Antiperiodic solutions for nonlinear asymmetric equations near resonance

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Abstract

We investigate the existence of solutions to second order scalar differential equations with asymmetric nonlinearities, subject to antiperiodic boundary conditions. Both resonance and nonresonance cases are examined, with the Landesman–Lazer conditions imposed in the resonant setting. The proofs rely on topological degree theory.

1 Introduction

We are interested in the T-antiperiodic problem associated with the scalar second order equation

$$\ddot{x} + g(t,x) = 0\,, \tag{1}$$

where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the growth conditions

$$\begin{aligned} \mu_1 \leqslant \liminf_{u\to+\infty} \frac{g(t,x)}{x} \leqslant \limsup_{x\to+\infty} \frac{g(t,x)}{x} \leqslant \mu_2\,, \\ \nu_1 \leqslant \liminf_{x\to-\infty} \frac{g(t,x)}{x} \leqslant \limsup_{x\to-\infty} \frac{g(t,x)}{x} \leqslant \nu_2\,, \end{aligned} \tag{2}$$

uniformly in $t \in [0,T]$, for some positive constants μ_1, μ_2, ν_1, and ν_2.

The investigation of (1) under T-antiperiodic boundary condition

$$(x(0), \dot{x}(0)) = -(x(T), \dot{x}(T))$$

shares certain similarities with other classical boundary value problems. For instance, let us recall some existence results associated with the T-periodic boundary condition $(x(0), \dot{x}(0)) = (x(T), \dot{x}(T))$, the Neumann and the Dirichlet boundary conditions, $\dot{x}(0) = 0 = \dot{x}(T)$ and $x(0) = 0 = x(T)$, respectively.

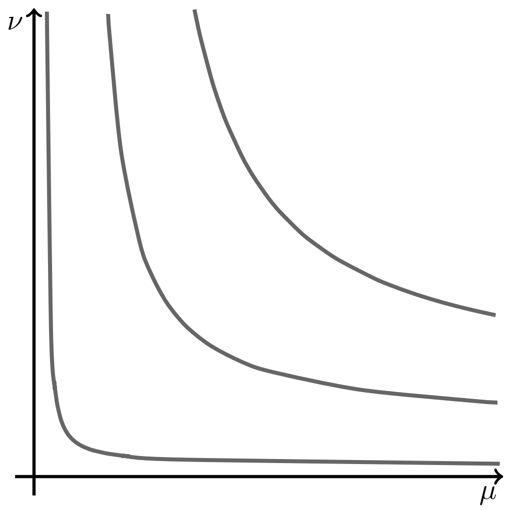

Figure 1: The Fučík spectrum for the periodic problem with $T = 2\pi$, or for the Neumann problem with $T = \pi$.

In 1969, Lazer and Leach [20] considered (1) with T-periodic boundary conditions. In that paper, $g(t,x) = \lambda x + r(t,x)$, where r is continuous, uniformly bounded and T-periodic in t and $\lambda = \left(\frac{2\pi n}{T}\right)^2$ for some positive integer n. They established that a sufficient condition for the existence of a T-periodic solution is the following: for every non-zero η satisfying $\ddot{\eta} + \lambda\eta = 0$,

$$\int_{\{\eta<0\}} \limsup_{x\to-\infty} r(t,x)\eta(t)dt + \int_{\{\eta>0\}} \liminf_{x\to+\infty} r(t,x)\eta(t)dt > 0\,. \tag{3}$$

The following year, Landesman and Lazer [19] introduced a similar condition for a Dirichlet problem associated with an elliptic operator. Since then, (3) is referred to as *Landesman–Lazer condition*. This condition is crucial for the nonlinearity to be kept sufficiently far from resonance. Their work has served as a foundation for numerous generalizations, see for example [3, 7, 9, 10, 11, 12, 15, 22, 24].

Some years later, Fučík [14] and Dancer [5, 6] introduced the so-called Fučík spectrum, defined as the set of points $(\mu,\nu) \in \mathbb{R}^2$ such that the asymmetric oscillator

$$\ddot{x} + \mu x^+ - \nu x^- = 0\,, \tag{4}$$

where $x^\pm = \max\{\pm x, 0\}$, has nontrivial T-periodic solutions.

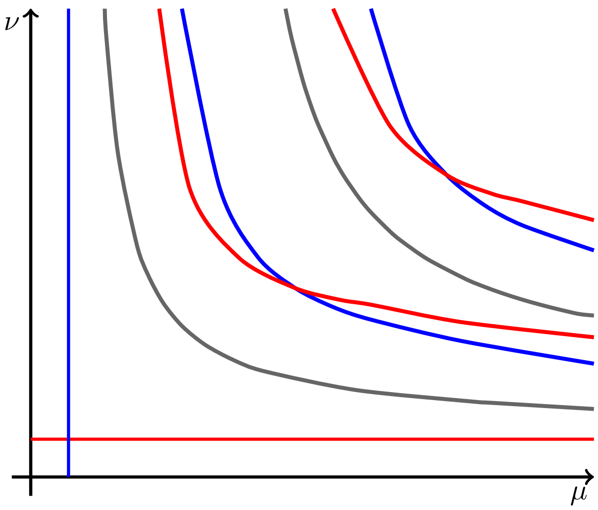

Figure 2: The Fučík spectrum for the Dirichlet problem.

In [8], it was shown that if the function g satisfies (2) and the rectangle $\mathcal{R} = [\mu_1, \mu_2] \times [\nu_1, \nu_2]$ does not intersect the Fučík spectrum Σ, then the equation (1) admits at least one T-periodic solution. This represents a typical *nonresonance* situation. See also [7, 18] for related results. When the set $\mathcal{R} \cap \Sigma$ consists of only one or both the vertices (μ_1, ν_1) and (μ_2, ν_2) of the rectangle, in order to avoid *resonance*, additional hypotheses are required. For instance, in [9, 10, 12, 22], the *double resonance* case was addressed by imposing Landesman–Lazer-type conditions on both sides, ensuring the existence of a T-periodic solution.

Concerning the Neumann and the Dirichlet problems associated with (1), we refer to [15, 21, 22, 24].

If compared with the literature available for periodic, Neumann and Dirichlet problems, in the study of the antiperiodic problems the number of references is considerably smaller. For instance, in [4], the existence of antiperiodic solutions for Liénard-type and Duffing-type differential equations with the p-Laplacian operator was established using degree theory. In [16], a resonant second order problem of the form $\ddot{x} = f(t, x, \dot{x})$ satisfying antiperiodic and periodic boundary conditions was analyzed. In [27], the authors explored the existence of antiperiodic solutions for a second order ordinary differential equation by using the interaction of the nonlinearity with the Fučík spectrum. In [23], antiperiodic oscillations are obtained for a forced

Duffing equation with negative linear stiffness, demonstrating how they develop multiple peaks under increasing forcing strength. For further related studies, we refer the reader to [1, 2, 17, 25, 26].

To the best of our knowledge, the antiperiodic problem associated with asymmetric scalar second order equations under resonance with respect to the Fučík spectrum has not yet been explored. In particular, Landesman–Lazer-type conditions have not been employed in such kind of problems. In the present paper, it is our aim to fill such a gap.

The paper is organized as follows. In Section 2, we analyze the Fučík spectrum corresponding to an antiperiodic problem and present some key properties. Then, in Section 3, we state and prove our main results for the antiperiodic problem under both nonresonance and double resonance situations.

