Applying the Poincaré–Birkhoff theorem to antiperiodic problems

Alessandro Fonda, Natnael Gezahegn Mamo,
Andrea Sfecci, and Wahid Ullah

Dedicated to Giuseppe Marino, on the occasion of his retirement

Abstract

We show how the Poincaré–Birkhoff theorem for Hamiltonian systems can be used to find multiple solutions of the antiperiodic problem. Applications are given to scalar second order differential equations whose nonlinearities provide a twist in the phase plane, among which those with a superlinear or sublinear behaviour at infinity.

1 Introduction

In recent years the Poincaré–Birkhoff theorem has found many applications to the search of periodic solutions of planar Hamiltonian systems. It is the aim of this short paper to show how the same theorem can be also used for detecting multiple solutions of the antiperiodic problem.

The so-called Poincaré's last geometric theorem [24] dates back to 1912 (historical accounts can be found in [10, 12, 22]). Originally stated for area-preserving homeomorphisms of an annulus

$$\mathcal{A} = \left\{(x, y) \in \mathbb{R}^2 : a \leqslant \sqrt{x^2 + y^2} \leqslant b\right\},$$

it can be restated on the strip $\mathcal{S} = \mathbb{R} \times [a, b]$ by the use of some modified polar coordinates (see [5, 26]). Different variants have been proposed, and also higher dimensional versions of it are now available for the Poincaré map of Hamiltonian flows (see, e.g., [17]).

To our knowledge, all the available applications of the Poincaré–Birkhoff theorem have been focalized on the periodic problem. Poincaré himself [24] stated this theorem in order to locate periodic solutions for the three body problem in Celestial Mechanics, followed by Birkhoff [1] who had in mind applications to Dynamics.

Consider now the antiperiodic problem associated with a planar Hamiltonian system, i.e.,

$$\begin{cases} J\dot{u} = \nabla_u H(t,u)\,, \\ u(T) = -u(0)\,. \end{cases} \tag{1}$$

Here $u = (q,p) \in \mathbb{R}^2$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ denotes the standard symplectic matrix. We will show how to adapt the version of the Poincaré–Birkhoff theorem on the annulus in order to obtain a multiplicity result for this problem. Several applications to second order differential equations will then directly follow in analogy with the corresponding results already known for the periodic problem.

The paper is organized as follows. In Section 2 we recall the Poincaré–Birkhoff theorem on a strip with varying boundaries in the version proposed by the first author jointly with Ureña [17]. This will be the starting point for our study of a peculiar boundary value problem in Section 3, where we search for solutions whose endpoint $u(T)$ is obtained by rotating the initial point $u(0)$ around the origin by a certain angle ϑ. The particular case $\vartheta = \pi$ will provide us the antiperiodic solutions we are looking for. In Section 4 we list a series of applications, mainly focused on second order differential equations.

2 The Poincaré–Birkhoff theorem on a strip

In this section we recall the version of the classical Poincaré–Birkhoff theorem proposed in [17]. Consider the problem

$$\begin{cases} J\dot{u} = \nabla_u H(t,u)\,, \\ u(0) = u(T)\,, \end{cases} \tag{2}$$

where $H : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with continuous partial gradient $\nabla_u H : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$. Writing $u = (q,p)$, here is the first assumption.

*A*1. The function $H(t,q,p)$ is 2π-periodic in q.

Let $a, b : \mathbb{R} \to \mathbb{R}$ be two continuous and 2π-periodic functions such that $a(q) < b(q)$ for every q, we define the sets

$$\Gamma_- = \{(q,p) \in \mathbb{R}^2 : p = a(q)\}, \qquad \Gamma_+ = \{(q,p) \in \mathbb{R}^2 : p = b(q)\}, \tag{3}$$

and

$$\mathcal{S} = \{(q,p) \in \mathbb{R}^2 : a(q) \leqslant p \leqslant b(q)\}. \tag{4}$$

Here is the second assumption.

