On the period map of planar Hamiltonian systems with separated variables

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Dedicated to Marco Sabatini on the occasion of his 70th birthday

Abstract. We consider planar Hamiltonian systems with Hamiltonian function of the type $H(x, y) = F(y) + G(x)$ having the origin as a global center, and generalize to this setting some known results on the period map. We are thus led to a characterization of isochronism for scalar second order equations involving the p-Laplacian operator.

1 Introduction

Let us start considering the scalar second order equation

$$x'' + g(x) = 0\,,$$

which is equivalent to a Hamiltonian system with Hamiltonian function $H(x, y) = \frac{1}{2}y^2 + G(x)$, where

$$G(x) = \int_0^x g(s)\, ds\,.$$

Assuming that the origin is a global center, the period map associating to any $E > 0$ the minimal period $\tau(E)$ of a solution with constant energy $H(x, y) = E$ is well defined. As a particular case, when the period map τ is constant, we have an *isochronous* equation and, correspondingly, an associated isochronous system: every nonconstant solution is periodic and all solutions share the same period.

A classical problem is the following.

Period map problem. *Given a positive function τ, recover g for which the period map of the corresponding system is precisely τ.*

The isochronism problem, i.e., such a problem for a constant period map τ, has been widely studied in literature, see e.g. [4, 8, 17, 20, 21, 23]. Urabe [26] proved that $g(x) = ax$ is the unique odd analytic function providing isochrononicity. Later on, it has been proved that, up to possible shifts, it is the unique isochronous function among polynomials [2, 9]. See also [1, 7, 10, 11, 13, 14, 15, 17, 19, 24, 27, 28] for other results. There are several other examples of isochronous functions g as, for instance,

$$g(x) = x - (x+1)^{-3} + 1\,, \qquad \text{on } \;]-1,+\infty[\,,$$

see [21, 23], or

$$g(x) = 1 - (1+2x)^{-1/2}\,, \qquad \text{on } \;]-1/2,+\infty[\,,$$

see [26]. As proved in [5] (see also [3, 6, 11]), a potential G is isochronous if and only if its graph arises by horizontally shearing the graph of a parabola.

The period map problem stated above can be formulated for the more general Hamiltonian system

$$x' = f(y)\,, \qquad y' = -g(x)\,, \tag{1}$$

with Hamiltonian function $H(x,y) = F(y) + G(x)$, where $G(x)$ has been defined above and

$$F(y) = \int_0^y f(\sigma)\,d\sigma\,.$$

These are what we call systems *with separated variables*. We will recall some known properties of the period map for these general systems in Section 2.

As a particular case, we have the system generated by a scalar equation of the type

$$\big(|x'|^{p-2}x'\big)' + g(x) = 0\,, \tag{2}$$

ruled by a p-Laplacian differential operator, with $p > 1$. In Section 3 we tackle the period map problem in this framework, extending the classical results. Our approach follows and completes the arguments in [16], where the case $p = 2$ was treated.

In Section 4 we concentrate our attention on the isochronism problem for planar systems of the type (1), and extend to a more general setting the results in [25].

Finally, in Section 5 we reconsider our main result from a different perspective, which could deserve further investigation.

2 Preliminaries

We will be interested in system (1) under suitable assumptions on f and g.

Definition 1. *The function* $f : \mathbb{R} \to \mathbb{R}$ *is said to be* admissible *if it is continuous and satisfies*

$$f(y)y > 0\,, \qquad \textit{for every } y \neq 0\,,$$

and

$$\lim_{|y|\to\infty} F(y) = +\infty\,.$$

Assuming both f and g to be admissible, the origin is the only equilibrium point for system (1), and all the other solutions surround it and are periodic. More precisely, for every $E > 0$ the set

$$\Omega(E) = \{(x, y) \in \mathbb{R}^2 : F(y) + G(x) \leq E\}\,,$$

depicted in Figure 1, is bounded and its boundary corresponds to a periodic solution of system (1), whose minimal period will be denoted by $\tau(E)$. The function $\tau :]0, +\infty[\to]0, +\infty[$ defined in this way will be called *the period map* of the system.

