On the existence of periodic solutions for damped asymmetric oscillators

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Abstract

We propose some new sufficient conditions for the existence of periodic solutions of an asymmetric oscillator with a positive damping term. Our results are complemented by an example where, in some situations, no periodic solutions may exist. This fact is well known in the undamped case, when the resonance phenomenon may appear. However, the damped case presents some unintuitive features which have not been so thoroughly studied in the literature, and the overall picture still has several aspects which need to be better understood.

1 Introduction

In this paper we want to study the periodic problem

$$\begin{cases} x'' + cx' + f(t,x) = e(t)\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,, \end{cases} \tag{1}$$

where c is a nonnegative real constant. We assume that the function f satisfies the Carathéodory conditions, and that $e \in L^1(0,T)$.

Looking back at the literature, the first studies in this field started in the sixties (see [14, 15, 17, 18]). These pioneering papers were the starting point of a vast literature, involving also Liénard and Rayleigh equations. One can mention for example the papers [16, 20, 21, 27] and the references therein. Most of these papers provide some sufficient conditions on the asymptotic behaviour of the quotient $f(t,x)/x$, with respect to the spectrum of the differential operator $\mathcal{L}x = -x''$ with T-periodic boundary conditions, in order to guarantee the existence of a solution to problem (1).

Let us first focus our attention on the linear problem

$$\begin{cases} x'' + cx' + \lambda(t)x = e(t)\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,, \end{cases} \tag{2}$$

with $\lambda(t)$ a positive function. Not only the cases $c = 0$ or $c > 0$ must be distinguished, but also the different situations arising when $\lambda(t)$ is constant or not.

Indeed, if $c = 0$, the phenomenon of *resonance* can occur. If $\lambda(t)$ is constant and coincides with $\left(\frac{2\pi N}{T}\right)^2$, for some integer N, then there exist functions $e(t)$ for which problem (2) has no solutions. Hence, in order to find sufficient conditions guaranteeing the existence of solutions of problem (2), one will have to worry about the interaction of $\lambda(t)$ with the spectrum

$$\sigma(\mathcal{L}) = \left\{ \left(\frac{2\pi N}{T} \right)^2 : N \in \mathbb{N} \right\}.$$

A huge literature has been devoted to finding *nonresonance conditions* in order to guarantee the existence of solutions of (1) when the quotient $f(t,x)/x$ asymptotically behaves like some $\lambda(t)$.

On the contrary, if $c > 0$, much less is known. Surely enough, if $\lambda(t)$ is constant, problem (2) always has a (unique) solution. But if $\lambda(t)$ is not constant, the situation can become very subtle. For example, we will show that there exist piecewise constant positive functions $\lambda(t)$ such that, for some function $e(t)$, problem (2) has no solution. The main issue will then be to find sufficient conditions guaranteeing the existence of solutions to problem (2), when $c > 0$.

The case when $\lambda(t)$ is constant enters in the wider class of problems of the type

$$\begin{cases} x'' + cx' + g(x) = e(t)\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,, \end{cases} \tag{3}$$

where the situation seems to be much simpler to be studied, due to the fact that some energy estimates on the solutions can be exploited. For example, the following statement can be obtained from [7]. See also [1, 9, 23, 25].

Theorem 1.1. *If* $c > 0$ *and*

$$\limsup_{x \to -\infty} g(x) < \frac{1}{T} \int_0^T e(t)\, dt < \liminf_{x \to +\infty} g(x)\,,$$

then problem (3) *has a solution.*

A generalization of problem (2) was proposed by Fučík [8] and Dancer [2] by introducing an *asymmetric nonlinearity*, like in

$$\begin{cases} x'' + cx' + \mu(t)x^+ - \nu(t)x^- = e(t)\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,. \end{cases} \tag{4}$$

Here, as usual, we adopt the notation $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

