among them we choose those that stabilize S.

c.s.	equations	$\sharp$ sing's	$\sharp$ lines	deg.locus	$\sharp$ sing. L-sets	stabilizer
$T_{21}(a,b,d,g)$	eq_1	1	21	Σ_1	720	Id
$T_{20}(a,b,d)$	eq_2	2	16	Σ_2	48	$S_2(g_2^{(20)})$
$T_{19}(a,b,g)$	eq_1 $2a-d=0$	1	15	Σ_1'	144	Id
$T_{18}(a,b)$	eq_3	3	12	Σ_3	12	$S_2(g_2^{(18)}) \times \left\langle g_3^{(18)} \right\rangle$
$T_{17}(a,b)$	$eq_2, d=0$	2	11	Σ_2'	12	Id
$T_{16}(a,b)$	eq_1 $d-g=0$ $2a+g=0$	1	10	Σ_1''	8	$S_2(g_2^{(16)})$
T_{15}	eq_4	4	9	–	24	$S_4(g_2^{(15)}, g_4^{(15)})$
T_{14}	eq_3 $2a+b=0$	3	8	–	2	$S_2(g_2^{(14)})$

where

$$\Sigma_1' : \ bg(a-g)(2a-g)(2a+b-g)=0, \qquad \Sigma_2' : \ ab(2a+b)=0, \qquad \Sigma_1'' : \ ab=0$$

and

$$g_2^{(20)} := \begin{pmatrix} 2d(a-d) & 0 & 0 & 0 \\ -(b-d)(a-d) & 2(a-d)^2 & -2(a-d)^2 & (a-d)(b+d) \\ -a(b-d) & 2a(a-2d) & -2(a-d)^2 & a(b+d) \\ 0 & 0 & 0 & 2d(a-d) \end{pmatrix}$$

$$g_2^{(18)} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}, \quad g_3^{(18)} := \begin{pmatrix} (a+b)(2a+b) & 0 & 0 & 0 \\ (a+b)a & -(a+b)^2 & (a+b)^2 & (a+b)^2 \\ -a^2 & (3a+2b)a & (a+b)^2 & -(a+b)a \\ (2a+b)a & -(2a+b)^2 & 0 & 0 \end{pmatrix}$$

$$g_2^{(16)} := \begin{pmatrix} 2a & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 \\ b-2a & 4a & -2a & -b-2a \\ 0 & 0 & 0 & 2a \end{pmatrix},$$

$$g_2^{(15)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_4^{(15)} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_2^{(14)} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$

The study of the orbit of the general element of each family $\mathbb{P}^s$ introduced above is similar to that described in the case of smooth c.ss. (see Section 4) and in the case of the family $T_{13}(p,q)$ (see Section 5, third example). Let S be the general element of one of the families $\mathbb{P}^s$ listed in the above table and G be the group of the permutations of the lines of S (hence preserving their incidence relations and singular points): since S is generic, we can assume that G depends only on $\mathbb{P}^s$.

Here the whole group G can be embedded in PGL_4 (while we recall that in the case of smooth c.ss. only an index two subgroup of $\mathbb{E}_6$ could be considered as a group of projectivities). This is essentially due to the fact that the map

$$G \ni g \;\mapsto\; g(L^*)$$

is a one–to–one correspondence between G and the set of singular L-sets of S. Indeed, one can directly verify that, taking the five lines of L^*, computing $\mathrm{res}\,(l_i, l_j)$ and the residual lines of each couple of incident lines arising in this way, one obtains all the 21 (resp. $16, \ldots, 8$) lines of S. This means that the choice of a singular L-set induces a labeling on all the lines of S, i.e. determines an element of G.
Then we can proceed as in the case of smooth c.ss. and define an anti–monomorphism $G \longrightarrow \mathrm{PGL}_4$ by $g \mapsto A_g^{-1}$, where A_g is the unique projectivity mapping L^* to $g(L^*)$.
In this way G acts on (an open subset of) $\mathbb{P}^s$ via the map

$$G \times \mathbb{P}^s \longrightarrow \mathbb{P}^s, \quad \text{given by} \quad (g, S) \mapsto A_g^{-1}(S)$$

so the intersection of the orbit $\mathcal{O}_S$ of S (under the action of PGL_4) with $\mathbb{P}^s$ is given by:

$$\mathcal{O}_S \cap \mathbb{P}^s = \mathcal{O}_S^G \cap \mathbb{P}^s = \mathcal{O}_S^{G/\mathrm{Stab}(S)} \cap \mathbb{P}^s.$$