2 Preliminaries

In this section, we discuss about the Fučík spectrum corresponding to the antiperiodic problem, and present some preliminary lemmas.

2.1 The Fučík spectrum

Consider the asymmetric oscillator under antiperiodic boundary conditions

$$\begin{cases} \ddot{x} + \mu x^+ - \nu x^- = 0\,, \\ x(0) + x(T) = 0\,, \quad \dot{x}(0) + \dot{x}(T) = 0\,. \end{cases} \tag{5}$$

If μ and ν are positive, the solutions of the differential equation in (5) are all periodic, with period

$$\mathcal{T}_{\mu,\nu} = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\,. \tag{6}$$

One particular solution is given by

$$\varphi_{\mu,\nu}(t) = \begin{cases} \frac{1}{\sqrt{\mu}} \sin\left(\sqrt{\mu}\, t\right) & \text{if } t \in \left[0, \frac{\pi}{\sqrt{\mu}}\right]\,, \\ -\frac{1}{\sqrt{\nu}} \sin\left(\sqrt{\nu}\left(t - \frac{\pi}{\sqrt{\mu}}\right)\right) & \text{if } t \in \left[\frac{\pi}{\sqrt{\mu}}, \mathcal{T}_{\mu,\nu}\right]\,, \end{cases} \tag{7}$$

extended by $\mathcal{T}_{\mu,\nu}$-periodicity to the whole $\mathbb{R}$. All the other solutions are of the form $x(t) = \rho\varphi_{\mu,\nu}(t-\theta)$ with $\rho \geqslant 0$ and $\theta \in \mathbb{R}$.

We define $\Sigma = \{(\mu,\nu) \in \mathbb{R}^2 : \text{ (5) has a nontrivial solution}\}$, the Fučík spectrum of the operator $-\ddot{x}$ under the antiperiodic boundary conditions. Easy computations show that

$$\Sigma = \bigcup_{k\in\mathbb{N}} \mathscr{C}_k \,,$$

where the set $\mathscr{C}_0$ consists of the two lines

$$\mathscr{C}_{0,1} = \left\{(\mu,\nu) \in \mathbb{R}^2 : \mu = \left(\frac{\pi}{T}\right)^2\right\} \,,$$
$$\mathscr{C}_{0,2} = \left\{(\mu,\nu) \in \mathbb{R}^2 : \nu = \left(\frac{\pi}{T}\right)^2\right\} \,,$$

while, for $k \geqslant 1$, $\mathscr{C}_k = \mathscr{C}_{k,1} \cup \mathscr{C}_{k,2}$, with

$$\mathscr{C}_{k,1} = \left\{(\mu,\nu) \in \mathbb{R}^2 : \mu > 0\,, \nu > 0\,,\ (k+1)\frac{\pi}{\sqrt{\mu}} + k\frac{\pi}{\sqrt{\nu}} = T\right\} \,,$$

and

$$\mathscr{C}_{k,2} = \left\{(\mu,\nu) \in \mathbb{R}^2 : \mu > 0\,, \nu > 0\,,\ k\frac{\pi}{\sqrt{\mu}} + (k+1)\frac{\pi}{\sqrt{\nu}} = T\right\} \,.$$

Notice that the curves $\mathscr{C}_{k,1}$ and $\mathscr{C}_{k-1,2}$, for $k \geqslant 1$, share the same horizontal asymptote $\nu = (k\pi/T)^2$.

It can be seen that Σ is a subset of the Fučík spectrum of the operator $-\ddot{x}$ under the corresponding Dirichlet boundary condition $x(0) = 0 = x(T)$.

We use the notations

$$m_{\mu,\nu} = \min\left\{\frac{\pi}{\sqrt{\mu}}, \frac{\pi}{\sqrt{\nu}}\right\} \,, \qquad M_{\mu,\nu} = \max\left\{\frac{\pi}{\sqrt{\mu}}, \frac{\pi}{\sqrt{\nu}}\right\} ,$$

and define the set $\mathcal{S} \subseteq \mathbb{R}^2$ as follows:

$$\mathcal{S} = \bigcup_{k\in\mathbb{N}} \mathcal{S}_k \,, \tag{8}$$

where, as depicted in Figure 3,

$$\mathcal{S}_k = \left\{(\mu,\nu) \in \mathbb{R}^2 : \mu > 0\,, \nu > 0\,,\ (k-1)\mathcal{T}_{\mu,\nu} + M_{\mu,\nu} < T < k\mathcal{T}_{\mu,\nu} + m_{\mu,\nu}\right\} .$$

We now examine the nontrivial solutions of problem (5) in three specific cases.

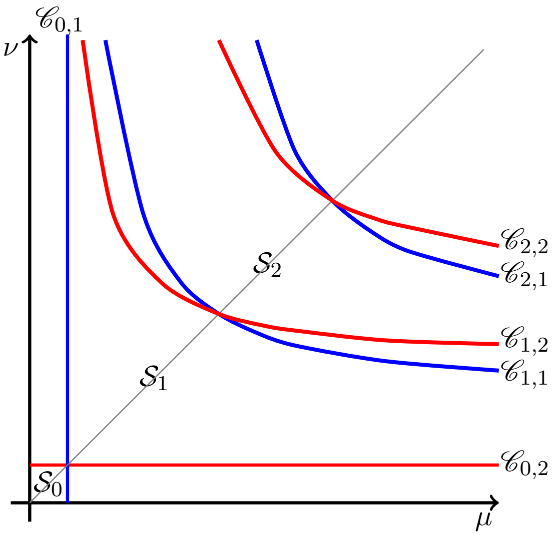

Figure 3: The Fučík spectrum for the antiperiodic problem and the sets $\mathcal{S}_k$.

(i) If $(\mu, \nu) \in \mathscr{C}_{k,1}$ with $\mu \neq \nu$, then the nontrivial solutions of problem (5) are of the type $x(t) = \rho\varphi_{\mu,\nu}(t)$ with $\rho > 0$. In particular, $x(0) = 0 = x(T)$ and $\dot{x}(0) > 0$.

(ii) If $(\mu, \nu) \in \mathscr{C}_{k,2}$ with $\mu \neq \nu$, then the nontrivial solutions of problem (5) can be written as $x(t) = \rho\varphi_{\mu,\nu}(t + \frac{\pi}{\sqrt{\mu}})$ with $\rho > 0$. In particular, $x(0) = 0 = x(T)$ and $\dot{x}(0) < 0$.

(iii) If $\mathscr{C}_{k,1} \cap \mathscr{C}_{k,2} = \{(\mu, \nu)\}$, i.e., $\mu = \nu = ((2k+1)\pi/T)^2$, it follows that the equation in (5) becomes linear and the nontrivial solutions are given by

$$x(t) = \rho \sin(\sqrt{\mu}(t - \theta)), \qquad \text{for any } \rho > 0 \text{ and } \theta \in \mathbb{R}.$$

2.2 Auxiliary results

In this section, we consider the problem

$$\begin{cases} \ddot{v} + \widehat{\mu}(t)v^+ - \widehat{\nu}(t)v^- = 0, \\ v(0) + v(T) = 0, \quad \dot{v}(0) + \dot{v}(T) = 0, \end{cases} \tag{9}$$

with the following hypothesis.