In the case $c = 0$, a crucial role for the existence of solutions of (4) is played by the so-called Fučík spectrum, namely the set

$$\Sigma = \bigcup_{N=0}^{\infty} \mathcal{C}_N \,,$$

where

$$\mathcal{C}_0 = \{(\mu, \nu) \in \mathbb{R} \times \mathbb{R} : \mu\nu = 0\} \,,$$

and, for $N \geq 1$,

$$\mathcal{C}_N = \left\{ (\mu, \nu) \in \,]0, +\infty[\, \times \,]0, +\infty[\,: \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{N} \right\} .$$

This happens to be the set of those (μ, ν) for which there exists a nontrivial solution to the problem

$$\begin{cases} x'' + \mu x^+ - \nu x^- = 0 \,, \\ x(0) = x(T) \,, \quad x'(0) = x'(T) \,. \end{cases}$$

Let $\mu_1, \mu_2, \nu_1, \nu_2$ be some constants such that

$$\mu_1 \leq \mu(t) \leq \mu_2 \,, \quad \nu_1 \leq \nu(t) \leq \nu_2 \,, \quad \text{for a.e. } t \in [0, T] \,, \tag{5}$$

and define the rectangle

$$\mathcal{R}_0 = [\mu_1, \mu_2] \times [\nu_1, \nu_2] \,.$$

It has been proved that if $\mathcal{R}_0$ is entirely contained in either the first or the third quadrant of the plane, and it has empty intersection with Σ, then problem (4) always has a solution (see, e.g., [4, 10]).

If $c > 0$, the problem has been first studied by Drabek and Invernizzi in [4]. They introduced the constants

$$\mu_j^c = \mu_j - c^2/4 \,, \quad \nu_j^c = \nu_j - c^2/4 \,, \quad j = 1, 2 \,, \tag{6}$$

and the rectangle

$$\mathcal{R}_c = [\mu_1^c, \mu_2^c] \times [\nu_1^c, \nu_2^c] \,.$$

Denoting by Q_i the i-th quadrant of the plane, with $i = 1, 2, 3, 4$, they proved that, if

$$\mathcal{R}_c \cup \mathcal{R}_0 \subseteq (Q_1 \cup Q_3) \setminus \Sigma \,,$$

then problem (4) always has a solution. The proof of this result was carried out through a study of the dynamics of the solutions in the phase plane, focusing the attention on the behaviour of their angular speed. We will be able to extend such a result in different directions, providing estimates involving both the radial and the angular speed of the solutions.

Variants and extensions of the result in [4] have been provided by several authors, see [5, 11, 16, 19, 20, 21, 22, 24, 26, 27] and the references therein.

In this paper we mainly focus our attention on the case $c > 0$, with the aim of providing further sufficient conditions in order to prove the existence of a solution to problem (1). We extend the previously known results in several directions, by the use of phase plane analysis. Nevertheless, as we will better explain later on, several aspects of the overall picture still remain unexplored.

In Section 2 we consider the general asymmetric case when the nonlinearity $f(t,x)$ has an at most linear growth in x. We will recall here a variant of the so-called Property P introduced by Habets and Metzen [10], which will be the guideline for the proof of our main theorem which generalizes in a single statement all the results in [10]. In order not to interrupt the exposition of our results, this proof will be postponed to Section 7.

In Sections 3 and 4 we concentrate on the symmetric case and compare our results with those obtained by a classical functional approach. In particular, in Corollary 4.4 we highlight a sufficient condition for the existence problem, new in the literature, which can be easily verified in practice. Then, in Section 5 we give an example showing that our assumptions are optimal, provided that the constant c is not too large.

In Section 6 we propose a generalization of Theorem 1.1 in the case when the function $f(t,x)$ in problem (1) is controlled by two multiples of *the same function* $g(x)$. The proof relies on phase plane analysis, combined with the application of the Brouwer fixed point theorem.

