An extension of the Poincaré–Birkhoff Theorem to systems involving Landesman–Lazer conditions

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Abstract

We provide multiplicity results for the periodic problem associated with Hamiltonian systems coupling a system having a Poincaré–Birkhoff twist-type structure with a system presenting some asymmetric nonlinearities, with possible one-sided superlinear growth. We investigate nonresonance, simple resonance and double resonance situations, by implementing some kind of Landesman–Lazer conditions.

1 Introduction

We consider the periodic problem associated with a system of the type

$$\begin{cases} \dot q = \partial_p \mathcal{H}(t,q,p) + \partial_p P(t,q,p,u)\,, \\ \dot p = -\partial_q \mathcal{H}(t,q,p) - \partial_q P(t,q,p,u)\,, \\ \ddot u + g(t,u) = -\partial_u P(t,q,p,u)\,, \end{cases} \tag{S}$$

where all the involved functions are continuous and T-periodic in t.

We will assume a Poincaré–Birkhoff setting for the planar system

$$\dot q = \partial_p \mathcal{H}(t,q,p)\,, \qquad \dot p = -\partial_q \mathcal{H}(t,q,p)\,, \tag{1}$$

while for the scalar equation

$$\ddot u + g(t,u) = 0 \tag{2}$$

the nonlinearity g will have an asymmetric behaviour combined with some Landesman–Lazer conditions.

In order to better understand our results, let us first present some historical premises. Concerning system (1), a modern version of the Poincaré–Birkhoff Theorem reads as follow, see [21].

Theorem 1. *Assume the Hamiltonian function $\mathcal{H}$ to be 2π-periodic in q. If there exist $a < b$ such that all the solutions of* (1) *starting with $p(0) \in [a,b]$ are defined on $[0,T]$ and satisfy*

$$\begin{cases} p(0) = a \quad \Longrightarrow \quad q(T) - q(0) < 0\,, \\ p(0) = b \quad \Longrightarrow \quad q(T) - q(0) > 0\,, \end{cases}$$

then there are at least two geometrically distinct T-periodic solutions such that $p(0) \in\,]a,b[\,$.

Since we assumed the 2π-periodicity in q of the Hamiltonian function $\mathcal{H}$, the T-periodic solutions can be collected in equivalence classes made of those solutions whose q-components differ by an integer multiple of 2π. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

Recently, there have been several generalizations of this theorem both in the planar case and in higher dimensions. In particular, system (1) has been coupled with some systems presenting different types of behaviours. In [15] the case of a linear nonresonant system was considered, while in [3] resonance with respect to one eigenvalue was addressed assuming an Ahmad–Lazer–Paul condition. In [12, 18] the possibility of dealing with lower and upper solutions has been tackled, and in [19, 20] the coupling with some isochronous oscillators has been faced.

As a particular case of an isochronous system, let us mention the classical asymmetric oscillator

$$\ddot{u} + \mu u^+ - \nu u^- = 0\,, \tag{3}$$

where $u^\pm = \max\{\pm u, 0\}$. It is well known, since the pioneering works by Fučík [22] and Dancer [4, 5], that equation (3) has nontrivial T-periodic solutions if and only if the couple (μ, ν) belongs to the so-called Fučík spectrum

$$\Sigma = \bigcup_{j\in\mathbb{N}} \mathscr{C}_j\,,$$

where

$$\mathscr{C}_0 = \{(\mu,\nu)\in\mathbb{R}^2 : \mu\nu = 0\}\,,$$

and, for $j \geq 1$,

$$\mathscr{C}_j = \left\{(\mu,\nu)\in\mathbb{R}^2 : \mu > 0\,, \nu > 0\,, \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{j}\right\}.$$

The set Σ then contains the two axes and an infinite number of curves $\mathscr{C}_j$ having a vertical asymptote $\mu = \overline{\mu}_j$ and a horizontal one $\nu = \overline{\nu}_j$ with $\overline{\mu}_j = \overline{\nu}_j = (j\pi/T)^2$.

Assume that the function g in (2) satisfies a linear growth assumption

$$\nu_1 \leq \liminf_{u\to-\infty} \frac{g(t,u)}{u} \leq \limsup_{u\to-\infty} \frac{g(t,u)}{u} \leq \nu_2\,,$$
$$\mu_1 \leq \liminf_{u\to+\infty} \frac{g(t,u)}{u} \leq \limsup_{u\to+\infty} \frac{g(t,u)}{u} \leq \mu_2\,,$$

uniformly in $t \in [0,T]$, for some positive constants μ_1, μ_2, ν_1, and ν_2. In [7], it was proved that if the rectangle $\mathcal{R} = [\mu_1,\mu_2]\times[\nu_1,\nu_2]$ does not intersect the Fučík spectrum Σ, then equation (2) has at least one T-periodic solution. This is a typical nonresonance situation. See also [6, 23, 31] for related results.

When the set $\mathcal{R}\cap\Sigma$ consists of only one or both the vertices (μ_1,ν_1) and (μ_2,ν_2) of the rectangle, some additional assumptions are needed. In [8, 9, 13], the so-called *double resonance* situation has been treated by assuming some Landesman–Lazer conditions on both sides (cf. [24, 25]), thus obtaining the existence of a T-periodic solution. (See also [2, 28] for an in-depth analysis on the Landesman–Lazer condition.)

The possibility of treating the case of an unbounded rectangle of the type $\mathcal{R} = [\mu_1, +\infty[\times[\nu_1, \nu_2]$ at a positive distance from the Fučík spectrum Σ was first faced by Fabry and Habets in [10] (see also [29]).

Usually, the proofs in the above quoted papers mainly rely on the use of some topological degree arguments. Typically, a homotopy with some nonresonant system needs to be constructed, and one has to find a priori bounds for the corresponding T-periodic problems. The Leray–Schauder degree theory can then be applied to obtain the existence of a solution. In this paper we propose a different approach. The strategy of the proofs is to wisely modify the original system so to be able to apply a suitable version of the Poincaré–Birkhoff Theorem. Then, we provide some a priori bounds so to ensure that the T-periodic solutions of the modified system are indeed solutions of the original one. In this way, we obtain our multiplicity results by combining assumptions of Poincaré–Birkhoff twist-type for equation (1) with the resonant/nonresonant assumptions we have mentioned above for the scalar equation (2).

The paper is organized as follows. In Section 2 we state our main results, focusing our attention on the case when (1) is a planar system and (2) is a scalar second order equation. The results are then extended to the higher dimensional setting in Section 4, where we also mention some other possible generalizations. The detailed proofs of the statements are carried out in Section 3.

2 Main results

Let us start by listing all our results related to system (S). The proofs will be postponed to the next section. Here are our main assumptions.

We first need some periodicity for the Hamiltonian function $\mathcal{H}$.

*A*1. The function $\mathcal{H}(t, q, p)$ is 2π-periodic in q.

We now introduce the *twist* assumption, adapted to our setting (cf. [12, 19, 20]).

*A*2. There are $a < b$ such that, for every C^1-function $\mathcal{U} : [0, T] \to \mathbb{R}$, all the solutions (q, p) of the system

$$\begin{cases} \dot{q} = \partial_p \mathcal{H}(t, q, p) + \partial_p P\big(t, q, p, \mathcal{U}(t)\big) , \\ \dot{p} = -\partial_q \mathcal{H}(t, q, p) - \partial_q P\big(t, q, p, \mathcal{U}(t)\big) , \end{cases}$$

starting with $p(0) \in [a, b]$, are defined on $[0, T]$ and satisfy

$$\begin{cases} p(0) = a \quad \Longrightarrow \quad q(T) - q(0) < 0 , \\ p(0) = b \quad \Longrightarrow \quad q(T) - q(0) > 0 . \end{cases} \tag{4}$$

Notice that the inequalities in (4) could also be reversed. Finally, a periodicity and boundedness condition is assumed for the coupling function.