*A*2. All the solutions $u = (q,p)$ of the Hamiltonian system in (2) starting with $u(0) \in \mathcal{S}$ are defined on $[0,T]$ and

$$\begin{cases} u(0) \in \Gamma_- & \Rightarrow \quad q(T) - q(0) < 0\,, \\ u(0) \in \Gamma_+ & \Rightarrow \quad q(T) - q(0) > 0\,. \end{cases} \tag{5}$$

The following result was proved in [17, Theorem 6.2].

Theorem 2.1. *Assume that A*1 *and A*2 *hold true. Then, problem* (2) *has at least two geometrically distinct solutions u such that $u(0) \in \mathring{\mathcal{S}}$. The same is true if both the inequalities in* (5) *are reversed.*

3 Our main problem

We are interested in the problem

$$\begin{cases} J\dot{z} = \nabla_z \mathcal{H}(t,z)\,, \\ z(T) = e^{i\vartheta} z(0)\,, \end{cases} \tag{6}$$

where $\vartheta \in \mathbb{R}$ is fixed, and we have used the complex notation to denote a clockwise rotation of angle ϑ in the plane. Here $\mathcal{H} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with continuous partial gradient $\nabla_z \mathcal{H} : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$. Notice that the antiperiodic problem (1) corresponds to the case $\vartheta = \pi$.

Let us first recall the definition of *angular rotation* around the origin of a given curve in the time interval $[t_1, t_2]$. Let $\phi : [t_1, t_2] \to \mathbb{R}^2$ be a continuous curve such that $\phi(t) \neq 0$ for every $t \in [t_1, t_2]$. Writing $\phi(t) = (\rho(t)\cos\theta(t), \rho(t)\sin\theta(t))$, where $\rho : \mathbb{R} \to]0, +\infty[$ and $\theta : \mathbb{R} \to \mathbb{R}$ are continuous functions, we define

$$\mathrm{Ang}(\phi; [t_1, t_2]) = \theta(t_2) - \theta(t_1)\,.$$

For a Jordan curve Γ, we use the notation $\mathcal{I}(\Gamma)$ to denote the bounded open planar region surrounded by Γ. Our main result is given below.

Theorem 3.1. *Let there exist two star-shaped Jordan curves* Γ_1 *and* Γ_2 *such that* $0 \in \mathcal{I}(\Gamma_1)$ *and* $\overline{\mathcal{I}(\Gamma_1)} \subseteq \mathcal{I}(\Gamma_2)$, *with the following property: every solution* $z(t)$ *of the Hamiltonian system in* (6) *starting with* $z(0) \in \overline{\mathcal{I}(\Gamma_2)} \setminus \mathcal{I}(\Gamma_1)$ *is defined on* $[0,T]$, *satisfies* $z(t) \neq 0$ *for all* $t \in [0,T]$, *and*

$$\begin{cases} z(0) \in \Gamma_1 \quad \Rightarrow \quad \mathrm{Ang}(z;[0,T]) > \vartheta\,, \\ z(0) \in \Gamma_2 \quad \Rightarrow \quad \mathrm{Ang}(z;[0,T]) < \vartheta\,. \end{cases} \tag{7}$$

Then, problem (6) *has at least two solutions with starting point* $z(0)$ *in* $\mathcal{I}(\Gamma_2) \setminus \overline{\mathcal{I}(\Gamma_1)}$. *The same is true if both the inequalities in* (7) *are reversed.*

Let us recall that assumption (7) is usually referred to as a *twist condition.*

Proof. Since the solutions $z(t)$ of the Hamiltonian system in (6) starting between Γ_1 and Γ_2 are defined on $[0,T]$ and do not pass through the origin, we can find a constant $\eta > 0$ such that, for all those solutions, one has $|z(t)| > 2\eta$, for every $t \in [0,T]$. We now modify the Hamiltonian function near the origin, using a smooth cutoff function. Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth function such that

$$\zeta(r) = \begin{cases} 0\,, & \text{if } r \leqslant \eta\,, \\ 1\,, & \text{if } r \geqslant 2\eta\,, \end{cases}$$

and consider the new Hamiltonian system

$$J\dot{z} = \nabla_z \widehat{\mathcal{H}}(t,z)\,, \tag{8}$$

with

$$\widehat{\mathcal{H}}(t,z) = \zeta(|z|)\,\mathcal{H}(t,z)\,.$$

Define the map $\Psi : \mathbb{R} \times \,]0,+\infty[\, \to \mathbb{R}^2$ by

$$\Psi(q,p) = \Big(\sqrt{2p}\,\cos q\,, -\sqrt{2p}\,\sin q\Big)\,.$$

Notice that q measures the angles in *clockwise* direction.