2 Linear growth - the asymmetric case

Let us assume that the function $f(t,x)$ has a linear growth in x, by introducing the following hypothesis.

Assumption (A). There exist constants μ_1, μ_2, ν_1, ν_2 for which

$$f(t,x) = \gamma_+(t,x)x^+ - \gamma_-(t,x)x^- + r(t,x)\,,$$

where

$$\mu_1 \leq \gamma_+(t,x) \leq \mu_2\,, \qquad \nu_1 \leq \gamma_-(t,x) \leq \nu_2\,,$$

and $r(t,x)$ is uniformly bounded.

The following definition is a variant of the one introduced by Habets and Metzen in [10].

Definition 2.1. *We say that the five-number row* $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ *satisfies the Property* (P) *if for every pair of functions* $\mu, \nu \in L^\infty(0,T)$ *satisfying* (5) *the problem*

$$\begin{cases} v'' + cv' + \mu(t)v^+ - \nu(t)v^- = 0 \\ v(0) = v(T)\,, \quad v'(0) = v'(T) \end{cases} \tag{7}$$

only has the trivial solution.

Such a property already appears, more or less implicitly, in [12, 13]. The following result is due to Habets and Metzen (see [10, Theorem 2] and the subsequent remark), cf. also [3, 6].

Theorem 2.2 (Habets–Metzen)**.** *Let Assumption* (A) *hold true with* $\mathcal{R}_0 \subseteq \mathring{Q}_1 \cup \mathring{Q}_3$*, assume* $c \geq 0$ *and that* $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ *satisfies the Property* (P)*. Then problem* (1) *has a solution, for every* $e(t)$*.*

In what follows we will always assume that both $\mu_j > c^2/4$ and $\nu_j > c^2/4$, so that the constants μ_j^c and ν_j^c introduced in (6) are positive. We define

$$\tau_1^c = \frac{\pi}{\sqrt{\mu_1^c}} + \frac{\pi}{\sqrt{\nu_1^c}}\,, \qquad \tau_2^c = \frac{\pi}{\sqrt{\mu_2^c}} + \frac{\pi}{\sqrt{\nu_2^c}}\,.$$

For every $\gamma \in \mathbb{R}$, we set

$$c_\gamma = c - 2\gamma\,,$$

and, recalling (6), define the constants

$$\bar\theta_{j,\gamma} = \arctan \frac{c_\gamma}{2\sqrt{\mu_j^c}}\,, \quad \hat\theta_{j,\gamma} = \arctan \frac{c_\gamma}{2\sqrt{\nu_j^c}}\,, \quad j = 1, 2\,,$$

and the functions

$$\mathcal{F}_+(\gamma) = \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2}}\ \exp\Big(-\frac{c_\gamma}{2\sqrt{\mu_1^c}}\Big(\frac{\pi}{2} - \bar\theta_{1,\gamma}\Big) - \frac{c_\gamma}{2\sqrt{\mu_2^c}}\Big(\frac{\pi}{2} + \bar\theta_{2,\gamma}\Big)\Big)\,,$$

$$\mathcal{F}_-(\gamma) = \sqrt{\frac{4\nu_2^c + c_\gamma^2}{4\nu_1^c + c_\gamma^2}}\ \exp\Big(-\frac{c_\gamma}{2\sqrt{\nu_1^c}}\Big(\frac{\pi}{2} - \hat\theta_{1,\gamma}\Big) - \frac{c_\gamma}{2\sqrt{\nu_2^c}}\Big(\frac{\pi}{2} + \hat\theta_{2,\gamma}\Big)\Big)\,.$$

Here is our main result, in this setting.

Theorem 2.3. *Assume* c*,* μ_1^c*,* ν_1^c *to be positive. If there exist a positive integer* N *and* $\gamma > 0$ *such that*

$$T \leq (N+1)\tau_2^c\,, \tag{8}$$

and

$$\mathcal{F}_+(\gamma)\mathcal{F}_-(\gamma) < e^{\gamma T/N}, \tag{9}$$

then $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ *satisfies the Property* (P)*.*

The proof is postponed to Section 7. Since condition (9) in the above theorem is a bit intricate, we now propose a more readable corollary, which also has the advantage of easily providing a whole series of possible applications.