*A*3. The function $P(t, q, p, u)$ is 2π-periodic in q and has a bounded gradient with respect to (q, p, u). In particular, there exists a constant $\overline{m}$ such that

$$|\partial_u P(t, q, p, u)| \le \overline{m} , \qquad \text{for every } (t, q, p, u) .$$

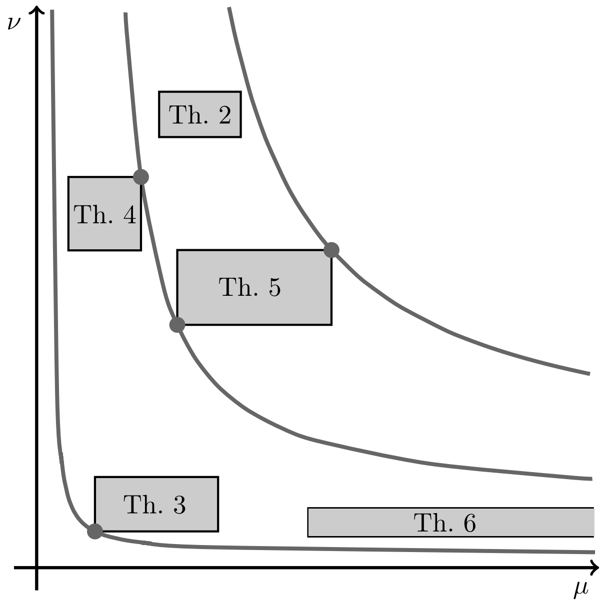

Figure 1: The Fučík curves and a sketch of the situation faced in each statement.

In order to state our results we distinguish two different situations: the case when the function g has an asymptotically linear growth and the case when it could have a one-sided superlinear growth. In Figure 1 we depict the settings considered in each of the statements given below, showing the positions of the rectangle $\mathcal{R}$ defined in the Introduction with respect to the Fučík spectrum.

2.1 Asymptotically linear growth

Here we assume the following linear growth assumption.

A4. There exist some positive constants μ_1, μ_2, ν_1, and ν_2 for which

$$\nu_1 \leq \liminf_{u\to-\infty} \frac{g(t,u)}{u} \leq \limsup_{u\to-\infty} \frac{g(t,u)}{u} \leq \nu_2 \,,$$
$$\mu_1 \leq \liminf_{u\to+\infty} \frac{g(t,u)}{u} \leq \limsup_{u\to+\infty} \frac{g(t,u)}{u} \leq \mu_2 \,,$$

uniformly in $t \in [0,T]$.

Let us state our main results in this setting. We start with a nonresonance situation.

Theorem 2 (Nonresonance)**.** *Let $A1 - A4$ hold true, and assume that there exists a positive integer N such that*

$$\frac{T}{N+1} < \frac{\pi}{\sqrt{\mu_2}} + \frac{\pi}{\sqrt{\nu_2}} \leq \frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}} < \frac{T}{N} \,. \tag{5}$$

Then there are at least two geometrically distinct T-periodic solutions of system (S), with $p(0) \in \,]a,b[\,$.

We now consider the situation of simple resonance from below.

Theorem 3 (Simple resonance from below)**.** *Let $A1-A4$ hold true, and assume that there exists a positive integer N such that*

$$\frac{T}{N+1} < \frac{\pi}{\sqrt{\mu_2}} + \frac{\pi}{\sqrt{\nu_2}} \le \frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}} = \frac{T}{N}\,.$$

Moreover, assume the existence of a constant C such that

$$\begin{aligned} g(t,u) &\le \nu_1 u + C \qquad \text{if } u \le 0\,,\\ g(t,u) &\ge \mu_1 u - C \qquad \text{if } u \ge 0\,. \end{aligned} \tag{6}$$

If for every non-zero function w such that $\ddot{w} + \mu_1 w^+ - \nu_1 w^- = 0$ one has

$$\int_{\{w<0\}} \liminf_{u\to-\infty} \big(\nu_1 u - g(t,u)\big)|w(t)|\,dt + \int_{\{w>0\}} \liminf_{u\to+\infty} \big(g(t,u) - \mu_1 u\big) w(t)\,dt > \overline{m} \int_0^T |w(t)|\,dt\,, \tag{7}$$

then the same conclusion of Theorem 2 holds.

As usual, we have used the notation

$$\{w<0\} = \{t \in [0,T] \,:\, w(t) < 0\}\,,$$

and similarly for $\{w > 0\}$. Symmetrically, we can consider the situation of simple resonance from above.

Theorem 4 (Simple resonance from above)**.** *Let $A1-A4$ hold true, and assume that there exists a positive integer N such that*

$$\frac{T}{N+1} = \frac{\pi}{\sqrt{\mu_2}} + \frac{\pi}{\sqrt{\nu_2}} \le \frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}} < \frac{T}{N}\,.$$

Moreover, assume the existence of a constant C such that

$$\begin{aligned} g(t,u) &\ge \nu_2 u - C \qquad \text{if } u \le 0\,,\\ g(t,u) &\le \mu_2 u + C \qquad \text{if } u \ge 0\,. \end{aligned} \tag{8}$$

If for every non-zero function v such that $\ddot{v} + \mu_2 v^+ - \nu_2 v^- = 0$ one has

$$\int_{\{v<0\}} \liminf_{u\to-\infty} \big(g(t,u) - \nu_2 u\big)|v(t)|\,dt + \int_{\{v>0\}} \liminf_{u\to+\infty} \big(\mu_2 u - g(t,u)\big) v(t)\,dt > \overline{m} \int_0^T |v(t)|\,dt\,, \tag{9}$$

then the same conclusion of Theorem 2 holds.

Finally, a double resonance situation is treated.

Theorem 5 (Double resonance)**.** *Let $A1 - A3$ hold true. Assume that there exist a positive integer N and positive constants $\mu_1, \mu_2, \nu_1, \nu_2$, and C such that conditions* (6) *and* (8) *hold, and*

$$\frac{T}{N+1} = \frac{\pi}{\sqrt{\mu_2}} + \frac{\pi}{\sqrt{\nu_2}} < \frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}} = \frac{T}{N}. \tag{10}$$

If for every non-zero function w such that $\ddot{w} + \mu_1 w^+ - \nu_1 w^- = 0$ one has (7) *and for every non-zero function v such that $\ddot{v} + \mu_2 v^+ - \nu_2 v^- = 0$ one has* (9)*, then the same conclusion of Theorem 2 holds.*

It will be sufficient to provide the proof of Theorem 5, since Theorems 2, 3, and 4 follow as direct consequences. Indeed, if e.g. we focus our attention on the values μ_1 and ν_1 in hypothesis (5), it is possible to find $\epsilon > 0$ such that

$$\frac{\pi}{\sqrt{\mu_1 - \epsilon}} + \frac{\pi}{\sqrt{\nu_1 - \epsilon}} = \frac{T}{N}.$$

Setting $\tilde{\mu}_1 = \mu_1 - \epsilon$ and $\tilde{\nu}_1 = \nu_1 - \epsilon$, then

$$\lim_{u \to -\infty} \big(\tilde{\nu}_1 u - g(t,u)\big) = +\infty, \qquad \lim_{u \to +\infty} \big(g(t,u) - \tilde{\mu}_1 u\big) = +\infty,$$

uniformly in t. So we easily verify that (7) holds. Concerning the constants μ_2 and ν_2, we can similarly implement the same argument.

2.2 One-sided superlinear growth

We now assume the following growth condition.

*A*5. There exist some positive constants μ_1, ν_1, and ν_2 satisfying

$$\nu_1 \le \liminf_{u \to -\infty} \frac{g(t,u)}{u} \le \limsup_{u \to -\infty} \frac{g(t,u)}{u} \le \nu_2,$$
$$\mu_1 \le \liminf_{u \to +\infty} \frac{g(t,u)}{u},$$

uniformly in $t \in [0, T]$.

Let us state our main results in this setting.

Theorem 6 (Nonresonance)**.** *Let $A1 - A3$ and $A5$ hold true, and assume that there exists a positive integer N such that*

$$\frac{T}{N+1} < \frac{\pi}{\sqrt{\nu_2}} < \frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}} < \frac{T}{N}. \tag{11}$$

Then the same conclusion of Theorem 2 holds.

As a consequence of the previous statement we have the following.

Corollary 7. *Let $A1 - A3$ hold true. Assume that there are two constants $\nu_2 \ge \nu_1 > 0$ and a positive integer N such that*

$$\nu_1 \le \liminf_{u \to -\infty} \frac{g(t,u)}{u} \le \limsup_{u \to -\infty} \frac{g(t,u)}{u} \le \nu_2,$$
$$\lim_{u \to +\infty} \frac{g(t,u)}{u} = +\infty, \tag{12}$$

uniformly in $t \in [0,T]$, *and*

$$\left(\frac{N\pi}{T}\right)^2 < \nu_1 \le \nu_2 < \left(\frac{(N+1)\pi}{T}\right)^2 .$$

Then the same conclusion of Theorem 2 holds.

Indeed, by (12), it is easy to find a sufficiently large constant μ_1 so to apply Theorem 6.