The quotient $G/\mathrm{Stab}(S)$ is clearly a group which freely acts on a suitable open subset of $\mathbb{P}^s$. Therefore

$$\sharp\,(\mathcal{O}_S \cap \mathbb{P}^s) = |G/\mathrm{Stab}(S)| = \frac{\sharp\,\{\text{sing. } L\text{-sets}\}}{|\mathrm{Stab}(S)|}$$

and this number can be computed from the above table.
Finally, since each stabilizer listed above is a finite group, then the orbit $\mathcal{O}_S$ of the general element $S \in \mathbb{P}^s$ has dimension 15.

c.s.	singularities	rational map	matrix	six points
T_1	$(1,0,0,0)$	$\begin{cases} x = vu^2 + w^3 \\ y = -v^3 \\ z = -v^2 w \\ t = -v^2 u \end{cases}$		$(1,0,0)^6$
T_2	$(0,0,0,1)$ $(1,0,0,0)$	$\begin{cases} x = vu^2 \\ y = v^2 u \\ z = uvw \\ t = -uw^2 - v^3 \end{cases}$	$\begin{pmatrix} y & x & 0 \\ -t & y & z \\ z & 0 & y \end{pmatrix}$	$(1,0,0)^2$ $(0,0,1)^4$
T_3	$(0,0,0,1)$ $(1,0,0,0)$ $(0,0,1,0)$	$\begin{cases} x = uw^2 \\ y = uvw \\ z = uvw + v^3 \\ t = u^2 w \end{cases}$	$\begin{pmatrix} y & x & 0 \\ 0 & y & t \\ z-y & 0 & y \end{pmatrix}$	$(1,0,0)^3$ $(0,0,1)^3$
T_4	$(0,0,0,1)$	$\begin{cases} x = (v-w)uv \\ y = (v-w)v^2 \\ z = (v-w)vw \\ t = u^2 v - u^2 w - uv^2 + uw^2 + v^3 \end{cases}$	$\begin{pmatrix} t & x & y \\ y+z & -y & 2x+t \\ y & 0 & x+y-z \end{pmatrix}$	$(1,0,0)^4$ $(0,0,1)$ $(1,0,1)$
T_5	$(0,0,0,1)$	$\begin{cases} \alpha = (uw + v^2) \\ \beta = w^2 \end{cases}$	$\begin{pmatrix} y & z & x \\ 0 & -y & z \\ -x & 0 & t \end{pmatrix}$	$(1,0,0)^5$ $(0,1,0)$
T_6	$(0,0,1,0)$ $(1,0,0,0)$	$\begin{cases} \alpha = w^2 \\ \beta = (u+v+w)\,u \end{cases}$	$\begin{pmatrix} 0 & -y & z \\ x+t & t & y \\ -x & 0 & t \end{pmatrix}$	$(1,0,0)$ $(0,1,0)^3$ $(1,-1,0)^2$
T_7	$(0,0,0,1)$ $(0,0,1,0)$ $(0,1,1,0)$	$\begin{cases} \alpha = (u-w)\,(v-w) \\ \beta = (2\,w-u-v)\,w \end{cases}$	$\begin{pmatrix} z & x+y & 2\,x+y \\ y & y & z \\ -x & 0 & t \end{pmatrix}$	$(1,0,0)^2$ $(0,1,0)^2$ $(1,1,1)^2$
T_8	$(0,0,0,1)$ $(0,0,1,1)$ $(1,0,0,1)$	$\begin{cases} \alpha = uw - v^2 + 2vw - 2w^2 \\ \beta = (u-w)\,(v-w) \end{cases}$	$\begin{pmatrix} 0 & z-y & z \\ x-y & z-t & y \\ x & -t & x-t \end{pmatrix}$	$(1,0,0)^2$ $(0,1,0)$ $(1,1,1)^3$
T_9	$(0,0,0,1)$	$\begin{cases} \alpha = uw - v^2 + vw - w^2 \\ \beta = (v-w)\,w \end{cases}$	$\begin{pmatrix} y & z & x+y \\ 0 & z-y & z \\ x & t & t \end{pmatrix}$	$(1,0,0)^4$ $(0,1,0)$ $(1,1,1)$
T_{10}	$(0,0,0,1)$	$\begin{cases} \alpha = -uw - v^2 + 2vw \\ \beta = (v-w)\,w \end{cases}$	$\begin{pmatrix} y & x-z & x \\ 0 & y & z \\ -x & 2x & t \end{pmatrix}$	$(1,0,0)^4$ $(0,1,0)$ $(1,1,1)$
T_{11}	$(0,0,1,0)$	$\begin{cases} \alpha = w^2 \\ \beta = u^2 + uv - uw - vw - w^2 \end{cases}$	$\begin{pmatrix} 0 & y & z \\ x-t & x-t & y-x \\ x & 0 & t \end{pmatrix}$	$(1,0,0)$ $(0,1,0)^3$ $(1,-1,0)^2$
T_{12}	$(0,1,1,0)$ $(0,0,0,1)$	$\begin{cases} \alpha = (u-v)\,(v-w) \\ \beta = uv - 2uw + w^2 \end{cases}$	$\begin{pmatrix} 0 & -y & z \\ y-z & y-z+t & x-2t \\ x & 0 & t \end{pmatrix}$	$(1,0,0)^2$ $(0,1,0)$ $(1,1,1)^3$
$T_{13}(p,q)$	$(0,0,0,1)$ $(0,1,1,0)$	$\begin{cases} \alpha = q\,(v-w)\,(2u-v-w) \\ \beta = (pu + qv - (p+q)w)\,w \end{cases}$	$\begin{pmatrix} (y-z)q & yq & (y-z)q \\ 3(y-z)q & (2y-x+z)q & xp \\ t-x & 0 & t \end{pmatrix}$	$(1,0,0)^2$ $(0,1,0)$ $(1,1,1)$ $(1,2,0)$ $(p+2q,p+q,3q)$