(H) The functions $\widehat{\mu}$, $\widehat{\nu} \in L^2(\mathbb{R})$ satisfy

$$\mu_1 \leqslant \widehat{\mu}(t) \leqslant \mu_2, \quad \nu_1 \leqslant \widehat{\nu}(t) \leqslant \nu_2, \tag{10}$$

for almost every $t \in \mathbb{R}$, all constants being positive.

Let us first recall the definition of "rotation number" of a planar curve around the origin. Assume that $s_1 < s_2$ and let $\phi : [s_1, s_2] \to \mathbb{R}^2$ be a continuous curve such that $\phi(t) \neq (0,0)$ for every $t \in [s_1, s_2]$. Writing $\phi(t) = (\rho(t)\cos\theta(t), \rho(t)\sin\theta(t))$, where $\rho : \mathbb{R} \to]0, +\infty[$ and $\theta : \mathbb{R} \to \mathbb{R}$ are continuous, we define

$$\mathrm{Rot}(\phi; [s_1, s_2]) = -\frac{\theta(s_2) - \theta(s_1)}{2\pi} .$$

In the following, when dealing with a solution x of (1), by a slight abuse of notation we will write $\mathrm{Rot}(x; [s_1, s_2])$ instead of $\mathrm{Rot}((x, \dot{x}); [s_1, s_2])$.

We first need the following result.

Proposition 2.1. *Assume* **(H)** *and let v be a nontrivial solution of the differential equation in* (9)*.*

1. *If* $\mathrm{Rot}(v; [a, b]) = N$ *for some $a < b$ and $N \in \mathbb{N}$, then*

$$N\mathcal{T}_{\mu_2,\nu_2} \leqslant b - a \leqslant N\mathcal{T}_{\mu_1,\nu_1} .$$

2. *If instead* $\mathrm{Rot}(v; [a, b]) = N + \frac{1}{2}$*, then*

$$N\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} \leqslant b - a \leqslant N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} .$$

Proof. The first part of the statement is rather standard (see, e.g., [12]), hence we omit the proof, for briefness. Let us prove the second part. Introducing the polar coordinates

$$(v, \dot{v}) = (\rho\cos\theta, \rho\sin\theta) ,$$

we see that

$$-\dot{\theta}(t) = \begin{cases} \widehat{\mu}(t)\cos^2\theta(t) + \sin^2\theta(t) , & \text{if } v(t) \geqslant 0 , \\ \widehat{\nu}(t)\cos^2\theta(t) + \sin^2\theta(t) , & \text{if } v(t) \leqslant 0 . \end{cases} \tag{11}$$

Notice that $-\dot{\theta}(t) > 0$ for every t. For definiteness, we assume $\theta(a) \in [-\frac{\pi}{2}, \frac{\pi}{2}[$; the case $\theta(a) \in [\frac{\pi}{2}, \frac{3\pi}{2}[$ can be treated similarly.

Set $\theta_0 = \theta(a)$. By assumption we have $\theta(b) = \theta_0 - (2N + 1)\pi$. Then we select $a \leqslant s_1 \leqslant s_2 < b$ such that $\theta(s_1) = -\pi/2$ and $\theta(s_2) = -\pi/2 - 2N\pi$, see Figure 4. By the first part of the statement, we deduce that

$$N\mathcal{T}_{\mu_2,\nu_2} \leqslant s_2 - s_1 \leqslant N\mathcal{T}_{\mu_1,\nu_1} . \tag{12}$$

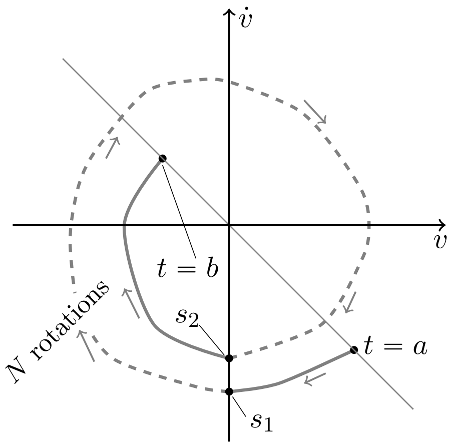

Figure 4: The rotating solution in the phase plane.

From (10) and (11), we get

$$\sin^2\theta(t) + \min\{\mu_1,\nu_1\}\cos^2\theta(t) \leqslant -\dot{\theta}(t) \leqslant \sin^2\theta(t) + \max\{\mu_2,\nu_2\}\cos^2\theta(t)\,,$$

which implies that

$$\frac{-\dot{\theta}(t)}{\sin^2\theta(t) + \max\{\mu_2,\nu_2\}\cos^2\theta(t)} \leqslant 1 \leqslant \frac{-\dot{\theta}(t)}{\sin^2\theta(t) + \min\{\mu_1,\nu_1\}\cos^2\theta(t)}\,.$$

Upon integration over $[a, s_1]$, we have

$$\int_{-\pi/2}^{\theta_0} \frac{d\theta}{\sin^2\theta + \max\{\mu_2,\nu_2\}\cos^2\theta} \leqslant s_1 - a \leqslant \int_{-\pi/2}^{\theta_0} \frac{d\theta}{\sin^2\theta + \min\{\mu_1,\nu_1\}\cos^2\theta}\,. \tag{13}$$

Similarly, integrating over $[s_2, b]$,

$$\int_{\theta_0-(2N+1)\pi}^{-\pi/2-2N\pi} \frac{d\theta}{\sin^2\theta + \max\{\mu_2,\nu_2\}\cos^2\theta} \leqslant b - s_2 \leqslant \int_{\theta_0-(2N+1)\pi}^{-\pi/2-2N\pi} \frac{d\theta}{\sin^2\theta + \min\{\mu_1,\nu_1\}\cos^2\theta}\,,$$

which is equivalent to write (since the integrand is π-periodic)

$$\int_{\theta_0}^{\pi/2} \frac{d\theta}{\sin^2\theta + \max\{\mu_2,\nu_2\}\cos^2\theta} \leqslant b - s_2 \leqslant \int_{\theta_0}^{\pi/2} \frac{d\theta}{\sin^2\theta + \min\{\mu_1,\nu_1\}\cos^2\theta}\,. \tag{14}$$

Summing in (13) and (14), we get

$$\int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sin^2\theta + \max\{\mu_2,\nu_2\}\cos^2\theta} \leqslant (s_1 - a) + (b - s_2) \leqslant \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sin^2\theta + \min\{\mu_1,\nu_1\}\cos^2\theta}\,, \tag{15}$$

and so

$$m_{\mu_2,\nu_2} = \frac{\pi}{\max\{\sqrt{\mu_2},\sqrt{\nu_2}\}} \leqslant (s_1 - a) + (b - s_2) \leqslant \frac{\pi}{\min\{\sqrt{\mu_1},\sqrt{\nu_1}\}} = M_{\mu_1,\nu_1}\,. \tag{16}$$

Adding the estimates in (12) and (16), we have

$$N\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} \leqslant b - a \leqslant N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1}\,, \tag{17}$$

which completes the proof. □

We recall the notation $\mathcal{R} = [\mu_1,\mu_2] \times [\nu_1,\nu_2]\,$.