2.3 Some examples

As a possible example of application we propose a system coupling a pendulum-type equation with a perturbed harmonic oscillator

$$\begin{cases} \ddot{q} + A\sin q = \partial_q \mathcal{P}(t,q,u)\,, \\ \ddot{u} + \mu u^+ - \nu u^- + h(u) = \partial_u \mathcal{P}(t,q,u)\,, \end{cases} \tag{13}$$

where A, μ, ν are positive,

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{N}\,,$$

for a certain positive integer N, and the function $\mathcal{P}(t,q,u)$ is T-periodic in t and continuously differentiable in (q,u). We assume the existence of the finite limits

$$\lim_{u\to-\infty} h(u) = h(-\infty)\,, \qquad \lim_{u\to+\infty} h(u) = h(+\infty)\,.$$

In particular, h is bounded, hence (6) holds, with $g(u) = \mu u^+ - \nu u^- + h(u)$.

As a consequence of Theorem 3 with $\mu_1 = \mu_2 = \mu$ and $\nu_1 = \nu_2 = \nu$, we have the following.

Corollary 8. *If* $\mathcal{P}(t,q,u)$ *is* 2π*-periodic in* q*, it has a bounded gradient with respect to* (q,u)*, and there exists a constant* $\overline{m}$ *such that*

$$|\partial_u \mathcal{P}(t,q,u)| \le \overline{m} < \frac{h(+\infty)\nu - h(-\infty)\mu}{\mu+\nu}\,, \quad \textit{for every } (t,q,u) \in \mathbb{R}^3\,, \tag{14}$$

then system (13) *has at least two geometrically distinct* T*-periodic solutions.*

Proof. It can be seen that the nontrivial solutions of the differential equation $\ddot{w} + \mu w^+ - \nu w = 0$ are of the type $w(t) = c\,\phi(t-\theta)$ with $c > 0$ and $\theta \in [0, \frac{T}{N}]$, where $\phi : \mathbb{R} \to \mathbb{R}$ is the $\frac{T}{N}$-periodic function such that

$$\phi(t) = \begin{cases} \frac{1}{\sqrt{\mu}} \sin\left(\sqrt{\mu}\,t\right) & \text{if } t \in \left[0, \frac{\pi}{\sqrt{\mu}}\right]\,, \\ -\frac{1}{\sqrt{\nu}} \sin\left(\sqrt{\nu}\left(t - \frac{\pi}{\sqrt{\mu}}\right)\right) & \text{if } t \in \left[\frac{\pi}{\sqrt{\mu}}, \frac{T}{N}\right]\,. \end{cases}$$

Noting that

$$\int_0^{\frac{\pi}{\sqrt{\mu}}} \phi(t)\,dt = \frac{2}{\mu}\,, \qquad \int_{\frac{\pi}{\sqrt{\mu}}}^{\frac{T}{N}} \phi(t)\,dt = \frac{2}{\nu}\,,$$

it is easily seen that (14) implies (7), hence Theorem 3 applies. ☐

As an application of Theorem 6 we suggest the following system

$$\begin{cases} \ddot{q} + A \sin q = \partial_q \mathcal{P}(t, q, u)\,, \\ \ddot{u} + \mu u^+ - \nu u^- + e^u |\sin u| + h(u) = \partial_u \mathcal{P}(t, q, u)\,, \end{cases} \tag{15}$$

where now the positive constants μ, ν satisfy

$$\frac{T}{N+1} < \frac{\pi}{\sqrt{\nu}} < \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} < \frac{T}{N}\,,$$

for a certain positive integer N. As above we assume that $A > 0$, h is bounded, and the function $\mathcal{P}(t, q, u)$ is T-periodic in t and continuously differentiable in (q, u). In this case $\nu_1 = \nu_2 = \nu$, $\mu_1 = \mu$,

$$g(u) = \mu u^+ - \nu u^- + e^u |\sin u| + h(u)\,,$$

and so $\limsup_{u \to +\infty} \frac{g(u)}{u} = +\infty$.

Corollary 9. *If $\mathcal{P}(t, q, u)$ is 2π-periodic in q and has a bounded gradient with respect to (q, u), then system (15) has at least two geometrically distinct T-periodic solutions.*

3 Proofs

The proofs of Theorems 5 and 6 are based on a result by the first author and Ullah (see [19, Theorem 1.1 and Corollary 3.3]), which we recall for the reader's convenience.

Theorem 10. *Let $A1 - A3$ hold true for the system*

$$\begin{cases} \dot{q} = \partial_p \mathcal{H}(t, q, p) + \partial_p P(t, q, p, u)\,, \\ \dot{p} = -\partial_q \mathcal{H}(t, q, p) - \partial_q P(t, q, p, u)\,, \\ \ddot{u} + \chi_1 u^+ - \chi_2 u^- = -\partial_u P(t, q, p, u)\,. \end{cases} \tag{16}$$

Moreover, assume that $\chi_1 > 0$, $\chi_2 > 0$, and

$$\frac{T}{\pi} \, \frac{\sqrt{\chi_1 \, \chi_2}}{\sqrt{\chi_1} + \sqrt{\chi_2}} \notin \mathbb{N}\,.$$

Then there are at least two geometrically distinct T-periodic solutions of system (16), with $p(0) \in \,]a, b[\,$.

The strategy of our proofs is as follows. We first modify system (S) and apply Theorem 10 so to obtain two periodic solutions of the modified system. Then, we prove that such periodic solutions are indeed solutions of the original system.

3.1 Proof of Theorem 5

First note that, as a consequence of (6) and (8), defining the continuous functions

$$\widehat{\zeta}(t, u) = \begin{cases} \max\left\{\mu_1\,,\, \min\left\{\frac{g(t,u)}{u}\,,\, \mu_2\right\}\right\} & \text{if } u \geq 1\,, \\ \max\{\mu_1\,,\, \min\{g(t, 1)\,,\, \mu_2\}\} & \text{if } u < 1\,, \end{cases}$$

$$\check{\zeta}(t,u) = \begin{cases} \max\left\{\nu_1\,,\, \min\left\{\frac{g(t,u)}{u}\,,\,\nu_2\right\}\right\} & \text{if } u \le -1\,, \\ \max\{\nu_1\,,\,\min\{g(t,-1)\,,\,\nu_2\}\} & \text{if } u > -1\,, \end{cases}$$

we can write

$$g(t,u) = \hat{\zeta}(t,u)u^+ - \check{\zeta}(t,u)u^- + h(t,u)\,,$$

with

$$\mu_1 \le \hat{\zeta}(t,u) \le \mu_2\,, \qquad \nu_1 \le \check{\zeta}(t,u) \le \nu_2\,,$$

for some bounded continuous function $h(t,u)$.

We modify system (S) as follows. For any $\rho > 1$, we define

$$g_\rho(t,u) = \hat{\zeta}_\rho(t,u)u^+ - \check{\zeta}_\rho(t,u)u^- + h(t,u)\,, \tag{17}$$

where

$$\hat{\zeta}_\rho(t,u) = \begin{cases} \hat{\zeta}(t,u) & \text{if } |u| \le \rho\,, \\ (\rho+1-|u|)\hat{\zeta}(t,u) + (|u|-\rho)\frac{1}{2}(\mu_1+\mu_2) & \text{if } \rho \le |u| \le \rho+1\,, \\ \frac{1}{2}(\mu_1+\mu_2) & \text{if } |u| \ge \rho+1\,, \end{cases}$$

and

$$\check{\zeta}_\rho(t,u) = \begin{cases} \check{\zeta}(t,u) & \text{if } |u| \le \rho\,, \\ (\rho+1-|u|)\check{\zeta}(t,u) + (|u|-\rho)\frac{1}{2}(\nu_1+\nu_2) & \text{if } \rho \le |u| \le \rho+1\,, \\ \frac{1}{2}(\nu_1+\nu_2) & \text{if } |u| \ge \rho+1\,. \end{cases}$$

Note that, by construction, we have

$$\mu_1 \le \hat{\zeta}_\rho(t,u) \le \mu_2\,, \qquad \nu_1 \le \check{\zeta}_\rho(t,u) \le \nu_2\,,$$

for every (t,u) and for every $\rho > 1$.

The modified system we are going to consider is

$$\begin{cases} \dot{q} = \partial_p \mathcal{H}(t,q,p) + \partial_p P(t,q,p,u)\,, \\ \dot{p} = -\partial_q \mathcal{H}(t,q,p) - \partial_q P(t,q,p,u)\,, \\ \ddot{u} + g_\rho(t,u) = -\partial_u P(t,q,p,u)\,. \end{cases} \tag{S_ρ}$$

The following proposition has a central role in the proof of Theorem 5.