We will now provide some useful formulas for the computation of the period map (see also, e.g., [10, 25]). Consider the restrictions

$$F_-, G_- :]-\infty, 0] \to [0, +\infty[\,, \qquad F_+, G_+ : [0, +\infty[\to [0, +\infty[\,,$$

for which $F_\pm(y) = F(y)$ and $G_\pm(x) = G(x)$. Since f and g are admissible, all these functions are invertible. We then define, for $E \geq 0$,

$$\ell_f(E) = F_+^{-1}(E) - F_-^{-1}(E)\,, \qquad \ell_g(E) = G_+^{-1}(E) - G_-^{-1}(E)\,.$$

For every solution of system (1) with energy E, we select some instants $t_0 < t_1$ such that $y(t) \geq 0$ for $t \in [t_0, t_1]$, and $y(t) \leq 0$ for $t \in [t_1, t_0 + \tau(E)]$. We focus our attention on the former interval, so that

$$y(t) = F_+^{-1}(E - G(x(t)))\,,$$

and so

$$-g(x(t)) = y'(t) = -g(x(t))\,x'(t)\,(F_+^{-1})'(E - G(x(t)))\,.$$

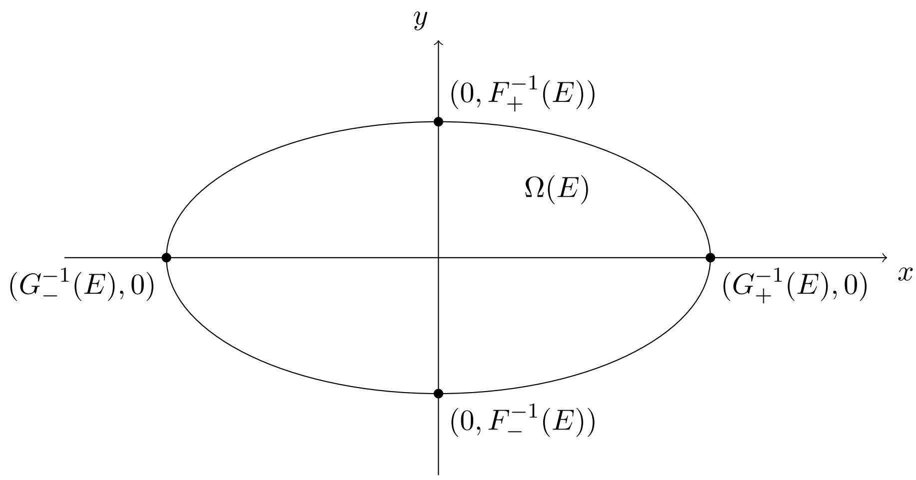

Figure 1: The set $\Omega(E)$ delimited by the level curve of energy E.

Since there exists a unique instant in $]t_0, t_1[$ at which $x(t)$ vanishes, an integration on $[t_0, t_1]$ gives

$$t_1 - t_0 = \int_{G_-^{-1}(E)}^{G_+^{-1}(E)} (F_+^{-1})'(E - G(s))\, ds\,.$$

Applying a similar reasoning in the interval $[t_1, t_0 + \tau(E)]$ and summing the two contributions, we get

$$\tau(E) = \int_{G_-^{-1}(E)}^{G_+^{-1}(E)} \ell_f'(E - G(s))\, ds\,. \tag{3}$$

Similarly, one can obtain the analogous formula

$$\tau(E) = \int_{F_-^{-1}(E)}^{F_+^{-1}(E)} \ell_g'(E - F(\sigma))\, d\sigma\,.$$

Further, by (3) we compute

$$\begin{aligned}
\tau(E) &= \int_{G_-^{-1}(E)}^{0} \ell_f'(E - G(s))\, ds + \int_0^{G_+^{-1}(E)} \ell_f'(E - G(s))\, ds \\
&= \int_0^E \ell_f'(E - u)(G_+^{-1})'(u)\, du - \int_0^E \ell_f'(E - u)(G_-^{-1})'(u)\, du \\
&= \int_0^E \ell_f'(E - u)(G_+^{-1}(u) - G_-^{-1}(u))'\, du\,,
\end{aligned}$$

and so

$$\tau(E) = \int_0^E \ell_f'(E - u)\ell_g'(u)\, du\,. \tag{4}$$

This last formula could also be obtained as $\tau(E) = a'(E)$, where $a(E)$ is the area of the set $\Omega(E)$, see for instance [12, Lemma 2.1]. We avoid the details, for briefness.