Corollary 2.4. *Assume c, μ_1^c, ν_1^c to be positive. If there exist a positive integer N and $\gamma > 0$ such that (8) holds and*

$$\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2} \cdot \frac{4\nu_2^c + c_\gamma^2}{4\nu_1^c + c_\gamma^2} \leq \exp\left(\frac{2\gamma T}{N} + \min\{c_\gamma \tau_1^c\,,\, c_\gamma \tau_2^c\}\right), \tag{10}$$

then $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ satisfies the Property (P).

Proof. We first notice that the function $f(x) = x\big(\frac{\pi}{2} - \arctan x\big)$ is strictly increasing. Hence, if $\gamma \leq c/2$ and so $c_\gamma \geq 0$, we get

$$\begin{aligned}\mathcal{F}_+(\gamma) &= \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2}}\ \exp\left(-c_\gamma \frac{\pi}{2\sqrt{\mu_2^c}}\right)\ \exp\left(f\left(\frac{c_\gamma}{2\sqrt{\mu_2^c}}\right) - f\left(\frac{c_\gamma}{2\sqrt{\mu_1^c}}\right)\right) \\ &\leq \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2}}\ \exp\left(-c_\gamma \frac{\pi}{2\sqrt{\mu_2^c}}\right).\end{aligned}$$

Similarly,

$$\mathcal{F}_-(\gamma) \leq \sqrt{\frac{4\nu_2^c + c_\gamma^2}{4\nu_1^c + c_\gamma^2}}\ \exp\left(-c_\gamma \frac{\pi}{2\sqrt{\nu_2^c}}\right),$$

so that

$$\mathcal{F}_+(\gamma)\mathcal{F}_-(\gamma) \leq \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2} \cdot \frac{4\nu_2^c + c_\gamma^2}{4\nu_1^c + c_\gamma^2}}\,\exp\left(-c_\gamma \frac{\tau_2^c}{2}\right).$$

Using (10), since $c_\gamma \tau_2^c \leq c_\gamma \tau_1^c$, we recover (9).

On the other hand, if $\gamma > c/2$ and so $c_\gamma < 0$, using now the strictly increasing function $g(x) = x\big(\frac{\pi}{2} + \arctan x\big)$, we get

$$\begin{aligned}\mathcal{F}_+(\gamma) &= \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2}}\ \exp\left(-c_\gamma \frac{\pi}{2\sqrt{\mu_1^c}}\right)\ \exp\left(g\left(\frac{c_\gamma}{2\sqrt{\mu_1^c}}\right) - g\left(\frac{c_\gamma}{2\sqrt{\mu_2^c}}\right)\right) \\ &\leq \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2}}\ \exp\left(-c_\gamma \frac{\pi}{2\sqrt{\mu_1^c}}\right).\end{aligned}$$

Similarly,

$$\mathcal{F}_-(\gamma) \leq \sqrt{\frac{4\nu_2^c + c_\gamma^2}{4\nu_1^c + c_\gamma^2}}\ \exp\left(-c_\gamma \frac{\pi}{2\sqrt{\nu_1^c}}\right),$$

so that

$$\mathcal{F}_+(\gamma)\mathcal{F}_-(\gamma) \leq \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2} \cdot \frac{4\nu_2^c + c_\gamma^2}{4\nu_1^c + c_\gamma^2}}\,\exp\left(-c_\gamma \frac{\tau_1^c}{2}\right).$$

Using (10), since now $c_\gamma \tau_1^c \leq c_\gamma \tau_2^c < 0$, we recover (9) also in this case. □

As previously announced, we now state four possible consequences.