Lemma 2.2. *Let assumption* **(H)** *hold. If $\mathcal{R} \subseteq \mathcal{S}$, then problem* (9) *only has the zero solution.*

Proof. By contradiction, assume that there is a nontrivial solution $v(t)$ for problem (9). Then $(v(t),\dot{v}(t)) \neq (0,0)$ for every $t \in [0,T]$, and there is an integer $K \geqslant 0$ such that $\mathrm{Rot}(v;[0,T]) = K + \frac{1}{2}$.

Recalling the definition of $\mathcal{S}$ in (8), we discuss two different cases.

Case 1: $\mathcal{R} \subseteq \mathcal{S}_0$.

From Proposition 2.1, we get

$$T < m_{\mu_2,\nu_2} \leqslant m_{\mu_2,\nu_2} + K\mathcal{T}_{\mu_2,\nu_2} \leqslant T\,,$$

which is impossible.

Case 2: $\mathcal{R} \subseteq \mathcal{S}_N$, for $N \geqslant 1$.

If $K \leqslant N-1$, then by Proposition 2.1, we have

$$T \leqslant K\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} \leqslant (N-1)\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} < T\,,$$

which is impossible. Similarly, if $K \geqslant N$, then we have

$$T \geqslant K\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} \geqslant N\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} > T\,,$$

which is again impossible.

This completes the proof of the lemma. □

We now consider the case when the rectangle $\mathcal{R}$ is contained in the closure of the set $\mathcal{S}$ and touches two curves of the Fučík spectrum.

Lemma 2.3. *Let μ_1, μ_2, ν_1, and ν_2 satisfy*

$$N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} = T = (N+1)\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2}\,, \tag{18}$$

for a certain integer $N \geqslant 0$. Suppose that problem (9) *admits a nontrivial solution v, with*

$$\begin{aligned} &\mu_1 \leqslant \widehat{\mu}(t) \leqslant \mu_2\,, \quad \text{a.e. on } I_+ = \{t \in [0,T] : v(t) > 0\}\,, \\ &\nu_1 \leqslant \widehat{\nu}(t) \leqslant \nu_2\,, \quad \text{a.e. on } I_- = \{t \in [0,T] : v(t) < 0\}\,. \end{aligned} \tag{19}$$

Then, either

$$\widehat{\mu}(t) = \mu_1 \text{ a.e. on } I_+ \quad \text{and} \quad \widehat{\nu}(t) = \nu_1 \text{ a.e. on } I_-\,,$$

or

$$\widehat{\mu}(t) = \mu_2 \text{ a.e. on } I_+ \quad \text{and} \quad \widehat{\nu}(t) = \nu_2 \text{ a.e. on } I_-\,.$$

Proof. Since v is a nontrivial solution, it satisfies $v^2(t) + \dot{v}^2(t) \neq 0$ for all $t \in [0,T]$, and $\mathrm{Rot}(v;[0,T]) = K + \frac{1}{2}$, for some integer $K \geqslant 0$. We claim that either $K = N$ or $K = N+1$. Indeed, if $K < N$, then by using Proposition 2.1 and (18), we have

$$T \leqslant K\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} < N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} = T\,,$$

which is impossible. Similarly, if $K > N+1$, we have

$$T \geqslant K\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} > (N+1)\mathcal{T}_{\mu_2,\nu_2} + m_{\mu_2,\nu_2} = T\,,$$

which is again impossible.

Let us analyze the alternative $K = N$, as the other situation follows similarly. We need to consider the following three cases.

<u>*Case 1.*</u> $\mu_1 < \nu_1$, i.e.,

$$T = N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\sqrt{\mu_1}} > N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\sqrt{\nu_1}}\,. \tag{20}$$

Claim. $v(0) = 0$ and $\dot{v}(0) > 0$.

Proof of the Claim. Assume by contradiction that $v(0) \neq 0$ or $v(0) = 0$ with $\dot{v}(0) < 0$.

We first discuss the case when $v(0) > 0$ or $v(0) = 0$ with $\dot{v}(0) < 0$. In this case, there exist $0 \leqslant t_0 < t_1 < \cdots < t_{2N} < T$ such that

$$v(t_i) = 0, \quad \dot{v}(t_{2i}) < 0\,, \quad \text{and} \quad \dot{v}(t_{2i+1}) > 0\,. \tag{21}$$

Introducing polar coordinates, recalling that $-\dot{\theta}(t) > 0$ for every t, we deduce that $\mathrm{Rot}(v; [t_i, t_{i+1}]) = 1/2$ for every i. We set $\theta(0) = \theta_0 \in [-\pi/2, \pi/2[\,$.

From Proposition 2.1, we have

$$t_{2N} - t_0 \leqslant N\mathcal{T}_{\mu_1,\nu_1}.$$

Combining this fact with the equality in (20) and recalling (11), we get

$$\begin{aligned}\frac{\pi}{\sqrt{\mu_1}} &\leqslant (t_0 - 0) + (T - t_{2N}) \\ &= \int_0^{t_0} \frac{-\dot{\theta}(t)dt}{\widehat{\mu}(t)\cos^2\theta(t) + \sin^2\theta(t)} + \int_{t_{2N}}^{T} \frac{-\dot{\theta}(t)dt}{\widehat{\nu}(t)\cos^2\theta(t) + \sin^2\theta(t)}\,,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{\theta_0-\pi}^{\theta_0} \frac{d\theta}{\mu_1\cos^2\theta + \sin^2\theta} \\ &\qquad \leqslant \int_{-\frac{\pi}{2}}^{\theta_0} \frac{d\theta}{\widetilde{\mu}(\theta)\cos^2\theta + \sin^2\theta} + \int_{\theta_0-\pi}^{-\frac{\pi}{2}} \frac{d\theta}{\widetilde{\nu}(\theta)\cos^2\theta + \sin^2\theta}\,,\end{aligned}$$

where $\widetilde{\mu}(\theta(t)) = \widehat{\mu}(t)$, and $\widetilde{\nu}(\theta(t)) = \widehat{\nu}(t)$. After splitting the left hand side integral, we thus obtain

$$\begin{aligned}&\int_{\theta_0-\pi}^{-\frac{\pi}{2}} \underbrace{\left(\frac{1}{\mu_1\cos^2\theta + \sin^2\theta} - \frac{1}{\widetilde{\nu}(\theta)\cos^2\theta + \sin^2\theta}\right)}_{>0,\ \text{since}\ \widetilde{\nu} \geqslant \nu_1 > \mu_1\,.} d\theta \\ &\qquad \leqslant \int_{-\frac{\pi}{2}}^{\theta_0} \underbrace{\left(\frac{1}{\widetilde{\mu}(\theta)\cos^2\theta + \sin^2\theta} - \frac{1}{\mu_1\cos^2\theta + \sin^2\theta}\right)}_{\leqslant 0,\ \text{since}\ \widetilde{\mu} \geqslant \mu_1\,.} d\theta\,.\end{aligned} \tag{22}$$

The left hand side integral is positive due to the fact that $\theta_0 < \frac{\pi}{2}$, and we get a contradiction.

The case $v(0) < 0$ can be treated similarly, thus completing the proof of the Claim.

Since $v(0) = 0$ and $\dot{v}(0) > 0$, we can select $0 = t_1 < \cdots < t_{2N+2} = T$ satisfying (21).