We will use the above formulas in the sequel.

3 The p-Laplacian equation

We consider the scalar differential equation (2), with $p > 1$. This equation can be written in the form of system (1) with $f(y) = |y|^{q-2}y$, i.e.,

$$x' = |y|^{q-2}y\,, \qquad y' = -g(x)\,, \tag{5}$$

where $q > 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Hence, the associated Hamiltonian function is

$$H(x,y) = \frac{1}{q}|y|^q + G(x)\,.$$

Here is our main result in this framework.

Theorem 2. *Let $\tau :]0,+\infty[\to]0,+\infty[$ be a continuously differentiable function, and set $\rho(E) = E^{1/p}\tau(E)$. Assume that the function ρ is strictly increasing and satisfies*

$$\lim_{E\to 0^+} \rho(E) = 0\,, \qquad \lim_{E\to 0^+} \rho'(E) = +\infty\,, \tag{6}$$

and

$$\liminf_{E\to +\infty} E\rho'(E) > 0\,. \tag{7}$$

Then, there exists a unique admissible odd function g for which τ is precisely the period map of system (5)*. Any other admissible function $\tilde{g}$ has the same property if and only if $\ell_{\tilde{g}}(E) = \ell_g(E)$, for every $E \geq 0$.*

Proof. By (4), if g is the function we are looking for, then necessarily

$$\tau(\mathcal{E}) = \int_0^{\mathcal{E}} \ell_f'(\mathcal{E}-u)\ell_g'(u)\,du = \frac{2}{q^{1/p}}\int_0^{\mathcal{E}} \frac{\ell_g'(u)}{(\mathcal{E}-u)^{1/p}}\,du\,,$$

for every $\mathcal{E} > 0$. So, we start seeking a function η such that $\eta(0) = 0$ and

$$\tau(\mathcal{E}) = \frac{2}{q^{1/p}}\int_0^{\mathcal{E}} \frac{\eta'(u)}{(\mathcal{E}-u)^{1/p}}\,du\,, \qquad \text{for every } \mathcal{E} > 0\,.$$

Assuming its existence, we must have

$$\begin{aligned}\int_0^E \frac{\tau(\mathcal{E})}{(E-\mathcal{E})^{1/q}}\,d\mathcal{E} &= \frac{2}{q^{1/p}}\int_0^E \left(\int_0^{\mathcal{E}} \frac{\eta'(u)}{(E-\mathcal{E})^{1/q}(\mathcal{E}-u)^{1/p}}\,du\right)d\mathcal{E}\\ &= \frac{2}{q^{1/p}}\int_0^E \left(\int_u^{E} \frac{1}{(E-\mathcal{E})^{1/q}(\mathcal{E}-u)^{1/p}}\,d\mathcal{E}\right)\eta'(u)\,du\,.\end{aligned}$$

Now, by the change of variable $v=(E-\mathcal{E})/(E-u)$,

$$\begin{aligned}\int_u^E \frac{d\mathcal{E}}{(E-\mathcal{E})^{1/q}(\mathcal{E}-u)^{1/p}} &= \int_0^1 \frac{dv}{v^{1/q}(1-v)^{1/p}} = B\left(\tfrac{1}{p},\tfrac{1}{q}\right)\\ &= B\left(\tfrac{1}{p},1-\tfrac{1}{p}\right) = \frac{\pi}{\sin(\pi/p)}\,,\end{aligned}$$

where we have used the Beta function

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt\,.$$

Notice that the above integral is independent of both u and E, and hence

$$\int_0^E \frac{\tau(\mathcal{E})}{(E-\mathcal{E})^{1/q}}\,d\mathcal{E} = \frac{2}{q^{1/p}}\int_0^E \frac{\pi}{\sin(\pi/p)}\eta'(u)\,du = \frac{2\pi}{q^{1/p}\sin(\pi/p)}\eta(E)\,.$$