Corollary 2.5. *Assume c, μ_1^c, ν_1^c to be positive and that*

$$\frac{\mu_2}{\mu_1}\frac{\nu_2}{\nu_1} < \exp(c\tau_2^c)\,.$$

Then $(c,\mu_1,\mu_2,\nu_1,\nu_2)$ satisfies the Property (P).

Proof. Fix a positive integer N satisfying (8). The result then follows from Corollary 2.4 choosing $\gamma > 0$ small enough. □

Corollary 2.6. *Assume c, μ_1^c, ν_1^c to be positive and that there exists a positive integer N satisfying* (8) *and*

$$\frac{\mu_2^c}{\mu_1^c}\frac{\nu_2^c}{\nu_1^c} < \exp\left(c\,\frac{T}{N}\right).$$

Then $(c,\mu_1,\mu_2,\nu_1,\nu_2)$ satisfies the Property (P).

Proof. It follows directly from Corollary 2.4 choosing $\gamma = c/2$. □

Corollary 2.7. *Assume c, μ_1^c, ν_1^c to be positive and that there exists a positive integer N satisfying* (8) *and*

$$\frac{\mu_2}{\mu_1}\frac{\nu_2}{\nu_1} < \exp\left(c\left[\frac{2T}{N} - \tau_1^c\right]\right).$$

Then $(c,\mu_1,\mu_2,\nu_1,\nu_2)$ satisfies the Property (P).

Proof. It follows from Corollary 2.4 taking $\gamma = c$. □

Corollary 2.8. *Assume c, μ_1^c, ν_1^c to be positive and that there exists a positive integer N for which*

$$\frac{T}{N+1} \le \frac{\pi}{\sqrt{\mu_2^c}} + \frac{\pi}{\sqrt{\nu_2^c}} \le \frac{\pi}{\sqrt{\mu_1^c}} + \frac{\pi}{\sqrt{\nu_1^c}} \le \frac{T}{N}\,.$$

Then $(c,\mu_1,\mu_2,\nu_1,\nu_2)$ satisfies the Property (P).

Proof. Taking the limit in (10) as $\gamma \to +\infty$, so that $c_\gamma \to -\infty$, the left hand side tends to 1, while

$$\exp\left(\frac{2\gamma T}{N} + \min\{c_\gamma\tau_1^c\,,\, c_\gamma\tau_2^c\}\right) = \exp\left(\frac{2\gamma T}{N} + c_\gamma\tau_1^c\right)$$
$$= \exp\left(2\gamma\left[\frac{T}{N} - \frac{\pi}{\sqrt{\mu_1^c}} - \frac{\pi}{\sqrt{\nu_1^c}}\right] + c\left[\frac{\pi}{\sqrt{\mu_1^c}} + \frac{\pi}{\sqrt{\nu_1^c}}\right]\right) \ge \exp(c\tau_1^c) > 1\,.$$

Then, for $\gamma > 0$ large enough we can apply Corollary 2.4. □

Notice that Corollaries 2.5, 2.6 correspond to [10, Theorem 11], [10, Theorem 10], respectively, and Corollary 2.8 to [10, Theorem 8]. Let us now show how our main result extends the one in [4].

Corollary 2.9. *Let Assumption (A) hold true, with both μ_1^c and ν_1^c positive. Assume c to be positive and*

$$\mathcal{R}_c \setminus \{(\mu_1^c, \nu_1^c), (\mu_2^c, \nu_2^c)\} \subseteq Q_1 \setminus \Sigma\,.$$

Then problem (1) *has a solution.*

Proof. We will prove that the Property (P) holds. The case $\mathcal{R}_c = \{(\mu_1^c, \nu_1^c)\}$ is easily settled by multiplying by v' the equation in (7) and integrating on the interval $[0, T]$. Otherwise, we have two possible cases. In the first one,

$$T \leq \frac{\pi}{\sqrt{\mu_2^c}} + \frac{\pi}{\sqrt{\nu_2^c}}\,,$$

and the conclusion follows by a standard argument, cf. Lemma 7.1.