We now introduce the following modified polar coordinates

$$v = \begin{cases} \dfrac{1}{\sqrt{\mu_1}}\, r\cos\theta & \text{if } v \geqslant 0\,, \\ \dfrac{1}{\sqrt{\nu_1}}\, r\cos\theta & \text{if } v \leqslant 0\,, \end{cases} \qquad \dot{v} = r\sin\theta\,,$$

and observe that

$$\dot{\theta} = \begin{cases} \sqrt{\mu_1}\ \dfrac{\ddot{v}v - \dot{v}^2}{\mu_1 v^2 + \dot{v}^2} & \text{if } v > 0\,, \\ \sqrt{\nu_1}\ \dfrac{\ddot{v}v - \dot{v}^2}{\nu_1 v^2 + \dot{v}^2} & \text{if } v < 0\,. \end{cases}$$

Then, for $i = 1, \ldots, N+1$, we have

$$\begin{aligned} \frac{\pi}{\sqrt{\mu_1}} &= \int_{t_{2i-1}}^{t_{2i}} \frac{\widehat{\mu}(t)v^2 + \dot{v}^2}{\mu_1 v^2 + \dot{v}^2} dt \\ &= \int_{t_{2i-1}}^{t_{2i}} \frac{(\widehat{\mu}(t) - \mu_1)v^2}{\mu_1 v^2 + \dot{v}^2} dt + \int_{t_{2i-1}}^{t_{2i}} \frac{\mu_1 v^2 + \dot{v}^2}{\mu_1 v^2 + \dot{v}^2} dt \\ &= \int_{t_{2i-1}}^{t_{2i}} \frac{(\widehat{\mu}(t) - \mu_1)v^2}{\mu_1 v^2 + \dot{v}^2} dt + (t_{2i} - t_{2i-1})\,. \end{aligned} \tag{23}$$

Similarly, for $j = 2, \ldots, N+1$, we have

$$\begin{aligned} \frac{\pi}{\sqrt{\nu_1}} &= \int_{t_{2j-2}}^{t_{2j-1}} \frac{\widehat{\nu}(t)v^2 + \dot{v}^2}{\nu_1 v^2 + \dot{v}^2} dt \\ &= \int_{t_{2j-2}}^{t_{2j-1}} \frac{(\widehat{\nu}(t) - \nu_1)v^2}{\nu_1 v^2 + \dot{v}^2} dt + \int_{t_{2j-2}}^{t_{2j-1}} \frac{\nu_1 v^2 + \dot{v}^2}{\nu_1 v^2 + \dot{v}^2} dt \\ &= \int_{t_{2j-2}}^{t_{2j-1}} \frac{(\widehat{\nu}(t) - \nu_1)v^2}{\nu_1 v^2 + \dot{v}^2} dt + (t_{2j-1} - t_{2j-2})\,. \end{aligned} \tag{24}$$

Summing (23) for $i = 1, \ldots, N+1$ and (24) for $j = 2, \ldots, N+1$, by (20) we obtain

$$T = \int_{I_+} \frac{(\widehat{\mu}(t) - \mu_1)v^2}{\mu_1 v^2 + \dot{v}^2} dt + \int_{I_-} \frac{(\widehat{\nu}(t) - \nu_1)v^2}{\nu_1 v^2 + \dot{v}^2} dt + T\,,$$

which implies that

$$\int_{I_+} \frac{(\widehat{\mu}(t) - \mu_1)v^2}{\mu_1 v^2 + \dot{v}^2} dt + \int_{I_-} \frac{(\widehat{\nu}(t) - \nu_1)v^2}{\nu_1 v^2 + \dot{v}^2} dt = 0\,.$$

Recalling (19), we get

$$\widehat{\mu}(t) = \mu_1 \text{ a.e. on } I_+ \quad \text{ and } \quad \widehat{\nu}(t) = \nu_1 \text{ a.e. on } I_-\,,$$

thus completing the proof in Case 1.

Case 2. $\mu_1 > \nu_1$, i.e.,

$$T = N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\sqrt{\nu_1}} > N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\sqrt{\mu_1}} \,. \tag{25}$$

A similar computation as in Case 1 shows that $v(0) = 0$ and $\dot v(0) < 0$. Then one can find $0 = t_0 < \cdots < t_{2N+1} = T$ satisfying (21) and obtain estimates as in (23) and (24), so that the conclusion follows similarly.

Case 3. $\mu_1 = \nu_1$, i.e.,

$$T = \frac{(2N+1)\pi}{\sqrt{\mu_1}} \,. \tag{26}$$

In this case,

$$\begin{aligned}
\frac{(2N+1)\pi}{\sqrt{\mu_1}} &= \int_0^T \frac{\widehat\mu(t)(v^+)^2 + \widehat\nu(t)(v^-)^2 + \dot v^2}{\mu_1 v^2 + \dot v^2} dt \\
&= \int_0^T \frac{(\widehat\mu(t) - \mu_1)(v^+)^2 + (\widehat\nu(t) - \mu_1)(v^-)^2}{\mu_1 v^2 + \dot v^2} dt + \int_0^T \frac{\mu_1 v^2 + \dot v^2}{\mu_1 v^2 + \dot v^2} dt \\
&= \int_0^T \frac{(\widehat\mu(t) - \mu_1)(v^+)^2 + (\widehat\nu(t) - \mu_1)(v^-)^2}{\mu_1 v^2 + \dot v^2} dt + T \,,
\end{aligned}$$

which implies that

$$\int_0^T \frac{(\widehat\mu(t) - \mu_1)(v^+)^2 + (\widehat\nu(t) - \mu_1)(v^-)^2}{\mu_1 v^2 + \dot v^2} dt = 0 \,,$$

leading to the same conclusion as in the previous cases. $\square$

3 Main results

We will need the following assumption.

(G) The function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ has the form

$$g(t,x) = \gamma_+(t,x)x^+ - \gamma_-(t,x)x^- + h(t,x) \,,$$

where γ_+, γ_- and h are continuous functions such that

$$\mu_1 \leqslant \gamma_+(t,x) \leqslant \mu_2 \,, \quad \nu_1 \leqslant \gamma_-(t,x) \leqslant \nu_2 \,,$$

for every $t \in [0,T]$ and $x \in \mathbb{R}$, the above constants $\mu_1, \mu_2, \nu_1, \nu_2$ all being positive, and h is uniformly bounded.

As above, we denote the rectangle $[\mu_1, \mu_2] \times [\nu_1, \nu_2]$ by $\mathcal{R}$.

Theorem 3.1 (Nonresonance)**.** *If assumption* **(G)** *holds with* $\mathcal{R} \subseteq \mathcal{S}$ *, then problem* (1) *has a solution.*

Proof. Let $\mathcal{R} \subseteq \mathcal{S}_k$, for some $k \in \mathbb{N}$, and set

$$\overline{\mu} = \frac{\mu_1 + \mu_2}{2}\,, \quad \overline{\nu} = \frac{\nu_1 + \nu_2}{2}\,.$$

Consider the following family of problems

$$\begin{cases} \ddot{x} + (1-\sigma)[\overline{\mu}x^+ - \overline{\nu}x^-] + \sigma g(t,x) = 0\,, \\ x(0) + x(T) = 0\,, \quad \dot{x}(0) + \dot{x}(T) = 0\,, \end{cases} \tag{27}$$

with $\sigma \in [0,1]$. We aim to show that there is a $r > 0$ such that, for every solution x of (27), one has $\|x\|_\infty \leqslant r$.