So, the continuous function $\eta :]0,+\infty[\, \to\,]0,+\infty[$ can be defined as

$$\eta(E) = \frac{q^{1/p}\sin(\pi/p)}{2\pi}\int_0^E \frac{\tau(\mathcal{E})}{(E-\mathcal{E})^{1/q}}\,d\mathcal{E}\,. \tag{8}$$

Setting

$$c_p = \frac{q^{1/p}\sin(\pi/p)}{2\pi}\,,$$

one has

$$\eta(E) = c_p\int_0^1 \frac{E^{1/p}\,\tau(Es)}{(1-s)^{1/q}}\,ds = c_p\int_0^1 \frac{\rho(Es)}{s^{1/p}(1-s)^{1/q}}\,ds\,,$$

so we deduce from the first assumption in (6) that $\lim_{E\to 0^+}\eta(E)=0$. Moreover, since $\rho(E)$ is strictly increasing, the function $\eta(E)$ is strictly increasing, as well.

Some elementary careful estimates guarantee that we are allowed to apply the Leibniz Integral Rule, providing us

$$\eta'(E) = c_p\int_0^1 \frac{s^{1/q}\rho'(Es)}{(1-s)^{1/q}}\,ds\,.$$

Then, assumption (7) gives us, by Fatou's Lemma,

$$\begin{aligned}\liminf_{E\to+\infty} E\,\eta'(E) &\geq c_p \liminf_{E\to+\infty} \int_0^1 \frac{Es\,\rho'(Es)}{s^{1/p}(1-s)^{1/q}}\,ds \\ &\geq c_p \int_0^1 \liminf_{E\to+\infty} \frac{Es\,\rho'(Es)}{s^{1/p}(1-s)^{1/q}}\,ds > 0\,,\end{aligned}$$

implying that $\lim_{E\to+\infty}\eta(E) = +\infty$. In conclusion, we have proved that the function $\eta :]0,+\infty[\,\to\,]0,+\infty[$ is invertible.

By the second assumption in (6) and Fatou's Lemma again we have that

$$\begin{aligned}\liminf_{E\to 0^+} \eta'(E) &\geq c_p \liminf_{E\to 0^+} \int_0^1 \frac{s^{1/q}\rho'(Es)}{(1-s)^{1/q}}\,ds \\ &\geq c_p \int_0^1 \liminf_{E\to 0^+} \frac{s^{1/q}\rho'(Es)}{(1-s)^{1/q}}\,ds = +\infty\,.\end{aligned}$$

This implies that the function $\eta^{-1} :]0,+\infty[\,\to\,]0,+\infty[$ can be extended by setting $\eta^{-1}(0) = 0$ so to obtain a continuously differentiable function, for which we maintain the same notation. We can now define $G : \mathbb{R} \to \mathbb{R}$ as the even function

$$G(s) = \begin{cases} \eta^{-1}(2s) & \text{if } s \geq 0\,, \\ \eta^{-1}(-2s) & \text{if } s < 0\,. \end{cases} \tag{9}$$

In such a way, G is continuously differentiable, and so we obtain the function we are looking for, by setting $g = G'$.

Since formula (8) prescribes the value of ℓ_g being equal to η, the conclusion of the proof easily follows. □

Remark 3. *The procedure followed in the first part of the above proof is inspired by [16], where the case* $p = 2$ *has been treated. However, the authors did not comment therein on the invertibility of the function* $\eta(E)$*. Indeed, in this respect, our result seems to be new in the literature even in the case* $p = 2$*.*

The assumptions of Theorem 2 surely hold true if $\tau(E)$ is a nondecreasing function. In particular, we immediately deduce the following.

Corollary 4. *For any positive constant* τ*, the unique admissible odd function* g *for which the p-Laplacian equation* (2) *is isochronous with associated period* τ *is*

$$g(x) = \left(\frac{2\pi_p}{\tau}\right)^2 |x|^{p-2}x\,,$$

where

$$\pi_p = \frac{2(p-1)^{1/p}}{p\sin(\pi/p)}\pi\,.$$

Any other admissible function $\tilde g$ has the same property if and only if

$$\ell_{\tilde g}(E) = \frac{\tau}{\pi_p}(pE)^{1/p}\,, \qquad \text{for every } E \geq 0\,.$$