Proposition 11. *There exists a constant $\bar{\rho} > 1$ such that, for all $\rho \ge \bar{\rho}$, every T-periodic solution (q,p,u) of (S_ρ) satisfies $||u||_\infty \le \bar{\rho}$.*

Proof. Assume by contradiction that for every positive integer m there is a $\rho_m \ge m$ and a T-periodic solution (q_m,p_m,u_m) of (S_{ρ_m}) such that $||u_m||_\infty > m$. Let $w_m = \frac{u_m}{||u_m||_\infty}$. Then, w_m is T-periodic and satisfies

$$\ddot{w}_m + \hat{\kappa}_m(t)w_m^+ - \check{\kappa}_m(t)w_m^- + \frac{h\big(t,||u_m||_\infty w_m\big)}{||u_m||_\infty} = -\frac{\partial_u P(t,q_m,p_m,||u_m||_\infty w_m)}{||u_m||_\infty}\,, \tag{18}$$

where

$$\hat{\kappa}_m(t) = \hat{\zeta}_{\rho_m}(t,||u_m||_\infty w_m(t)) \quad \text{and} \quad \check{\kappa}_m(t) = \check{\zeta}_{\rho_m}(t,||u_m||_\infty w_m(t))\,.$$

Notice that $\mu_1 \le \hat{\kappa}_m(t) \le \mu_2$ and $\nu_1 \le \check{\kappa}_m(t) \le \nu_2$.

From the differential equation (18) and the properties of $\widehat{\kappa}_m$, $\check{\kappa}_m$ and h, the sequence $(w_m)_m$ is bounded in $W^{2,2}(0,T)$; therefore there exists a function w such that, up to a subsequence, $w_m \to w$ in $C^1([0,T])$. Since the sequences $(\widehat{\kappa}_m)_m$ and $(\check{\kappa}_m)_m$ are bounded, we can suppose that, up to a subsequence, they converge weakly in $L^2(0,T)$ to some functions $\widehat{\kappa}$ and $\check{\kappa}$, respectively with $\mu_1 \leq \widehat{\kappa}(t) \leq \mu_2$ and $\nu_1 \leq \check{\kappa}(t) \leq \nu_2$, almost everywhere on $[0,T]$. So, $||w||_\infty = 1$, and passing to the weak limit in (18), it solves

$$\ddot{w} + \widehat{\kappa}(t)w^+ - \check{\kappa}(t)w^- = 0\,.$$

By [8, Lemma 3] either

$$\widehat{\kappa}(t) = \mu_1 \text{ a.e. on } \{w > 0\} \quad \text{and} \quad \check{\kappa}(t) = \nu_1 \text{ a.e. on } \{w < 0\}\,,$$

or

$$\widehat{\kappa}(t) = \mu_2 \text{ a.e. on } \{w > 0\} \quad \text{and} \quad \check{\kappa}(t) = \nu_2 \text{ a.e. on } \{w < 0\}\,.$$

Let us consider the first case, the second one being treated similarly. So,

$$\ddot{w} + \mu_1 w^+ - \nu_1 w^- = 0\,. \tag{19}$$

The nontrivial solution $(w(t), \dot{w}(t))$ of (19) makes exactly N rotations around the origin as t varies from 0 to T. This is also true for $(w_m(t), \dot{w}_m(t))$, if m is large enough, and so also for $(u_m(t), \dot{u}_m(t))$.

We now write $(u_m, \dot{u}_m)$ in the following modified polar coordinates:

$$u_m = \begin{cases} \dfrac{1}{\sqrt{\mu_1}}\, r_m \cos\theta_m\,, & \text{if } u_m \geq 0\,, \\ \dfrac{1}{\sqrt{\nu_1}}\, r_m \cos\theta_m\,, & \text{if } u_m \leq 0\,, \end{cases} \qquad \dot{u}_m = r_m \sin\theta_m\,.$$

We compute the derivatives

$$\dot{\theta}_m = \begin{cases} \sqrt{\mu_1}\, \dfrac{\ddot{u}_m u_m - \dot{u}_m^2}{\mu_1 u_m^2 + \dot{u}_m^2} & \text{if } u_m > 0\,, \\ \sqrt{\nu_1}\, \dfrac{\ddot{u}_m u_m - \dot{u}_m^2}{\nu_1 u_m^2 + \dot{u}_m^2} & \text{if } u_m < 0\,. \end{cases}$$

Since the couple $(u_m, \dot{u}_m)$ performs N rotations around the origin in the interval $[0,T]$, we have

$$\begin{aligned} \frac{\pi N}{\sqrt{\mu_1}} &= \int_{\{u_m>0\}} \frac{\big[\widehat{\zeta}_{\rho_m}(t,u_m)u_m + h(t,u_m) + \partial_u P(t,q_m,p_m,u_m)\big]u_m + \dot{u}_m^2}{\mu_1 u_m^2 + \dot{u}_m^2} \\ &= \int_{\{u_m>0\}} \frac{\big[(\widehat{\zeta}_{\rho_m}(t,u_m) - \mu_1)u_m + h(t,u_m) + \partial_u P(t,q_m,p_m,u_m)\big]u_m}{\mu_1 u_m^2 + \dot{u}_m^2} \\ &\qquad + meas(\{u_m > 0\})\,, \end{aligned}$$

where *meas* denotes the Lebesgue measure. Similarly,

$$\begin{aligned} \frac{\pi N}{\sqrt{\nu_1}} &= \int_{\{u_m<0\}} \frac{\big[(\check{\zeta}_{\rho_m}(t,u_m) - \nu_1)u_m + h(t,u_m) + \partial_u P(t,q_m,p_m,u_m)\big]u_m}{\nu_1 u_m^2 + \dot{u}_m^2} \\ &\qquad + meas(\{u_m < 0\})\,. \end{aligned}$$

So, setting

$$b_m(t,u_m) := \big(\widehat{\zeta}_{\rho_m}(t,u_m) - \mu_1\big)u_m^+ - \big(\check{\zeta}_{\rho_m}(t,u_m) - \nu_1\big)u_m^- + h(t,u_m)\,,$$

summing the above two identities and using (10), we get

$$\int_0^T \frac{\big[b_m(t,u_m) + \partial_u P(t,q_m,p_m,u_m)\big]u_m}{\mu_1(u_m^+)^2 + \nu_1(u_m^-)^2 + \dot u_m^2} = 0\,.$$

Recalling that $w_m = \frac{u_m}{||u_m||_\infty}$, we have

$$\int_0^T \frac{\big[b_m(t,u_m) + \partial_u P(t,q_m,p_m,u_m)\big]w_m}{\mu_1(w_m^+)^2 + \nu_1(w_m^-)^2 + \dot w_m^2} = 0\,.$$

Since $\mu_1(w^+(t))^2 + \nu_1(w^-(t))^2 + \dot w(t)^2$ is positive and constant in t, and

$$\lim_{m\to\infty} \big(\mu_1(w_m^+)^2 + \nu_1(w_m^-)^2 + \dot w_m^2\big) = \mu_1(w^+)^2 + \nu_1(w^-)^2 + \dot w^2\,,$$

uniformly in $[0,T]$, by Fatou's Lemma,

$$\int_0^T \liminf_m \frac{\big[b_m(t,u_m) + \partial_u P(t,q_m,p_m,u_m)\big]w_m}{\mu_1(w_m^+)^2 + \nu_1(w_m^-)^2 + \dot w_m^2} \le 0\,.$$

So, it has to be

$$\int_0^T \liminf_m \big[b_m(t,u_m) + \partial_u P(t,q_m,p_m,u_m)\big]w_m \le 0\,.$$

Then,

$$\int_0^T \liminf_m b_m(t,u_m)w_m \le \overline{m}\int_0^T |w(t)|\,dt\,,$$

so equivalently

$$\int_0^T \liminf_m \big[g_{\rho_m}(t,u_m(t)) - \big(\mu_1 u_m^+(t) - \nu_1 u_m^-(t)\big)\big]w_m(t)\,dt \le \overline{m}\int_0^T |w(t)|\,dt\,. \quad (20)$$

Let us now fix $t \in [0,T]$ such that $w(t) < 0$; so $w_m(t) < 0$ and $u_m(t) < 0$, for sufficiently large m. We claim that

$$\liminf_m \big[\nu_1 u_m(t) - g_{\rho_m}(t,u_m(t))\big] \ge \liminf_{u\to-\infty} \big[\nu_1 u - g(t,u)\big]\,. \quad (21)$$

In order to prove this, we consider some different cases.