Table 1

$$E_1 = [0,0,1,0,0,0], \quad G_4 = [0,0,0,0,0,1], \quad E_2 = [0,0,0,1,0,0]$$
$$G_3 = [1,1,0,0,-1,-1], \quad E_3 = [0,1,-1,1,-1,0]$$

$$\begin{array}{rl}
G_5 = & [c^2, ce, 0, 0, ce, e^2] \\
G_6 = & [c^2, cf, 0, 0, cf, f^2] \\
F_{24} = & [0,0,0, bc - c^2 + ef, -cd + ef + c^2, c(e+f-d)] \\
F_{14} = & [0, bc + c^2 + ef, -c^2 - cd + ef, 0, 0, (-b+e+f)c] \\
F_{34} = & [0, -bc + c^2 - ef, c^2 + cd - ef, -bc + c^2 - ef, c^2 + cd - ef, c(d+b)] \\
F_{13} = & [bc - c^2 + ef, bc - c^2 + ef, (c-e)(c-f), 0, bc - ce - cf + 2ef, bc - ce - cf + 2ef] \\
F_{15} = & [c(bc - c^2 + ef), e(bc - c^2 + ef), (c-e)(cd - cf - ef), 0, c^2(e-f-b), ce(e-f-b)] \\
F_{16} = & [c(bc - c^2 + ef), f(bc - c^2 + ef), (c-f)(cd - ce - ef), 0, c^2(f-e-b), cf(f-e-b)] \\
F_{25} = & [(c^2 + cd - ef)c, (d-e+f)c^2, 0, -(c+e)(bc - cf + ef), (c^2 + cd - ef)e, (d-e+f)ce] \\
F_{26} = & [(c^2 + cd - ef)c, (d-f+e)c^2, 0, -(c+f)(bc - ce + ef), (c^2 + cd - ef)f, (d-f+e)cf] \\
F_{23} = & [c^2 + cd - ef, -cd + ce + cf + 2\,ef, 0, (c+e)(c+f), -c^2 - cd + ef, cd - ce - cf - 2\,ef] \\
F_{35} = & [c(bc - cd + 2\,ef), c^2 f - c^2 d + bce + e^2 f, (c-e)(cd - cf - ef), \\
 & \quad (c+e)(bc - cf + ef), c^2 f - bc^2 - cde + e^2 f, ce(2f - d - b)] \\
F_{36} = & [c(bc - cd + 2\,ef), c^2 e - c^2 d + bcf + ef^2, (c-f)(cd - ce - ef), \\
 & \quad (c+f)(bc - ce + ef), c^2 e - bc^2 - cdf + ef^2, cf(2e - b - d)] \\
E_5 = & [0, (c-f)(bc - cf + ef)(cd - cf - ef), (f-c)(cd - cf - ef)^2, -(c+f)(bc - cf + ef)^2, \\
 & \quad (c+f)(bc - cf + ef)(cd - cf - ef), 2(bc - cf + ef)(cd - cf - ef)f] \\
E_6 = & [0, (c-e)(bc - ce + ef)(cd - ce - ef), (e-c)(cd - ce - ef)^2, -(c+e)(bc - ce + ef)^2, \\
 & \quad (c+e)(bc - ce + ef)(cd - ce - ef), 2(bc - ce + ef)(cd - ce - ef)e] \\
F_{56} = & [c^2(bc - cd + 2ef), -bc^2 d + bc^2 f + bc^2 e - c^2 ef + fe^2 c - cdef + bcef + f^2 ec + e^2 f^2, \\
 & \quad (cd - ce - ef)(cd - cf - ef), -(bc - cf + ef)(bc - ce + ef), \\
 & \quad c^2 ef - c^2 df - c^2 de + bc^2 d + fe^2 c + f^2 ec + cdef - bcef - e^2 f^2, (bc - cd + 2ef)ef] \\
E_4 = & [2(c^2 + cd - ef)(bc - c^2 + ef)c^2, (c^2 + cd - ef)(bc - c^2 + ef)(c^2 + ce + cf - ef), \\