Proof. It is well known that system (5) with $g(x) = a|x|^{p-2}x$ is isochronous with period $2\pi_p/\sqrt{a}$ (see, e.g., [18]). Hence, choosing $a = (2\pi_p/\tau)^2$, we have isochronism with period τ. Theorem 2 guarantees the uniqueness claimed in the first part of the statement, and any other admissible function $\tilde g$ has this isochronism property if and only if $\ell_{\tilde g}(E) = \ell_g(E)$, for every $E \geq 0$. From (8), we can compute

$$\ell_g(E) = \frac{q^{1/p}\sin(\pi/p)}{2\pi}\int_0^E \frac{\tau}{(E-\mathcal{E})^{1/q}}\,d\mathcal{E} = \frac{\tau}{\pi_p}(pE)^{1/p}\,,$$

yielding the conclusion. □

Remark 5. *If $p = 2$, we obtain $\ell_g(E) = \frac{\tau}{\pi}\sqrt{2E}$. As already mentioned in the Introduction, this fact was emphasized in [3, 5, 6, 11], where the authors said that a potential G is isochronous if and only if its graph arises by horizontally shearing the graph of a parabola $y = c\,x^2$. For a general $p > 1$ we have proved that the same property holds replacing the parabola by the curve $y = c\,|x|^p$.*

Remark 6. *The function $\tau(E) = E^\alpha$ satisfies the assumptions of Theorem 2 provided that $-1/p < \alpha < 1/q$. Notice that this period function is precisely the outcome for the p-Laplacian equation (2) when $g(x) = c\,|x|^{r-2}x$, for a suitable constant $c = c(\alpha) > 0$ and*

$$r = r(\alpha) = \frac{p}{1+\alpha p}\,.$$

Indeed, for such an equation we compute, using (4),

$$\tau(E) = 4\,q^{-1/p}\,r^{-1+1/r}\,c^{-1/r}\,B\big(\tfrac{1}{r},\tfrac{1}{q}\big)\,E^{\frac{1}{r}-\frac{1}{p}},$$

providing us with the value of the constant

$$c(\alpha) = \big[4\,q^{-1/p}\,B\big(\alpha+\tfrac{1}{p},\tfrac{1}{q}\big)\big]^{\frac{p}{\alpha p+1}}\left(\frac{p}{\alpha p+1}\right)^{1-\frac{p}{\alpha p+1}}.$$

Notice that, taking $\alpha = -1/p$, recalling (8) we see that the function $\eta(E)$ is constant, hence not invertible. On the other hand, taking $\alpha = 1/q$, we *find $\eta(E) = pc_pE$, so that in (9) one has $G(s) = 2|s|/(pc_p)$, which is not differentiable at 0. These facts show that the assumptions of Theorem 2 are in some sense optimal.*

Remark 7. *If we do not assume (7), then it could be that $\lim_{E\to+\infty}\eta(E) = \ell \in\,]0,+\infty[$. In this case the function G would be defined only on $]-\frac{\ell}{2},\frac{\ell}{2}[$, with*

$$\lim_{s\to-\frac{\ell}{2}^+} G(s) = \lim_{s\to\frac{\ell}{2}^-} G(s) = +\infty\,,$$

hence we would have a singular potential.

4 Isochronism in systems with separated variables

As usual, let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and consider two admissible functions f and g. Our aim in this section is to show that the isochronism of the two systems

$$y' = |x|^{p-2}x\,, \qquad x' = -f(y) \tag{10}$$

and

$$x' = |y|^{q-2}y\,, \qquad y' = -g(x) \tag{11}$$

guarantees the isochronism of system

$$x' = f(y)\,, \qquad y' = -g(x)\,. \tag{12}$$

The following theorem generalizes [25, Theorem 2.5], where the case $p = q = 2$ has been treated.