Case 1. If $|u_m(t)| \le \rho_m$ then, recalling the definition of g_ρ in (17), we have

$$\nu_1 u_m(t) - g_{\rho_m}(t,u_m(t)) = \nu_1 u_m(t) - g(t,u_m(t))\,,$$

and we easily conclude, since $u_m(t) \to -\infty$.

Case 2a. If $|u_m(t)| \geq \rho_m + 1$ and $\nu_1 < \nu_2$ then, since h is bounded and $u_m(t) \to -\infty$, we get

$$\lim_m \big[\nu_1 u_m(t) - g_{\rho_m}(t, u_m(t))\big] = \lim_m \big[\tfrac{\nu_1 - \nu_2}{2} u_m(t) - h(t, u_m(t))\big] = +\infty\,.$$

Case 2b. If $|u_m(t)| \geq \rho_m + 1$ and $\nu_1 = \nu_2$, the identities $\check{\zeta}(t,u) = \check{\zeta}_{\rho_m}(t,u) = \nu_1$ hold and we simply have

$$\nu_1 u_m(t) - g_{\rho_m}(t, u_m(t)) = \nu_1 u_m(t) - g(t, u_m(t))\,.$$

So, (21) follows also in this case.

Case 3. If $\rho_m < |u_m(t)| < \rho_m + 1$ we get

$$\begin{aligned}\nu_1 u_m(t) - g_{\rho_m}(t, u_m(t)) &= \big[\nu_1 - \check{\zeta}_{\rho_m}(t, u_m(t))\big] u_m(t) - h(t, u_m(t)) \\ &\geq \big[\nu_1 - \min\big\{\check{\zeta}(t, u_m(t))\,,\, \tfrac{\nu_1+\nu_2}{2}\big\}\big] u_m(t) - h(t, u_m(t))\,.\end{aligned}$$

If $\min\big\{\check{\zeta}(t, u_m(t))\,,\, \tfrac{\nu_1+\nu_2}{2}\big\} = \check{\zeta}(t, u_m(t))$ we have

$$\nu_1 u_m(t) - g_{\rho_m}(t, u_m(t)) \geq \nu_1 u_m(t) - g(t, u_m(t))\,,$$

otherwise we get

$$\nu_1 u_m(t) - g_{\rho_m}(t, u_m(t)) \geq \tfrac{\nu_1 - \nu_2}{2} u_m(t) - h(t, u_m(t))\,,$$

and we can apply one of the previous arguments.

The claim is thus proved.

From (21) we deduce, for every $t \in [0,T]$ with $w(t) < 0$,

$$\begin{aligned}&\liminf_m \big[g_{\rho_m}(t, ||u_m||_\infty w_m(t)) - \big(\mu_1 u_m^+(t) - \nu_1 u_m^-(t)\big)\big] w_m(t) \\ &\qquad \geq \liminf_{u \to -\infty} \big(\nu_1 u - g(t,u)\big)|w(t)|\,.\end{aligned}$$

Similarly, if $w(t) > 0$ for some t, then $w_m(t) > 0$ and $u_m(t) > 0$ for sufficiently large m, and we can prove that

$$\begin{aligned}&\liminf_m \big[g_{\rho_m}(t, ||u_m||_\infty w_m(t)) - \big(\mu_1 u_m^+(t) - \nu_1 u_m^-(t)\big)\big] w_m(t) \\ &\qquad \geq \liminf_{u \to +\infty} \big(g(t,u) - \mu_1 u\big) w(t)\,.\end{aligned}$$

Finally, by (20),

$$\begin{aligned}&\overline{m} \int_0^T |w(t)|\, dt \\ &\geq \int_{\{w<0\}} \liminf_{u\to-\infty} \big(\nu_1 u - g(t,u)\big)|w(t)|\, dt + \int_{\{w>0\}} \liminf_{u\to+\infty} \big(g(t,u) - \mu_1 u\big) w(t)\, dt,\end{aligned}$$

a contradiction with (7), thus proving Proposition 11. □

Now we are ready to conclude the proof of Theorem 5. We fix $\rho > \overline{\rho}$ and notice that g_ρ can be written as

$$g_\rho(t,u) = \chi_1 u^+ - \chi_2 u^- + p_\rho(t,u)\,,$$

where $\chi_1 = \frac{1}{2}(\mu_1 + \mu_2)$, $\chi_2 = \frac{1}{2}(\nu_1 + \nu_2)$, and the function p_ρ is bounded. By applying Theorem 10 with $P(t, q, p, u)$ replaced by

$$P_\rho(t, q, p, u) := P(t, q, p, u) + \int_0^u p_\rho(t, s)ds\,,$$

we conclude that system (S_ρ) has at least two geometrically distinct T-periodic solutions, with $p(0) \in \,]a, b[\,$. By Proposition 11, these solutions are indeed solutions of the original system (S).

The proof of Theorem 5 is thus completed. $\square$

3.2 Proof of Theorem 6

Without loss of generality we can suitably modify the constants μ_1, ν_1 and ν_2 so to have

$$\begin{aligned} &\nu_2 u - C \le g(t, u) \le \nu_1 u + C && \text{if } u \le 0\,, \\ &\mu_1 u - C \le g(t, u) && \text{if } u \ge 0\,, \end{aligned}$$

for a certain constant $C > 0$.

Similarly as in the previous proof, defining the continuous functions

$$\widehat{\zeta}(t, u) = \begin{cases} \max\left\{\mu_1\,,\, \frac{g(t,u)}{u}\right\} & \text{if } u \ge 1\,, \\ \max\{\mu_1\,,\, g(t, 1)\} & \text{if } u < 1\,, \end{cases}$$

$$\check{\zeta}(t, u) = \begin{cases} \max\left\{\nu_1\,,\, \min\left\{\frac{g(t,u)}{u}\,,\, \nu_2\right\}\right\} & \text{if } u \le -1\,, \\ \max\{\nu_1\,,\, \min\{g(t, -1)\,,\, \nu_2\}\} & \text{if } u > -1\,, \end{cases}$$

we can write

$$g(t, u) = \widehat{\zeta}(t, u)u^+ - \check{\zeta}(t, u)u^- + h(t, u)\,,$$

with

$$\mu_1 \le \widehat{\zeta}(t, u)\,, \qquad \nu_1 \le \check{\zeta}(t, u) \le \nu_2\,,$$

and $h(t, u)$ continuous and bounded.

For any $\rho > 1$, define

$$g_\rho(t, u) = \widehat{\zeta}_\rho(t, u)u^+ - \check{\zeta}_\rho(t, u)u^- + h(t, u)\,,$$

where

$$\widehat{\zeta}_\rho(t, u) = \begin{cases} \widehat{\zeta}(t, u) & \text{if } |u| \le \rho\,, \\ (\rho + 1 - |u|)\widehat{\zeta}(t, u) + (|u| - \rho)\mu_1 & \text{if } \rho \le |u| \le \rho + 1\,, \\ \mu_1 & \text{if } |u| \ge \rho + 1\,, \end{cases}$$

and

$$\check{\zeta}_\rho(t, u) = \begin{cases} \check{\zeta}(t, u) & \text{if } |u| \le \rho\,, \\ (\rho + 1 - |u|)\check{\zeta}(t, u) + (|u| - \rho)\frac{1}{2}(\nu_1 + \nu_2) & \text{if } \rho \le |u| \le \rho + 1\,, \\ \frac{1}{2}(\nu_1 + \nu_2) & \text{if } |u| \ge \rho + 1\,. \end{cases}$$

Notice that

$$\mu_1 \le \widehat{\zeta}_\rho(t, u)\,, \qquad \nu_1 \le \check{\zeta}_\rho(t, u) \le \nu_2\,,$$

for every $(t,u) \in [0,T] \times \mathbb{R}$ and $\rho > 1$. We now consider the modified system

$$\begin{cases} \dot q = \partial_p \mathcal{H}(t,q,p) + \partial_p P(t,q,p,u)\,, \\ \dot p = -\partial_q \mathcal{H}(t,q,p) - \partial_q P(t,q,p,u)\,, \\ \ddot u + g_\rho(t,u) = -\partial_u P(t,q,p,u)\,. \end{cases} \tag{$\widetilde{S}_\rho$}$$

We first need an a priori bound for the minimum distance from the origin in the phase plane.