Assume by contradiction that for every positive integer n there exist $\sigma_n \in [0,1]$ and a solution x_n of (27), with $\sigma = \sigma_n$, such that $\|x_n\|_\infty > n$. Passing to a subsequence we can assume that $(\sigma_n)_n$ converges to some $\overline{\sigma} \in [0,1]$. Set $v_n = \frac{x_n}{||x_n||_\infty}$. Then,

$$\begin{cases} \ddot{v}_n + \widehat{\mu}_n(t)v_n^+ - \widehat{\nu}_n(t)v_n^- + \sigma_n \frac{h(t,||x_n||_\infty v_n)}{||x_n||_\infty} = 0\,, \\ v_n(0) + v_n(T) = 0\,, \quad \dot{v}_n(0) + \dot{v}_n(T) = 0\,, \end{cases} \tag{28}$$

where

$$\widehat{\mu}_n(t) = (1-\sigma_n)\overline{\mu} + \sigma_n \gamma_+(t, ||x_n||_\infty v_n(t))\,,$$
$$\widehat{\nu}_n(t) = (1-\sigma_n)\overline{\nu} + \sigma_n \gamma_-(t, ||x_n||_\infty v_n(t))\,.$$

Notice that $\mu_1 \leqslant \widehat{\mu}_n(t) \leqslant \mu_2$ and $\nu_1 \leqslant \widehat{\nu}_n(t) \leqslant \nu_2$.

From the differential equation in (28) and the properties of $\widehat{\mu}_n$, $\widehat{\nu}_n$ and h, the sequence $(v_n)_n$ is bounded in $H^2(0,T)$, therefore there exists a function v such that, up to a subsequence, $v_n \to v$ in $C^1([0,T])$ and weakly in $H^2(0,T)$. In particular $||v||_\infty = 1$. Since the sequences $(\widehat{\mu}_n)_n$ and $(\widehat{\nu}_n)_n$ are bounded, we can suppose that, up to a subsequence, they converge weakly in $L^2(0,T)$ to some functions $\widehat{\mu}$ and $\widehat{\nu}$, respectively, with $\mu_1 \leqslant \widehat{\mu}(t) \leqslant \mu_2$ and $\nu_1 \leqslant \widehat{\nu}(t) \leqslant \nu_2$, almost everywhere on $[0,T]$. Passing to the weak limit in (28), v solves

$$\begin{cases} \ddot{v} + \widehat{\mu}(t)v^+ - \widehat{\nu}(t)v^- = 0\,, \\ v(0) + v(T) = 0\,, \quad \dot{v}(0) + \dot{v}(T) = 0\,, \end{cases} \tag{29}$$

for almost every $t \in [0,T]$, which is a contradiction with Lemma 2.2.

To conclude the proof of the theorem, recalling the notation (6), we define $\zeta = \mathcal{T}_{\overline{\mu},\overline{\nu}}$ and consider the curve

$$\mathscr{C} = \left\{ (\mu,\nu) : \mu > 0, \nu > 0, \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \zeta \right\} ,$$

which is contained in $\mathcal{S}_k$ for some k. This curve connects the point $(\overline{\mu},\overline{\nu})$ with $((\frac{2\pi}{\zeta})^2, (\frac{2\pi}{\zeta})^2)$. We parametrize this part of the curve by a continuous map $\xi : [0,1] \to \mathbb{R}^2$ as

$$\xi(\sigma) = (\overline{\mu}(\sigma), \overline{\nu}(\sigma)) ,$$

with $\xi(0) = (\overline{\mu},\overline{\nu})$, $\xi(1) = ((\frac{2\pi}{\zeta})^2, (\frac{2\pi}{\zeta})^2)$.

By Lemma 2.2, for every $\sigma \in [0,1]$, the problem

$$\begin{cases} \ddot{v} + \overline{\mu}(\sigma)v^+ - \overline{\nu}(\sigma)v^- = 0 , \\ v(0) + v(T) = 0 , \quad \dot{v}(0) + \dot{v}(T) = 0 \end{cases} \tag{30}$$

only has the zero solution.

The proof of the theorem can be now completed by a standard application of the Leray–Schauder topological degree theory. □

Remark 3.2. *Notice that the growth conditions in* (2) *imply assumption* **(G)** *after slightly changing the involved constants. Hence, being $\mathcal{S}$ an open set, in the assumption of Theorem 3.1, we can replace assumption* **(G)** *with* (2).

Next, we examine the case when $\mathring{\mathcal{R}} \subseteq \mathcal{S}_N$ for some $N \geqslant 1$, and $\mathcal{R} \cap \Sigma = \{(\mu_1,\nu_1),(\mu_2,\nu_2)\}$. This situation arises when the resonance condition (18) is satisfied.

Theorem 3.3 (Double Resonance)**.** *Assume* **(G)** *and the existence of a positive integer N such that* (18) *holds. If for every non-zero T-antiperiodic solution ψ of $\ddot{\psi} + \mu_1\psi^+ - \nu_1\psi^- = 0$ one has*

$$\int_{\{\psi<0\}} \limsup_{x\to-\infty} \big(g(t,x) - \nu_1 x\big)\psi(t)\, dt + \int_{\{\psi>0\}} \liminf_{x\to+\infty} \big(g(t,x) - \mu_1 x\big)\psi(t)\, dt > 0 , \tag{31}$$

and for every non-zero T-antiperiodic solution χ of $\ddot{\chi} + \mu_2\chi^+ - \nu_2\chi^- = 0$ one has

$$\int_{\{\chi<0\}} \limsup_{x\to-\infty} \big(\nu_2 x - g(t,x)\big)\chi(t)\, dt + \int_{\{\chi>0\}} \liminf_{x\to+\infty} \big(\mu_2 x - g(t,x)\big)\chi(t)\, dt > 0 , \tag{32}$$

then problem (1) *has a solution.*

In the above, we have used the standard notation

$$\{\psi < 0\} = \{t \in [0,T] \,:\, \psi(t) < 0\}\,,$$

and similarly for $\{\psi > 0\}$.

Proof. Let us set

$$\overline{\mu} = \frac{\mu_1 + \mu_2}{2}\,, \quad \overline{\nu} = \frac{\nu_1 + \nu_2}{2}\,,$$

and consider the family of problems (27), with $\sigma \in [0,1]$.

Claim. There is a $r > 0$ such that, for every solution x of (27) one has $\|x\|_\infty \leqslant r$.

In order to prove the Claim, assume by contradiction that for every positive integer n there exist $\sigma_n \in [0,1]$ and a solution x_n of (27), with $\sigma = \sigma_n$, such that $\|x_n\|_\infty > n$. Notice that $\sigma_n \neq 0$.

Passing to a subsequence we can assume that $(\sigma_n)_n$ converges to some $\overline{\sigma} \in [0,1]$. Set $v_n = \frac{x_n}{||x_n||_\infty}$. Then, v_n solves (28), and arguing as in the proof of Theorem 3.1, the sequence $(v_n)_n$ converges, up to a subsequence, to a function v in $C^1([0,T])$ and weakly in $H^2(0,T)$. This function is such that $||v||_\infty = 1$, and it solves (29) for almost every $t \in [0,T]$.