Writing $u = (q,p)$, we define the function $H : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H(t,u) = \begin{cases} \widehat{\mathcal{H}}\Big(t, \Psi\Big(q - \dfrac{\vartheta t}{T}\,, p\Big)\Big) + \dfrac{\vartheta p}{T}\,, & \text{if } p \geqslant 0\,, \\ \dfrac{\vartheta p}{T}\,, & \text{if } p < 0\,, \end{cases}$$

and consider the system

$$J\dot{u} = \nabla_u H(t,u)\,. \tag{9}$$

The function H satisfies Assumption A1.

By construction, $u = (q, p)$ is a solution of (9) with $p(t) > 2\eta^2$ for every $t \in [0, T]$ if and only if

$$z(t) = \Psi\Big(q(t) - \frac{\vartheta t}{T}\,, p(t)\Big) \tag{10}$$

is a solution of (8) with $|z(t)| > 2\eta$ for every $t \in [0, T]$. In this case, z is also a solution of $J\dot{z} = \nabla_z \mathcal{H}(t, z)$.

We now assume that the inequalities in (7) hold. The case when they are reversed can be treated similarly. Let us introduce the curves Γ_- and Γ_+ such that $\Gamma_1 = \Psi(\Gamma_-)$, and $\Gamma_2 = \Psi(\Gamma_+)$. We parametrize them as in (3) so to introduce the set $\mathcal{S}$ as in (4).

Let u be a solution of system (9) starting with $u(0) \in \mathcal{S}$ and consider z as in (10). Then, z is a solution of (8) and $z(0) \in \mathcal{I}(\Gamma_2) \setminus \overline{\mathcal{I}(\Gamma_1)}$. Moreover, since

$$\mathrm{Ang}(z; [0, T]) = \theta(T) - \theta(0) = -(q(T) - \vartheta) + q(0) = q(0) - q(T) + \vartheta\,,$$

by (7) we have

$$u(0) \in \Gamma_- \quad \Rightarrow \quad z(0) \in \Gamma_1 \quad \Rightarrow \quad q(T) - q(0) < 0\,,$$

and

$$u(0) \in \Gamma_+ \quad \Rightarrow \quad z(0) \in \Gamma_2 \quad \Rightarrow \quad q(T) - q(0) > 0\,.$$

Having verified A2, we can thus apply Theorem 2.1 to the modified system (9) and find two geometrically distinct solutions $u_j = (q_j, p_j)$ of (9) such that $u_j(T) = u_j(0)$ and $u_j(0) \in \mathring{\mathcal{S}}$, with $j = 1, 2$.

Setting

$$z_j(t) = \Psi\Big(q_j(t) - \frac{\vartheta t}{T}\,, p_j(t)\Big)\,,$$

one obtains two solutions of (8) such that $z_j(T) = e^{i\vartheta} z_j(0)$ and $z_j(0)$ belongs to $\mathcal{I}(\Gamma_2) \setminus \overline{\mathcal{I}(\Gamma_1)}$. By construction, such solutions satisfy $|z_j(t)| > 2\eta$ for every $t \in [0, T]$, hence they are solutions of (6). The proof is thus completed. $\square$

Remark 3.2. *Let $u : [0, T] \to \mathbb{R}^2$ be a solution of problem* (1)*, and assume H to be defined on $\mathbb{R} \times \mathbb{R}^2$ and to satisfy*

$$H(t + T, -u) = H(t, u)\,, \qquad \textit{for every } (t, u) \in \mathbb{R} \times \mathbb{R}^2\,.$$

Then, extending u to the whole real line by requiring that

$$u(t+T) = -u(t)\,, \qquad \textit{for every } t \in \mathbb{R}\,,$$

it is easily verified that u solves $J\dot{u}(t) = \nabla_u H(t, u(t))$ for every $t \in \mathbb{R}$. This solution $u : \mathbb{R} \to \mathbb{R}^2$ is what is usually called a T-antiperiodic solution.