Proposition 12. *There exist constants $\bar\rho, R > 1$ such that, for all $\rho \ge \bar\rho$, any T-periodic solution (q,p,u) of $(\widetilde{S}_\rho)$ satisfies*

$$\min\{u^2(t) + \dot u^2(t) \mid t \in [0,T]\} \le R^2.$$

Proof. Assume by contradiction that for every positive integer m there is a $\rho_m > m$ and a T-periodic solution (q_m,p_m,u_m) $(\widetilde{S}_{\rho_m})$ such that $\min\{u_m^2 + \dot u_m^2\} > m^2$. We introduce some modified polar coordinates

$$u_m = \begin{cases} \frac{1}{\sqrt{\mu_1}}\, r_m \cos\theta_m & \text{if } u_m \ge 0\,, \\ \frac{1}{\sqrt{\nu_i}}\, r_m \cos\theta_m & \text{if } u_m \le 0\,, \end{cases} \qquad \dot u_m = r_m \sin\theta_m\,,$$

for $i = 1$ or 2, and observe that

$$\dot\theta_m = \begin{cases} \sqrt{\mu_1}\, \dfrac{\ddot u_m u_m - \dot u_m^2}{\mu_1 u_m^2 + \dot u_m^2} & \text{if } u_m > 0\,, \\ \sqrt{\nu_i}\, \dfrac{\ddot u_m u_m - \dot u_m^2}{\nu_i u_m^2 + \dot u_m^2} & \text{if } u_m < 0\,. \end{cases}$$

Let K_m be the integer number of rotations performed by the T-periodic solution $(u_m(t), \dot u_m(t))$ around the origin as t varies from 0 to T.

Notice that

$$\frac{\big(h(t,u_m(t)) + \partial_u P(t,q_m(t),p_m(t),u_m(t))\big)u_m(t)}{\mu_1 u_m^2(t) + \dot u_m^2(t)} \to 0\,,$$
$$\frac{\big(h(t,u_m(t)) + \partial_u P(t,q_m(t),p_m(t),u_m(t))\big)u_m(t)}{\nu_i u_m^2(t) + \dot u_m^2(t)} \to 0\,, \quad i = 1,2\,,$$

uniformly in $t \in [0,T]$.

Let us fix $\varepsilon > 0$ such that

$$N\left(\frac{\pi}{\sqrt{\mu_1}} + \frac{\pi}{\sqrt{\nu_1}}\right) < T - 2\varepsilon\,, \tag{22}$$

integrating $-\dot\theta_m(t)$ on $\{u_m > 0\}$ and $\{u_m < 0\}$, respectively, we get for m large

$$\begin{aligned} K_m\pi = \sqrt{\mu_1}\Bigg[\int_{\{u_m>0\}} & \frac{\widehat\zeta_{\rho_m}(t,u_m)u_m^2 + \dot u_m^2}{\mu_1 u_m^2 + \dot u_m^2} \\ &+ \int_{\{u_m>0\}} \frac{\big(h(t,u_m) + \partial_u P(t,q_m,p_m,u_m)\big)u_m}{\mu_1 u_m^2 + \dot u_m^2}\Bigg] \\ \ge \sqrt{\mu_1}\big(\, & meas\,(\{u_m > 0\}) - \varepsilon\big)\,, \end{aligned}$$

$$
\begin{aligned}
K_m\pi &= \sqrt{\nu_1}\Bigg[\int_{\{u_m<0\}} \frac{\breve{\zeta}_{\rho_m}(t,u_m)u_m^2+\dot u_m^2}{\nu_1 u_m^2+\dot u_m^2} \\
&\qquad + \int_{\{u_m<0\}} \frac{\big(h(t,u_m)+\partial_u P(t,q_m,p_m,u_m)\big)u_m}{\nu_1 u_m^2+\dot u_m^2}\Bigg] \\
&\geq \sqrt{\nu_1}\big(\, meas\,(\{u_m<0\}) - \varepsilon\big)\,.
\end{aligned}
$$

Hence, we obtain

$$
K_m\left(\frac{\pi}{\sqrt{\mu_1}}+\frac{\pi}{\sqrt{\nu_1}}\right) \geq T - 2\varepsilon\,,
$$

so that from (22) we deduce $K_m > N$. Similarly, for sufficiently small $\overline{\varepsilon} > 0$ and large m, we obtain

$$
\begin{aligned}
K_m\pi &= \sqrt{\nu_2}\Bigg[\int_{\{u_m<0\}} \frac{\breve{\zeta}_{\rho_m}(t,u_m)u_m^2+\dot u_m^2}{\nu_2 u_m^2+\dot u_m^2} \\
&\qquad + \int_{\{u_m<0\}} \frac{\big(h(t,u_m)+\partial_u P(t,q_m,p_m,u_m)\big)u_m}{\nu_2 u_m^2+\dot u_m^2}\Bigg] \\
&\leq \sqrt{\nu_2}\big(1+\overline{\varepsilon}\big)\ meas\,\big(\{u_m<0\}\big) \\
&< \frac{(N+1)\pi}{T}\ meas\,\big(\{u_m<0\}\big) \\
&< (N+1)\pi\,.
\end{aligned}
$$

So, we get $K_m < N+1$.

Hence, it has to be $N < K_m < N+1$, a contradiction. □

We now prove an a priori estimate for $\|u\|_\infty$.

Proposition 13. *There exists a constant $\tilde\rho \geq \bar\rho$ such that, for all $\rho \geq \tilde\rho$, every T-periodic solution (q,p,u) of $(\widetilde{S}_\rho)$ satisfies $||u||_\infty \leq \tilde\rho$.*

Proof. Let us choose ε satisfying

$$
0 < \varepsilon < \frac{\pi}{\sqrt{\nu_2}}(N+1) - T\,,
$$

and fix $R_1 \geq R$ such that

$$
\left|\frac{\big[h(t,u)+\partial_u P(t,q,p,u)\big]u}{\nu_2 u^2+v^2}\right| \leq \frac{\varepsilon}{T} \qquad \text{if } u^2+v^2 \geq R_1^2\,. \tag{23}
$$

We consider a solution of $(\widetilde{S}_\rho)$ such that $u(t)^2+\dot u(t)^2 \geq R_1^2$ for every t in an interval $[\tau_1,\tau_2]$, with $\tau_2-\tau_1 \leq T$. Moreover, we assume that the trajectory $(u,v)=(u,\dot u)$ performs K complete rotations around the origin in this interval. Introducing the modified polar coordinates

$$
u(t) = \frac{1}{\sqrt{\nu_2}} r(t)\cos\theta(t)\,, \quad \dot u(t) = r(t)\sin\theta(t) \qquad \text{if } u(t) \leq 0\,,
$$

we have

$$\frac{\pi K}{\sqrt{\nu_2}} = \int_{[\tau_1,\tau_2]\cap\{u<0\}} \frac{\check{\zeta}_\rho(t,u)u^2 + \dot{u}^2}{\nu_2 u^2 + \dot{u}^2} + \frac{h(t,u)u + \partial_u P(t,q,p,u)u}{\nu_2 u^2 + \dot{u}^2}$$
$$\le T + \varepsilon < \frac{\pi}{\sqrt{\nu_2}}(N+1)\,,$$

so that

$$K < N+1\,. \tag{24}$$

We will now construct a curve Γ which guides $(u(t),v(t)) = (u(t),\dot{u}(t))$ in the phase plane. This curve will have the shape of a spiral performing $N+2$ rotations around the origin and will have image in $\{u^2+v^2 > R_1^2\}$, see Figure 2b in the case $N=1$. For this purpose, we define two continuous functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ such that

$$g_1(u) < g_\rho(t,u) + \partial_u P(t,q,p,u) < g_2(u)\,, \tag{25}$$

and we denote by G_1 and G_2 the corresponding primitives. Notice that it is possible to choose the functions g_1 and g_2 so to have

$$\lim_{|u|\to\infty} G_1(u) = \lim_{|u|\to\infty} G_2(u) = +\infty\,.$$

Letting

$$H_i(u,v) = \frac{1}{2}v^2 + G_i(u)\,, \qquad i = 1,2\,,$$

we can assume that the value R_1 introduced in (23) is large enough so to have that any region $\{H_i(u,v) \le E\}$, with $i=1,2$, is star-shaped with respect to the origin, whenever it contains the ball $\{u^2+v^2 \le R_1^2\}$. We choose $(x_1,0)$, with $x_1 > R_1$, as a starting point. The first rotation of Γ is obtained by gluing together the following level curves