We apply Lemma 2.3 and consider, for definiteness, the first alternative in (19), i.e.,

$$\widehat{\mu}(t) = \mu_1 \text{ a.e. on } I_+ \quad \text{ and } \quad \widehat{\nu}(t) = \nu_1 \text{ a.e. on } I_-\,,$$

the second one being treated similarly. So, v is a solution of

$$\begin{cases} \ddot{v} + \mu_1 v^+ - \nu_1 v^- = 0\,, \\ v(0) + v(T) = 0\,, \quad \dot{v}(0) + \dot{v}(T) = 0\,. \end{cases} \tag{33}$$

Since we are assuming that (18) holds, from the first equality we deduce that $\mathrm{Rot}(v;[0,T]) = N + \frac{1}{2}$. This is also true for $(v_n(t), \dot{v}_n(t))$, if n is large enough, and so also for $(x_n(t), \dot{x}_n(t))$.

Let us write $(x_n, \dot{x}_n)$ in the following modified polar coordinates:

$$x_n = \begin{cases} \dfrac{1}{\sqrt{\mu_1}}\, r_n \cos\theta_n\,, & \text{if } x_n \geqslant 0\,, \\ \dfrac{1}{\sqrt{\nu_1}}\, r_n \cos\theta_n\,, & \text{if } x_n \leqslant 0\,, \end{cases} \qquad \dot{x}_n = r_n \sin\theta_n\,.$$

We compute the derivatives

$$\dot\theta_n = \begin{cases} \sqrt{\mu_1}\,\dfrac{\ddot x_n x_n - \dot x_n^2}{\mu_1 x_n^2 + \dot x_n^2} & \text{if } x_n > 0\,, \\ \sqrt{\nu_1}\,\dfrac{\ddot x_n x_n - \dot x_n^2}{\nu_1 x_n^2 + \dot x_n^2} & \text{if } x_n < 0\,. \end{cases}$$

Since the couple $(x_n(t), \dot x_n(t))$ performs precisely $2N+1$ half rotations around the origin in the interval $[0, T]$, two distinct cases arise.

<u>*Case 1.*</u> There exist

$$0 < t_0^n < t_1^n < \cdots < t_{2N}^n < T\,,$$

and $\alpha_n \in]0, \pi[$, satisfying

$$\begin{aligned} &\theta_n(0) = \alpha_n - \frac{\pi}{2}\,, \\ &x_n(t_l^n) = 0\,, && \text{for } l = 0, 1, 2, \dots, 2N\,, \\ &\dot x_n(t_{2j}^n) < 0\,, && \text{for } j = 0, 1, 2, \dots, N\,, \\ &\dot x_n(t_{2j-1}^n) > 0\,, && \text{for } j = 1, 2, \dots, N\,. \end{aligned}$$

<u>*Case 2.*</u> There exist

$$0 < t_1^n < \cdots < t_{2N}^n < t_{2N+1}^n < T\,,$$

and $\alpha_n \in]0, \pi[$, satisfying

$$\begin{aligned} &\theta_n(0) = -\alpha_n - \frac{\pi}{2}\,, \\ &x_n(t_l^n) = 0\,, && \text{for } l = 1, 2, \dots, 2N\,, \\ &\dot x_n(t_{2j}^n) < 0\,, && \text{for } j = 1, 2, \dots, N\,, \\ &\dot x_n(t_{2j-1}^n) > 0\,, && \text{for } j = 1, 2, \dots, N\,. \end{aligned}$$

We focus our analysis on Case 1, as the second one can be treated anal-

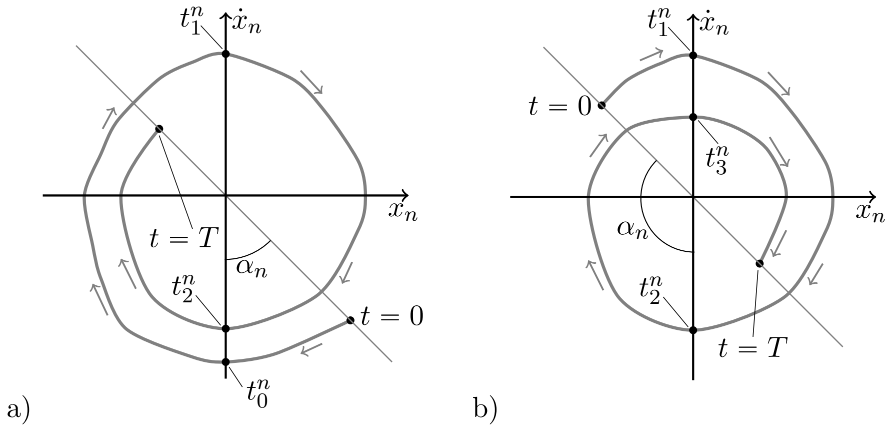

Figure 5: a) Case 1 for $N = 1$. b) Case 2 for $N = 1$.

ogously. We have the following time estimate:

$$
\begin{aligned}
\frac{\alpha_n}{\sqrt{\mu_1}} &= \frac{\theta_n(0) - \theta_n(t_0^n)}{\sqrt{\mu_1}} = \int_0^{t_0^n} \frac{-\dot{\theta}_n}{\sqrt{\mu_1}} \\
&= \int_0^{t_0^n} \frac{[(1-\sigma_n)\overline{\mu} x_n + \sigma_n g(t, x_n)]x_n + \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} \\
&\geqslant \int_0^{t_0^n} \frac{[(1-\sigma_n)\mu_1 x_n + \sigma_n g(t, x_n)]x_n + \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} \\
&= \int_0^{t_0^n} \frac{\mu_1 x_n^2 + \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} + \sigma_n \int_0^{t_0^n} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \\
&= t_0^n + \sigma_n \int_0^{t_0^n} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \,.
\end{aligned} \tag{34}
$$

A similar computation gives the following time estimates for $j = 1, \dots, N$:

$$
\begin{aligned}
\frac{\pi}{\sqrt{\nu_1}} &\geqslant \int_{t_{2j-2}^n}^{t_{2j-1}^n} \frac{\nu_1 x_n^2 + \dot{x}_n^2}{\nu_1 x_n^2 + \dot{x}_n^2} + \sigma_n \int_{t_{2j-2}^n}^{t_{2j-1}^n} \frac{(g(t, x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2} \\
&= (t_{2j-1}^n - t_{2j-2}^n) + \sigma_n \int_{t_{2j-2}^n}^{t_{2j-1}^n} \frac{(g(t, x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2} \,,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\frac{\pi}{\sqrt{\mu_1}} &\geqslant \int_{t_{2j-1}^n}^{t_{2j}^n} \frac{\mu_1 x_n^2 + \dot{x}_n^2}{\mu_1 x_n^2 + \dot{x}_n^2} + \sigma_n \int_{t_{2j-1}^n}^{t_{2j}^n} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \\
&= (t_{2j}^n - t_{2j-1}^n) + \sigma_n \int_{t_{2j-1}^n}^{t_{2j}^n} \frac{(g(t, x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \,,
\end{aligned} \tag{36}
$$

and

$$\frac{\pi-\alpha_n}{\sqrt{\nu_1}} \geqslant \int_{t_{2N}^n}^{T} \frac{\nu_1 x_n^2 + \dot{x}_n^2}{\nu_1 x_n^2 + \dot{x}_n^2} + \sigma_n \int_{t_{2N}^n}^{T} \frac{(g(t,x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2}$$
$$= (T - t_{2N}^n) + \sigma_n \int_{t_{2N}^n}^{T} \frac{(g(t,x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2}\,. \qquad (37)$$