4 Examples of applications

In this section we provide some applications of Theorem 3.1 to the search of antiperiodic solutions for some scalar second order differential equations. We thus focus our attention on the problem

$$\begin{cases} \ddot{x} + f(t,x) = 0\,, \\ x(T) = -x(0)\,, \quad \dot{x}(T) = -\dot{x}(0)\,, \end{cases} \tag{11}$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Similar applications could be stated for equations involving a p-Laplacian, a relativistic, or a mean curvature differential operator, but for the sake of simplicity we prefer not to deal with these problems.

Notice that, in view of Remark 3.2, if f is defined on $\mathbb{R} \times \mathbb{R}$ and satisfies

$$f(t+T, -x) = -f(t,x)\,, \qquad \text{for every } (t,x) \in \mathbb{R} \times \mathbb{R}\,, \tag{12}$$

then any solution $x : [0,T] \to \mathbb{R}$ of (11) can be extended on the whole real line by setting

$$x(t+T) = -x(t)\,, \qquad \text{for every } t \in \mathbb{R}\,,$$

so to get an antiperiodic solution of the equation $\ddot{x} + f(t,x) = 0$.

We now propose some examples, which are based on different behaviours of the nonlinearity $f(t,x)$, providing the twist condition needed in order to apply Theorem 3.1 to problem (6) with $\vartheta = \pi$, when choosing the Hamiltonian function

$$\mathcal{H}(t,x,y) = \frac{y^2}{2} + \int_0^x f(t,s)\,ds\,.$$

We will not enter into the details of the proofs, since they directly follow from the corresponding results for the periodic problem.

4.1 Linear growth at infinity

In the following theorem the origin is an equilibrium point in the phase plane. We refer to [2, 4, 7, 19, 23, 25, 27] for the corresponding periodic problem.

Theorem 4.1. *Assume that there are positive constants $\lambda_0, \lambda_\infty$ such that*

$$\lim_{x\to 0} \frac{f(t,x)}{x} = \lambda_0\,, \qquad \lim_{|x|\to\infty} \frac{f(t,x)}{x} = \lambda_\infty\,,$$

uniformly in $t \in [0,T]$. If

$$\text{either } \frac{\pi}{\sqrt{\lambda_0}} < T < \frac{\pi}{\sqrt{\lambda_\infty}}\,, \text{ or } \frac{\pi}{\sqrt{\lambda_\infty}} < T < \frac{\pi}{\sqrt{\lambda_0}}\,, \tag{13}$$

then problem (11) *has at least two nontrivial solutions.*

Under the above assumptions, in the phase plane, the solutions “near the origin” make a complete rotation in a time approximately equal to $2\pi/\sqrt{\lambda_0}$, while for those “near infinity” the approximate time needed for a rotation is $2\pi/\sqrt{\lambda_\infty}$. Due to the symmetries of the associated linear equations $\ddot{x}+\lambda_0 x = 0$ and $\ddot{x}+\lambda_\infty x = 0$, assumption (13) provides the needed twist condition (7), possibly with reversed inequalities. Theorem 4.1 can be generalized in several ways, following the ideas in the above cited papers on the periodic problem.

4.2 Superlinear growth

We now analyze the possible situation when $f(t,x)$ has a superlinear growth at infinity. For the corresponding periodic problem we refer to [14, 20, 21].