$$\mathcal{L}_{1,down} = \{(u,v)\in\mathbb{R}^2 : H_2(u,v) = H_2(x_1,0)\,,\ v\le 0\}\,,$$
$$\mathcal{L}_{1,up} = \{(u,v)\in\mathbb{R}^2 : H_1(u,v) = H_1(\xi_1,0)\,,\ v\ge 0\}\,,$$

where $(\xi_1,0)$ is the intersection of $\mathcal{L}_{1,down}$ with the negative x-semiaxis. Setting $\mathcal{L}_1 = \mathcal{L}_{1,down}\cup\mathcal{L}_{1,up}$ we can choose x_1 sufficiently large so to have $\mathcal{L}_1 \subseteq \{u^2+v^2 > R_1^2\}$, cf. Figure 2a. Then, denoting by $(x_2,0)$ the intersection of $\mathcal{L}_{1,up}$ with the positive x-semiaxis, iterating the above procedure we can construct the set $\mathcal{L}_2 = \mathcal{L}_{2,down}\cup\mathcal{L}_{2,up}$ as the second rotation of Γ in the plane, by defining

$$\mathcal{L}_{2,down} = \{(u,v)\in\mathbb{R}^2 : H_2(u,v) = H_2(x_2,0)\,,\ v\le 0\}\,,$$
$$\mathcal{L}_{2,up} = \{(u,v)\in\mathbb{R}^2 : H_1(u,v) = H_1(\xi_2,0)\,,\ v\ge 0\}\,,$$

where $(\xi_2,0)$ is the intersection of $\mathcal{L}_{2,down}$ with the negative x-semiaxis.

Similarly we may construct $\mathcal{L}_3, \mathcal{L}_4, \dots, \mathcal{L}_{N+2}$. The curve we are looking for is $\Gamma = \mathcal{L}_1\cup\cdots\cup\mathcal{L}_{N+2}$. Finally, let us fix $\tilde{\rho} \ge \bar{\rho}$ so to have $\Gamma \subseteq \{u^2+v^2 < \tilde{\rho}^2\}$. Notice that $\tilde{\rho} > R_1$.

Given a solution (q,p,u) of $(\widetilde{S}_\rho)$ we can compute

$$\frac{d}{dt}H_i(u(t),\dot{u}(t)) = \dot{u}(t)\ddot{u}(t) + g_i(u(t))\dot{u}(t)$$
$$= \dot{u}(t)\big(g_i(u(t)) - g_\rho(t,u(t)) - \partial_u P(t,q(t),p(t),u(t))\big)\,,$$

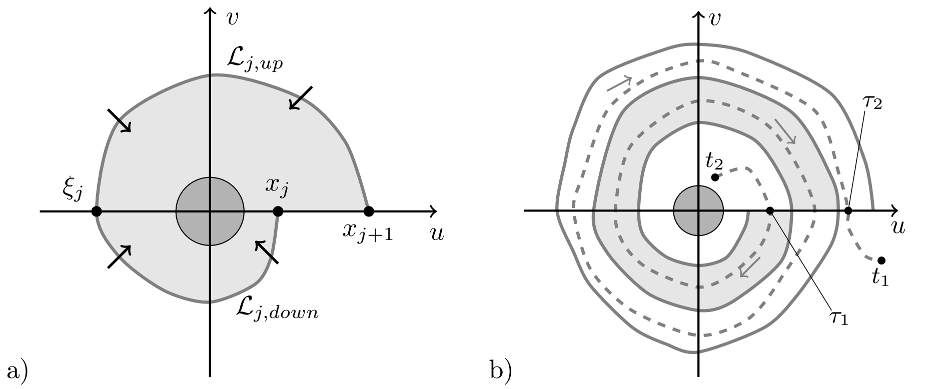

Figure 2: a) The construction of the set $\mathcal{L}_j$. b) The curve guiding the trajectories of system $(\widetilde{S}_\rho)$, in the case $N = 1$.

so that, recalling (25), we get

$$\frac{d}{dt}H_1(u(t), \dot{u}(t)) < 0 \quad \text{if } \dot{u}(t) > 0\,,$$
$$\frac{d}{dt}H_2(u(t), \dot{u}(t)) < 0 \quad \text{if } \dot{u}(t) < 0\,.$$

As a consequence, for any solution (q, p, u) of $(\widetilde{S}_\rho)$, if $(u(t_0), \dot{u}(t_0)) \in \Gamma$ at a certain t_0, then the trajectory $(u(t), \dot{u}(t))$ must cross Γ *"from the outside to the inside"* at $t = t_0$ (cf. Figure 2a).

Let us now assume by contradiction that there is a T-periodic solution (q, p, u) of $(\widetilde{S}_\rho)$, with $\rho \geq \tilde{\rho}$ such that $|u(t_1)| > \tilde{\rho}$ for a certain $t_1 \in [0, T]$. Then, from Proposition 12, we have the existence of $t_2 \in [t_1 - T, t_1]$ such that

$$u(t_2)^2 + \dot{u}(t_2)^2 \leq R_1^2 \leq \tilde{\rho}^2 \leq u(t_1)^2 + \dot{u}(t_1)^2\,.$$

Hence, the trajectory $(u(t), \dot{u}(t))$ must complete at least $N+1$ complete rotations guided by the curve Γ in the interval $[t_2, t_1]$, see Figure 2b. More precisely, we can find an interval $[\tau_1, \tau_2]$, with $\tau_2 - \tau_1 \leq T$, such that

$$R_1^2 \leq u(t)^2 + \dot{u}(t)^2 \leq \tilde{\rho}^2\,, \qquad \text{for every } t \in [\tau_1, \tau_2]\,,$$

and the solution performs exactly $K = N + 1$ rotations in the interval $[\tau_1, \tau_2]$. We thus get a contradiction, since we proved in (24) that $K < N + 1$. Hence, the proof of Proposition 13 is completed. □

Now fix $\rho > \tilde{\rho}$. We can conclude by the same argument as in the proof of Theorem 5. Indeed, Theorem 10 applies with $\chi_1 = \mu_1$ and $\chi_2 = \frac{1}{2}(\nu_1 + \nu_2)$, so that system $(\widetilde{S}_\rho)$ has at least two geometrically distinct T-periodic solutions such that $p(0) \in \,]a, b[\,$. By Proposition 13, these solutions are indeed solutions of the original system (S).

The proof of Theorem 6 is thus concluded. □

4 Further extensions and generalizations

1. The scalar p-Laplace operator. When the second order differential operator $\ddot u$ is replaced by a scalar p-Laplacian operator $\frac{d}{dt}(|\dot u|^{p-2}\dot u)$ one could try to make use of [20, Theorem 4.1] dealing with a planar system ruled by a (p,q)-homogeneous Hamiltonian function. This will be the argument of a future investigation.

2. Higher order systems - I. We first remark that we can consider system (S) in higher dimensions, i.e.,

$$\begin{cases} \dot q = \nabla_p \mathcal{H}(t,q,p) + \nabla_p P(t,q,p,u)\,, \\ \dot p = -\nabla_q \mathcal{H}(t,q,p) - \nabla_q P(t,q,p,u)\,, \\ \ddot u + g(t,u) = -\partial_u P(t,q,p,u)\,, \end{cases} \tag{S_M}$$

with $q = (q_1,\dots,q_M)$ and $p = (p_1,\dots,p_M)$. All the involved functions are continuous and T-periodic in t.

Assumptions $A1 - A3$ can be adapted as follows.

$A1'$. The function $\mathcal{H}(t,q,p)$ is 2π-periodic in q_i for every $i \in \{1,\dots,M\}$.

$A2'$. Given the rectangle

$$\mathcal{D} = [a_1,b_1] \times \cdots \times [a_M,b_M]\,,$$

there exists an M-tuple $\sigma = (\sigma_1,\dots,\sigma_M) \in \{-1,1\}^M$ such that for every C^1-function $\mathcal{U} : [0,T] \to \mathbb{R}$, all the solutions (q,p) of system

$$\begin{cases} \dot q = \nabla_p \mathcal{H}(t,q,p) + \nabla_p P(t,q,p,\mathcal{U}(t))\,, \\ \dot p = -\nabla_q \mathcal{H}(t,q,p) - \nabla_q P(t,q,p,\mathcal{U}(t))\,, \end{cases}$$

starting with $p(0) \in \mathcal{D}$, are defined on $[0,T]$ and, for every $i \in \{1,\dots,M\}$,

$$\begin{cases} p_i(0) = a_i \quad \Longrightarrow \quad \sigma_i(q_i(T) - q_i(0)) < 0\,, \\ p_i(0) = b_i \quad \Longrightarrow \quad \sigma_i(q_i(T) - q_i(0)) > 0\,. \end{cases}$$

$A3'$. The function $P(t,q,p,u)$ is 2π-periodic in q_i for every index $i \in \{1,\dots,M\}$, and has a bounded gradient with respect to (q,p,u). In particular, there exists a constant $\overline{m}$ such that

$$|\partial_u P(t,q,p,u)| \le \overline{m}\,, \qquad \text{for every } (t,q,p,u)\,.$$

In this new setting we can rephrase Theorem 5.