 & \quad (c^2 + cd - ef)^2 (c-e)(c-f), -(c+e)(c+f)(bc - c^2 + ef)^2, \\
 & \quad -(c^2 + cd - ef)(c^2 - ce - cf - ef)(bc - c^2 + ef), 2(c^2 + cd - ef)(bc - c^2 + ef)ef] \\
F_{45} = & [0, (c+f)(c-f)(cd - cf - ef)(bc - c^2 + ef), (c-f)^2(c^2 + cd - ef)(cd - cf - ef), \\
 & \quad (c+f)^2(bc - cf + ef)(-bc + c^2 - ef), -(c+f)(c-f)(c^2 + cd - ef)(bc - cf + ef), \\
 & \quad -(c+f)(c-f)(bc^2 + bcf + c^2 d - 2c^2 f - cdf + 2ef^2)e] \\
F_{46} = & [0, (c+e)(c-e)(cd - ce - ef)(bc - c^2 + ef), (c-e)^2(c^2 + cd - ef)(cd - ce - ef), \\
 & \quad (c+e)^2(bc - ce + ef)(-bc + c^2 - ef), -(c+e)(c-e)(c^2 + cd - ef)(bc - ce + ef), \\
 & \quad -(c+e)(c-e)(bc^2 + bce + c^2 d - 2c^2 e - cde + 2e^2 f)f] \\
G_1 = & [(bc - c^2 + ef)^2(bc - cd + 2ef), (bc - c^2 + ef)(bc - ce - cf + 2ef)(bc - cd + 2ef), 0, \\
 & \quad 2(bc - c^2 + ef)(bc - cf + ef)(bc - ce + ef), \\
 & \quad -(bc - c^2 + ef)(b^2c^2 + bc^2 d - bc^2 e - bc^2 f - c^2 de - c^2 df + 2c^2 ef + 2cdef - 2e^2 f^2), \\
 & \quad -(bc - ce - cf + 2ef)(b^2c^2 + bc^2 d - bc^2 e - bc^2 f - c^2 de - c^2 df + 2c^2 ef + 2cdef - 2e^2 f^2)] \\
F_{12} = & [(c^2 + cd - ef)(bc - cd + 2ef)^2(-bc + c^2 - ef), \\
 & \quad (bc - cd + 2ef)(-2bcef - 2e^2 f^2 + (-b - d + 2\,e)c^2 f + (b+d)(d-e)c^2)(bc - c^2 + ef), 0, 0, \\
 & \quad (c^2 + cd - ef)(bc - cd + 2ef)(2cdef - 2e^2 f^2 - (b + d - 2e)c^2 f - (-b+e)(b+d)c^2), \\
 & \quad -(bc^2 d - bc^2 e - bc^2 f - 2bcef + c^2 d^2 - c^2 de - c^2 df + 2c^2 ef - 2e^2 f^2) \\
 & \quad (b^2c^2 + bc^2 d - bc^2 e - bc^2 f - c^2 de - c^2 df + 2c^2 ef + 2cdef - 2e^2 f^2)] \\
G_2 = & [(c^2 + cd - ef)^2(bc - cd + 2\,ef), \\
 & \quad -(c^2 + cd - ef)(bc^2 d - bc^2 e - bc^2 f - 2bcef + c^2 d^2 - c^2 de - c^2 df + 2c^2 ef - 2e^2 f^2), \\
 & \quad 2(c^2 + cd - ef)(cd - ce - ef)(cd - cf - ef), 0, \\
 & \quad -(c^2 + cd - ef)(cd - ce - cf - 2ef)(bc - cd + 2\,ef), \\
 & \quad (cd - ce - cf - 2ef)(bc^2 d - bc^2 e - bc^2 f - 2bcef + c^2 d^2 - c^2 de - c^2 df + 2c^2 ef - 2e^2 f^2)]
\end{array}$$

Table 2

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