Theorem 4.2. *Assume the existence of $c_0 > 0$ and $\delta > 0$ such that*

$$|x| \leqslant \delta \quad \Rightarrow \quad |f(t,x)| \leqslant c_0|x|\,.$$

If moreover

$$\lim_{|x|\to\infty} \frac{f(t,x)}{x} = +\infty\,,$$

uniformly in $t \in [0,T]$, then problem (11) *has infinitely many solutions. More precisely, there exists a positive integer k_0 such that, for any $k \geqslant k_0$, there are at least two solutions of problem* (11) *performing exactly $2k+1$ clockwise half-rotations around the origin in the time interval $[0,T]$.*

In the above theorem, the origin is an equilibrium point in the phase plane. The solutions near the origin rotate with a bounded angular speed, while those far from the origin rotate with an angular speed which becomes larger and larger. This difference gives the twist condition required by Theorem 3.1.

We now consider the special case $f(t,x) = g(x) - e(t)$ and the corresponding problem

$$\begin{cases} \ddot{x} + g(x) = e(t)\,, \\ x(T) = -x(0)\,, \quad \dot{x}(T) = -\dot{x}(0)\,. \end{cases} \tag{14}$$

Unlike the above, we assume g to be defined on an open interval $]A, B[$, where

$$-\infty \leqslant A < 0 < B \leqslant +\infty\,.$$

Notice that there could be one or two singularities at the endpoints of the interval. Let $G(x) = \int_0^x g(s)\,ds$ be a primitive of g. Here is a variant of Theorem 4.2 (see [8, 11, 15] for the periodic problem).

Theorem 4.3. *If*

$$\lim_{x\to A^+} \frac{g(x)}{x} = \lim_{x\to B^-} \frac{g(x)}{x} = +\infty\,,$$

and

$$\lim_{x\to A^+} G(x) = \lim_{x\to B^-} G(x) = +\infty\,,$$

then the same conclusion of Theorem 4.2 holds for problem (14).

Notice that in the above statement the origin is not assumed to be an equilibrium in the phase plane. The main difference with Theorem 4.2 is that now the solutions of initial value problems are globally defined on $[0, T]$ (see [11]). This fact also permits us to extend after truncation the function g to the whole real line, so to be able to apply the Poincaré–Birkhoff theorem.

4.3 Sublinear growth

A substantially different situation arises when the nonlinearity $f(t, x)$ in problem (11) has a sublinear growth at infinity. Our theorem stated below can be proved following the lines of [9, 13, 16], where the existence of subharmonic solutions for the periodic problem was treated. We assume $f(t, x)$ to be defined for every $(t, x) \in \mathbb{R} \times \mathbb{R}$, i.e., $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Theorem 4.4. *Assume the existence of $c > 0$ and $d > 0$ such that*

$$|x| \geqslant d \quad \Rightarrow \quad xf(t,x) \geqslant c|x| \,.$$

If moreover

$$\lim_{|x|\to\infty} \frac{f(t,x)}{x} = 0 \,,$$

uniformly in $t \in \mathbb{R}$, then there exists a $\tau_0 > 0$ such that, for every $\tau \geqslant \tau_0$, there are at least two solutions of

$$\begin{cases} \ddot{x} + f(t,x) = 0 \,, \\ x(\tau) = -x(0) \,, \quad \dot{x}(\tau) = -\dot{x}(0) \,, \end{cases}$$

performing exactly one clockwise half-rotation around the origin in the time interval $[0, \tau]$.

The situation here is more involved due to the fact that large amplitude solutions in the phase plane rotate with a very small angular speed. This is why we have to assume τ sufficiently large in order to get the twist assumption so to be able to apply Theorem 3.1.

Notice that, when the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the symmetry property (12) for a given $T > 0$, the above theorem can be applied taking $\tau = nT$, for a sufficiently large integer n. As a consequence, we can find a positive integer n_0 such that, for any odd integer $n \geqslant n_0$, there are at least two nT-antiperiodic solutions of the equation $\ddot{x} + f(t,x) = 0$. These are the analogs of the subharmonic solutions in the periodic case.

4.4 Small perturbations

Let us first consider the antiperiodic problem for the pendulum equation, i.e.,

$$\begin{cases} \ddot{x} + \alpha \sin x = e(t) \,, \\ x(T) = -x(0) \,, \quad \dot{x}(T) = -\dot{x}(0) \,, \end{cases} \tag{15}$$

where $\alpha > 0$ is a given constant, and $e : [0, T] \to \mathbb{R}$ is any continuous function. The following statement finds its counterpart for the periodic problem in [6, 12, 18].