Theorem 14. *Let $A1' - A3'$ hold true and assume that there exist a positive integer N and some positive constants μ_1,μ_2,ν_1,ν_2, and C such that* (6), (8) *and* (10) *hold. If for every non-zero function w such that $\ddot w + \mu_1 w^+ - \nu_1 w^- = 0$ one has* (7) *and for every non-zero function v such that $\ddot v + \mu_2 v^+ - \nu_2 v^- = 0$ one has* (9)*, then there are at least $M+1$ geometrically distinct T-periodic solutions of system (S_M), with $p(0)$ belonging to the interior of $\mathcal{D}$.*

In the same spirit Theorem 6 can be rewritten as follows.

Theorem 15. *Let $A1' - A3'$ and $A5$ hold true. If there exist a positive integer N such that (11) holds, then there are at least $M+1$ geometrically distinct T-periodic solutions of system (S_M), with $p(0)$ belonging to the interior of $\mathcal{D}$.*

Condition $A2'$ can be replaced by some different types of twist conditions (see, e.g., [14, 15]). We do not enter in such details for briefness.

4. Higher order systems - II. Similar results can also be obtained for systems of the type

$$\begin{cases} \dot q = \nabla_p \mathcal{H}(t,q,p) + \nabla_p P(t,q,p,u)\,, \\ \dot p = -\nabla_q \mathcal{H}(t,q,p) - \nabla_q P(t,q,p,u)\,, \\ \ddot u_j + g_j(t,u_j) = -\partial_{u_j} P(t,q,p,u)\,, \quad j = 1,\dots,L\,, \end{cases} \tag{26}$$

where now $u = (u_1,\dots,u_L)$. If the functions g_j satisfy the assumptions of Theorems 14 or 15, for some positive constants $\mu_{1,j}$, $\mu_{2,j}$, $\nu_{1,j}$, $\nu_{2,j}$, the same conclusions hold. The proofs are still carried out by applying [19, Theorem 1.1]. Notice that the integer N in (10) and (11) could depend upon j, as well.

Still more, concerning the last equations in (26), notice that we have focused our attention on scalar second order differential equations. All our results could be extended to the setting of planar systems, as it has been shown in [13].

5. Neumann boundary conditions. Similar results could be stated for Neumann-type boundary value problems associated with (S_M), i.e.,

$$p(0) = 0 = p(T)\,, \quad \dot u(0) = 0 = \dot u(T)\,,$$

in the spirit of [16, 17, 20, 26]. It is worth to be noticed that, in this case, the twist condition is unnecessary. We address the reader to [1, 27, 30] for related results involving the Landesman–Lazer condition in this setting. We do not enter in details for briefness.

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References

[1] A. Boscaggin and M. Garrione, Resonant Sturm–Liouville boundary value problems for differential systems in the plane, J. Anal. Appl. 35 (2016), 41–59.

[2] H. Brézis and L. Nirenberg, Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 5 (1978), 225–326.

[3] F. Chen and D. Qian, An extension of the Poincaré–Birkhoff theorem for Hamiltonian systems coupling resonant linear components with twisting components, J. Differential Equations 321 (2022), 415–448.

[4] E.N. Dancer, Boundary-value problems for weakly nonlinear ordinary differential equations, Bull. Australian Math. Soc. 15 (1976), 321–328.

[5] E.N. Dancer, Proofs of the results in "Boundary-value problems for weakly nonlinear ordinary differential equations", Rend. Istit. Mat. Univ. Trieste 42 (2010), 31–57.

[6] P. Drábek, Landesman–Lazer condition for nonlinear problems with jumping nonlinearities, J. Differential Equations 85 (1990), 186–199.

[7] P. Drábek and S. Invernizzi, On the periodic BVP for the forced Duffing equation with jumping nonlinearity, Nonlinear Anal. 10 (1986), 642–650.

[8] C. Fabry, Landesman–Lazer conditions for periodic boundary value problems with asymmetric nonlinearities, J. Differential Equations 116 (1995), 405–418.

[9] C. Fabry and A. Fonda, Periodic solutions of nonlinear differential equations with double resonance, Ann. Mat. Pura Appl. 157 (1990), 99–116.

[10] C. Fabry and P. Habets, Periodic solutions of second order differential equations with superlinear asymmetric nonlinearities, Arch. Math. 60 (1993), 266–276.

[11] A.F. Filippov, Differential Equations with Discontinuous Righthand Sides, Kluwer, Dordrecht, 1988.

[12] A. Fonda, M. Garzón and A. Sfecci, An extension of the Poincaré–Birkhoff Theorem coupling twist with lower and upper solutions, J. Math. Anal. Appl. 528 (2023), Paper No. 127599, 33 pp.

[13] A. Fonda and M. Garrione, Double resonance with Landesman–Lazer conditions for planar systems of ordinary differential equations, J. Differential Equations 250 (2011), 1052–1082.

[14] A. Fonda and P. Gidoni, An avoiding cones condition for the Poincaré–Birkhoff Theorem, J. Differential Equations 262 (2017), 1064–1084.

[15] A. Fonda and P. Gidoni, Coupling linearity and twist: an extension of the Poincaré–Birkhoff theorem for Hamiltonian systems, NoDEA Nonlinear Differential Equations Appl. 27 (2020), Paper No. 55, 26 pp.

[16] A. Fonda, N. G. Mamo, F. Obersnel and A. Sfecci, Multiplicity results for Hamiltonian systems with Neumann-type boundary conditions, NoDEA Nonlinear Differential Equations Appl., doi.org/10.1007/s00030-023-00913-4.

[17] A. Fonda and R. Ortega, A two-point boundary value problem associated with Hamiltonian systems on a cylinder, Rend. Circ. Mat. Palermo 72 (2023), 3931–3947.

[18] A. Fonda and W. Ullah, Periodic solutions of Hamiltonian systems coupling twist with generalized lower/upper solutions, J. Differential Equations 379 (2024), 148–174.

[19] A. Fonda and W. Ullah, Periodic solutions of Hamiltonian systems coupling twist with an isochronous center, Differential Integral Equations, 37 (2024), 323–336.

[20] A. Fonda and W. Ullah, Boundary value problems associated with Hamiltonian systems coupled with positively (p, q) homogeneous systems, Preprint (2023).

[21] A. Fonda and A.J. Ureña, A higher dimensional Poincaré–Birkhoff theorem for Hamiltonian flows, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017), 679–698.

[22] S. Fučík, Boundary value problems with jumping nonlinearities, Časopis Pèst. Mat. 101 (1976), 69–87.

[23] M.-Y. Jiang, A Landesmam–Lazer theorem for periodic solutions of the resonant asymmetric p-Laplacian equation, Acta Math. Sin. 21 (2005), 1219–1228.

[24] E.M. Landesman and A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970), 609–623.

[25] A.C. Lazer and D.E. Leach, Bounded perturbations of forced harmonic oscillators at resonance, Ann. Mat. Pura Appl. 82 (1969), 49–68.

[26] N.G. Mamo, Neumann-type boundary value problem associated with Hamiltonian systems, Topol. Methods Nonlinear Anal., to appear.

[27] R. Marvulli and A. Sfecci, Landesman–Lazer type conditions for scalar one-sided superlinear nonlinearities with Neumann boundary conditions, Adv. Differential Equations 28 (2023), 247–286.

[28] J. Mawhin, Landesman–Lazer's type problems for nonlinear equations, Confer. Sem. Mat. Univ. Bari 147 (1977), 22 pp.

[29] J. Mawhin, and J.R. Ward, Periodic solutions of some forced Liénard differential equations at resonance, Arch. Math. 41 (1983), 337–351.

[30] A. Sfecci, Double resonance in Sturm–Liouville planar boundary value problems, Topol. Methods Nonlinear Anal. 55 (2020), 655–680.

[31] P. Tomiczek, Potential Landesman–Lazer type conditions and the Fučík spectrum, Electron. J. Differential Equations 94 (2005), 1–12.

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