Theorem 8. *Let system (10) be isochronous of period $\mathcal{T}_f$. Then, system (11) is isochronous of period $\mathcal{T}_g$ if and only if system (12) is isochronous of period*

$$\tau = \frac{\mathcal{T}_f\mathcal{T}_g}{2\pi_p}\,.$$

Proof. Since system (10) is isochronous, by Corollary 4 we have that

$$\ell_f(E) = \frac{\mathcal{T}_f}{\pi_q}(qE)^{1/q}\,,$$

and so

$$\ell_f'(E) = \frac{\mathcal{T}_f}{\pi_q(qE)^{1/p}}\,.$$

From (3), the period map of system (12) is then given by

$$\tau(E) = \int_{G_-^{-1}(E)}^{G_+^{-1}(E)} \ell_f'(E - G(s))\, ds = \frac{\mathcal{T}_f}{\pi_p q^{1/p}} \int_{G_-^{-1}(E)}^{G_+^{-1}(E)} \frac{ds}{[E - G(s)]^{1/p}}\,,$$

where we have used the fact that $\pi_q = \pi_p$.

Let us denote by $\mathcal{T}_g(E)$ the period map of system (11). Again, using (3) with $f(y) = |y|^{q-2}y$, we get

$$\mathcal{T}_g(E) = \frac{2}{q^{1/p}} \int_{G_-^{-1}(E)}^{G_+^{-1}(E)} \frac{ds}{[E - G(s)]^{1/p}} = \frac{2\pi_p \tau(E)}{\mathcal{T}_f}\,.$$

The conclusion easily follows. □

5 Final remarks

In the proof of Theorem 2, a crucial step in order to obtain formula (8) was the observation that there exists a function h for which the integral

$$\int_u^E \ell_f'(\mathcal{E} - u) h(E - \mathcal{E})\, d\mathcal{E}$$

is independent of both u and E. Precisely, $h(x) = x^{-1/q}$. When dealing with the isochronous problem, an alternative approach could be the following: after writing (4) as

$$\tau = \int_0^E \ell_f'(E - u)\ell_g'(u)\, du = \ell_f' * \eta',$$

we perform a unilateral Laplace transform so to obtain

$$\mathcal{L}(\tau) = \mathcal{L}(\ell_f')\mathcal{L}(\eta')\,,$$

i.e.,

$$\frac{\tau}{s} = s\mathcal{L}(\ell_f)(s)\, s\mathcal{L}(\eta)(s),$$

whence

$$\mathcal{L}(\eta)(s) = \frac{\tau}{s^3 \mathcal{L}(\ell_f)(s)}.$$

We claim that the right-hand side in the above equality is well defined for any $s \in \mathbb{C}$ having positive real part. Indeed, for any $M > 0$ the function $\ell_f \chi_{[0,M]}$ is positive and increasing in $[0, M]$. As a consequence of a result by Pólya [22], this fact implies that $\mathcal{L}(\ell_f \chi_{[0,M]})$ can vanish only in the left half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z \le 0\}$. Since $\mathcal{L}(\ell_f \chi_{[0,M]})(s) \to \mathcal{L}(\ell_f)(s)$ for every $s \in \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ as a consequence of the Dominated Convergence Theorem, the claim follows.

We then find the formula

$$\eta = \mathcal{L}^{-1}\left(\frac{\tau}{s^3\mathcal{L}(\ell_f)(s)}\right).$$

As an example, when $f(y) = |y|^{q-2}y$, we have that $\ell_f(E) = 2(qE)^{1/q}$, hence

$$\mathcal{L}(\ell_f)(s) = \frac{2\Gamma(1/q)}{q^{1/p}s^{1+1/q}},$$

where Γ is the Euler Gamma function. It follows that

$$\eta = \mathcal{L}^{-1}\left(\frac{\tau q^{1/p}s^{1+1/q}}{2s^3\Gamma(1/q)}\right) = \frac{\tau q^{1/p}}{2\Gamma(1/q)}\mathcal{L}^{-1}\left(\frac{1}{s^{1+1/p}}\right).$$

Since

$$\mathcal{L}^{-1}\left(\frac{1}{s^{1+1/p}}\right)(x) = \frac{x^{1/p}}{\Gamma(1+1/p)},$$

we obtain

$$\eta(E) = \frac{\tau p q^{1/p}E^{1/p}}{2\Gamma(1/q)\Gamma(1/p)}.$$

Recalling the well-known property of the Gamma function

$$\Gamma(1/p)\Gamma(1/q) = B(1/p, 1/q) = \frac{\pi}{\sin(\pi/p)},$$

we finally obtain

$$\eta(E) = \frac{\tau}{\pi_p}(pE)^{1/p},$$

corresponding to the formula in the statement of Corollary 4.

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