Theorem 4.5. *If $T > \pi/\sqrt{\alpha}$, then there exists $\varepsilon > 0$ such that problem* (15) *has at least two solutions, provided that $\|e\|_\infty \leqslant \varepsilon$.*

Notice that, when $e = 0$, the pendulum equation has a center at the origin in the phase plane, and the period of the solutions near the origin can be approximated by $2\pi/\sqrt{\alpha}$, after linearization, while the period of the solutions approaching the two heteroclinic orbits become larger and larger. This difference creates the twist condition needed in order to apply Theorem 3.1 since, by symmetry, the time needed to perform a complete rotation is exactly twice the time needed to perform half a rotation. The result then follows using the fact that the twist persists under small perturbations.

The above situation can be generalized in many different ways. The nonlinearity $\alpha \sin x$ can be replaced by any function $g(x)$ for which the equation $\ddot{x} + g(x) = 0$ has a non-isochronous center at the origin in the phase plane, and there exist two periodic solutions $x_1(t)$, $x_2(t)$ such that, writing

$$(x_j(t), \dot{x}_j(t)) = (\rho_j(t)\cos(\theta_j(t)), \rho_j(t)\sin(\theta_j(t)))\,, \quad j = 1, 2\,,$$

the angular speeds satisfy

$$\dot{\theta}_2(t) < \frac{\pi}{T} < \dot{\theta}_1(t)\,, \qquad \text{for every } t \in \mathbb{R}\,.$$

A simpler situation is encountered if g is assumed to be an *odd* function. In this case, taking two periodic solutions $x_1(t)$, $x_2(t)$ of the equation $\ddot{x} + g(x) = 0$ with minimal periods τ_1, τ_2, respectively, the twist condition will be satisfied if there exists a natural number k such that

$$\text{either } \big(k + \tfrac{1}{2}\big)\tau_1 < T < \big(k + \tfrac{1}{2}\big)\tau_2\,, \text{ or } \big(k + \tfrac{1}{2}\big)\tau_2 < T < \big(k + \tfrac{1}{2}\big)\tau_1\,.$$

Acknowledgements

The research contained in this paper was carried out within the framework of DEG1 (Differential Equations Group Of North-East) and it has been partly supported by the Italian PRIN Project 2022ZXZTN2 *Nonlinear differential problems with applications to real phenomena.* A. Fonda and A. Sfecci also acknowledge INdAM (Istituto Nazionale di Alta Matematica).

References

[1] G.D. Birkhoff, Dynamical Systems, Amer. Math. Soc., New York, 1927.

[2] A. Boscaggin, Subharmonic solutions of planar Hamiltonian systems: a rotation number approach, Adv. Nonlin. Stud. 11 (2011), 77–103.

[3] A. Boscaggin, Periodic solutions to superlinear planar Hamiltonian systems, Port. Math. 69 (2012), 127–140.

[4] A. Boscaggin and M. Garrione, Resonance and rotation numbers for planar Hamiltonian systems: multiplicity results via the Poincaré–Birkhoff theorem, Nonlinear Anal. 74 (2011), 4166–4185.

[5] M. Brown and W.D. Neumann, Proof of the Poincaré–Birkhoff fixed point theorem, Michigan Math. J. 24 (1977), 21–31.

[6] P. Buttazzoni and A. Fonda, Periodic perturbations of scalar second order differential equations, Discrete Contin. Dyn. Syst. 3 (1997), 451–455.

[7] F. Dalbono and C. Rebelo, Poincaré–Birkhoff fixed point theorem and periodic solutions of asymptotically linear planar Hamiltonian systems, Turin Fortnight Lectures on Nonlinear Analysis (2001), Rend. Sem. Mat. Univ. Politec. Torino 60 (2002), 233–263 (2003).

[8] T. Ding and F. Zanolin, Periodic solutions of Duffing's equations with superquadratic potential, J. Differential Equations 95 (1992), 240–258.