Since the solution completes $2N+1$ half rotations in time T, addition in (34), (35), (36) and (37) for $j = 1, \ldots, N$ leads to

$$\begin{aligned}
\sigma_n\Big[\int_{\{x_n<0\}} &\frac{(g(t,x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2} + \int_{\{x_n>0\}} \frac{(g(t,x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2}\Big] + T \\
&\leqslant N\mathcal{T}_{\mu_1,\nu_1} + \frac{\alpha_n}{\sqrt{\mu_1}} + \frac{\pi-\alpha_n}{\sqrt{\nu_1}} \\
&\leqslant N\mathcal{T}_{\mu_1,\nu_1} + \frac{\alpha_n}{\min\{\sqrt{\mu_1},\sqrt{\nu_1}\}} + \frac{\pi-\alpha_n}{\min\{\sqrt{\mu_1},\sqrt{\nu_1}\}} \\
&= N\mathcal{T}_{\mu_1,\nu_1} + \frac{\pi}{\min\{\sqrt{\mu_1},\sqrt{\nu_1}\}} \\
&= N\mathcal{T}_{\mu_1,\nu_1} + M_{\mu_1,\nu_1} = T\,,
\end{aligned}$$

where the last step follows from (18). Being $\sigma_n \neq 0$, we get

$$\int_{\{x_n<0\}} \frac{(g(t,x_n) - \nu_1 x_n)x_n}{\nu_1 x_n^2 + \dot{x}_n^2} + \int_{\{x_n>0\}} \frac{(g(t,x_n) - \mu_1 x_n)x_n}{\mu_1 x_n^2 + \dot{x}_n^2} \leqslant 0\,,$$

which can be written as

$$\int_0^T \frac{\big[g(t,x_n) - (\mu_1 x_n^+ - \nu_1 x_n^-)\big]x_n}{\mu_1 (x_n^+)^2 + \nu_1 (x_n^-)^2 + \dot{x}_n^2} \leqslant 0\,.$$

Recalling that $v_n = \frac{x_n}{||x_n||_\infty}$, we have

$$\int_0^T \frac{\big[g(t,x_n) - (\mu_1 x_n^+ - \nu_1 x_n^-)\big]v_n}{\mu_1 (v_n^+)^2 + \nu_1 (v_n^-)^2 + \dot{v}_n^2} \leqslant 0\,.$$

Since $\mu_1(v^+(t))^2 + \nu_1(v^-(t))^2 + \dot{v}(t)^2$ is positive and constant in t, and

$$\lim_{n\to\infty} \big(\mu_1 (v_n^+)^2 + \nu_1 (v_n^-)^2 + \dot{v}_n^2\big) = \mu_1 (v^+)^2 + \nu_1 (v^-)^2 + \dot{v}^2\,,$$

uniformly in $[0,T]$, by Fatou's Lemma,

$$\int_0^T \liminf_n \frac{\big[g(t,x_n) - (\mu_1 x_n^+ - \nu_1 x_n^-)\big]v_n}{\mu_1 (v_n^+)^2 + \nu_1 (v_n^-)^2 + \dot{v}_n^2} \leqslant 0\,.$$

So, it has to be

$$\int_0^T \liminf_n \big[g(t,x_n(t)) - (\mu_1 x_n^+(t) - \nu_1 x_n^-(t))\big]v_n(t)\,dt \leqslant 0\,. \qquad (38)$$

Let us now fix $t \in [0,T]$ such that $v(t) < 0$; so $v_n(t) < 0$ for sufficiently large n, and $\lim_n x_n(t) = -\infty$, hence

$$\liminf_n [\nu_1 x_n(t) - g(t, x_n(t))] \geqslant \liminf_{x\to-\infty} [\nu_1 x - g(t,x)]\,,$$

which implies that, for every $t \in [0,T]$ with $v(t) < 0$, we have

$$\begin{aligned}\liminf_n \big[g(t, \|x_n\|_\infty v_n(t)) - (\mu_1 x_n^+(t) - \nu_1 x_n^-(t))\big]v_n(t) \\ \geqslant \liminf_{x\to-\infty}[\nu_1 x - g(t,x)]|v(t)| = \limsup_{x\to-\infty}[g(t,x) - \nu_1 x]v(t)\,.\end{aligned}$$

Similarly, if $v(t) > 0$ for some t, then $v_n(t) > 0$ for sufficiently large n, and $\lim_n x_n(t) = +\infty$, hence

$$\begin{aligned}\liminf_n \big[g(t, \|x_n\|_\infty v_n(t)) - (\mu_1 x_n^+(t) - \nu_1 x_n^-(t))\big]v_n(t) \\ \geqslant \liminf_{x\to+\infty}[g(t,x) - \mu_1 x]v(t)\,.\end{aligned}$$

Thus, by (38),

$$\int_{\{v<0\}} \limsup_{x\to-\infty} \big(g(t,x) - \nu_1 x\big)v(t)\,dt + \int_{\{v>0\}} \liminf_{x\to+\infty} \big(g(t,x) - \mu_1 x\big)v(t)\,dt \leqslant 0\,, \tag{39}$$

a contradiction with (31), thus proving the Claim. The proof of Theorem 3.3 can now be completed arguing as in the proof of Theorem 3.1. □

Remark 3.4. *A similar existence result holds for the so-called simple resonance, i.e., the case when* $\mathring{\mathcal{R}} \subseteq \mathcal{S}$*, and* $\mathcal{R} \cap \Sigma = \{(\mu_1, \nu_1)\}$ *or* $\mathcal{R} \cap \Sigma = \{(\mu_2, \nu_2)\}$*. Clearly enough, in this case, the Landesman–Lazer condition will be imposed only on one side.*

Remark 3.5. *Notice that Theorem 3.3 generalizes Theorem 3.1. Indeed, if we focus our attention on* $\mathcal{R} \subseteq \mathcal{S}$*, it is possible to find* $\epsilon_1, \epsilon_2 > 0$ *such that*

$$N\mathcal{T}_{\mu_1-\epsilon_1,\nu_1-\epsilon_1} + M_{\mu_1-\epsilon_1,\nu_1-\epsilon_1} = T = (N+1)\mathcal{T}_{\mu_2+\epsilon_2,\nu_2+\epsilon_2} + m_{\mu_2+\epsilon_2,\nu_2+\epsilon_2}\,.$$

Setting $\tilde\mu_1 = \mu_1 - \epsilon_1$*,* $\tilde\mu_2 = \mu_2 + \epsilon_2$*,* $\tilde\nu_1 = \nu_1 - \epsilon_1$ *and* $\tilde\nu_2 = \nu_2 + \epsilon_2$*, we have*

$$\lim_{x\to-\infty} \big(g(t,x) - \tilde\nu_1 x\big) = -\infty\,, \qquad \lim_{x\to+\infty} \big(g(t,x) - \tilde\mu_1 x\big) = +\infty\,,$$

and

$$\lim_{x\to-\infty} \big(\tilde\nu_2 x - g(t,x)\big) = -\infty\,, \qquad \lim_{x\to+\infty} \big(\tilde\mu_2 x - g(t,x)\big) = +\infty\,,$$

uniformly in t*, from which we easily verify that the Landesman–Lazer conditions* (31) *and* (32)*, respectively, hold.*

Remark 3.6. *The results contained in this paper could be rephrased in a* L^2*-Carathéodory setting. We avoid the details, for briefness.*

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