[9] T. Ding and F. Zanolin, Subharmonic solutions of second order nonlinear equations: a time-map approach, Nonlinear Anal. 20 (1993), 509–532.

[10] A. Fonda, Periodic solutions of Hamiltonian systems with symmetries, in: Topological Methods for Delay and Ordinary Differential Equations with Applications to Continuum Mechanics, Springer, 2024, pp. 1–19.

[11] A. Fonda, R.F. Manasevich and F. Zanolin, Subharmonic solutions for some second order differential equations with singularities, SIAM J. Math. Anal. 24 (1993), 1294–1311.

[12] A. Fonda, M. Sabatini and F. Zanolin, Periodic solutions of perturbed Hamiltonian systems in the plane by the use of the Poincaré–Birkhoff theorem, Topol. Methods Nonlinear Anal. 40 (2012), 29–52.

[13] A. Fonda, Z. Schneider and F. Zanolin, Periodic oscillations for a nonlinear suspension bridge model, J. Comp. Appl. Math. 52 (1994), 113–140.

[14] A. Fonda and A. Sfecci, Periodic solutions of weakly coupled superlinear systems, J. Differential Equations 260 (2016), 2150–2162.

[15] A. Fonda and A. Sfecci, Multiple periodic solutions of Hamiltonian systems confined in a box, Discrete Contin. Dyn. Syst. 37 (2017), 1425–1436.

[16] A. Fonda and R. Toader, Subharmonic solutions of Hamiltonian systems displaying some kind of sublinear growth, Adv. Nonlinear Anal. 8 (2019), 583–602.

[17] A. Fonda and A.J. Ureña, A higher dimensional Poincaré–Birkhoff theorem for Hamiltonian flows, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), 679–698.

[18] A. Fonda and F. Zanolin, Periodic oscillations of forced pendulums with a very small length, Proc. Royal Soc. Edinburgh 127A (1997), 67–76.

[19] P. Gidoni and A. Margheri, Lower bounds on the number of periodic solutions for asymptotically linear planar Hamiltonian systems, Discrete Contin. Dyn. Syst. 39 (2019), 585–606.

[20] P. Hartman, On boundary value problems for superlinear second order differential equations, J. Differential Equations 26 (1977), 37–53.

[21] H. Jacobowitz, Periodic solutions of $x'' + f(t,x) = 0$ via the Poincaré–Birkhoff theorem, J. Differential Equations 20 (1976), 37–52.

[22] P. Le Calvez, About Poincaré–Birkhoff theorem, Publ. Mat. Urug. 13 (2011), 61–98.

[23] A. Margheri, C. Rebelo and F. Zanolin, Maslov index, Poincaré–Birkhoff theorem and periodic solutions of asymptotically linear planar Hamiltonian systems, J. Differential Equations 183 (2002), 342–367.

[24] H. Poincaré, Sur un théorème de géométrie, Rend. Circ. Mat. Palermo 33 (1912), 375–407.

[25] D. Qian and P.J. Torres and P. Wang, Periodic solutions of second order equations via rotation numbers, J. Differential Equations 266 (2019), 4746–4768.

[26] C. Rebelo, A note on the Poincaré–Birkhoff fixed point theorem and periodic solutions of planar systems, Nonlinear Anal. 29 (1997), 291–311.

[27] C. Zanini, Rotation numbers, eigenvalues, and the Poincaré–Birkhoff theorem, J. Math. Anal. Appl. 279 (2003), 290–307.

Authors' addresses:

A. Fonda, N.G. Mamo, A. Sfecci, and W. Ullah
Dipartimento di Matematica, Informatica e Geoscienze
Università degli Studi di Trieste
P.le Europa 1, 34127 Trieste, Italy

e-mail: a.fonda@units.it, natnaelgezahegn.mamo@phd.units.it,
asfecci@units.it, wahid.ullah@phd.units.it

Mathematics Subject Classification: 34B15

Keywords: Antiperiodic solutions; Poincaré–Birkhoff theorem; Hamiltonian systems.