In the second case, there exists a positive integer N such that

$$\frac{T}{N+1} \leq \frac{\pi}{\sqrt{\mu_2^c}} + \frac{\pi}{\sqrt{\nu_2^c}} \leq \frac{\pi}{\sqrt{\mu_1^c}} + \frac{\pi}{\sqrt{\nu_1^c}} \leq \frac{T}{N}\,.$$

Hence, by Corollary 2.8 and Theorem 2.2 we easily conclude. □

Now, in order to state a dual version of Theorem 2.3, we introduce the functions

$$\widetilde{\mathcal{F}}_+(\gamma) = \sqrt{\frac{4\mu_1^c + c_\gamma^2}{4\mu_2^c + c_\gamma^2}}\, \exp\left(-\frac{c_\gamma}{2\sqrt{\mu_2^c}}\left(\frac{\pi}{2} - \bar{\theta}_{2,\gamma}\right) - \frac{c_\gamma}{2\sqrt{\mu_1^c}}\left(\frac{\pi}{2} + \bar{\theta}_{1,\gamma}\right)\right),$$

$$\widetilde{\mathcal{F}}_-(\gamma) = \sqrt{\frac{4\nu_1^c + c_\gamma^2}{4\nu_2^c + c_\gamma^2}}\, \exp\left(-\frac{c_\gamma}{2\sqrt{\nu_2^c}}\left(\frac{\pi}{2} - \hat{\theta}_{2,\gamma}\right) - \frac{c_\gamma}{2\sqrt{\nu_1^c}}\left(\frac{\pi}{2} + \hat{\theta}_{1,\gamma}\right)\right).$$

Here is the corresponding result.

Theorem 2.10. *Assume c, μ_1^c, ν_1^c to be positive. If there exist a positive integer N and $\gamma > 0$ such that*

$$T \geq N\tau_1^c\,, \tag{11}$$

and

$$\widetilde{\mathcal{F}}_+(\gamma)\widetilde{\mathcal{F}}_-(\gamma) > e^{\gamma T/(N+1)},$$

then $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ satisfies the Property (P).

The proof, being rather similar to the one of Theorem 2.3, will just be sketched at the end of Section 7, for the sake of briefness. We now state the dual version of Corollary 2.4.

Corollary 2.11. *Assume c, μ_1^c, ν_1^c to be positive. If there exist a positive integer N and $\gamma > 0$ such that* (11) *holds and*

$$\frac{4\mu_1^c + c_\gamma^2}{4\mu_2^c + c_\gamma^2} \cdot \frac{4\nu_1^c + c_\gamma^2}{4\nu_2^c + c_\gamma^2} \geq \exp\left(\frac{2\gamma T}{N+1} + \max\{c_\gamma \tau_1^c\,,\ c_\gamma \tau_2^c\}\right),$$

then $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ satisfies the Property (P).

As an immediate consequence, similarly as above, we have a whole series of corollaries. After noticing that the dual version of Corollary 2.8 remains the same, we collect the dual versions of Corollaries 2.5, 2.6, and 2.7 in a single statement.

Corollary 2.12. *Assume c, μ_1^c, ν_1^c to be positive and that there exists a positive integer N satisfying* (11) *and, either*

$$\frac{\mu_2^c}{\mu_1^c}\frac{\nu_2^c}{\nu_1^c} > \exp\left(c\,\frac{T}{N+1}\right),$$

or

$$\frac{\mu_2}{\mu_1}\frac{\nu_2}{\nu_1} > \exp\left(c \cdot \min\left\{\tau_2^c\,,\ \frac{2T}{N+1} - \tau_1^c\right\}\right).$$

Then $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ satisfies the Property (P).

3 Linear growth - the symmetric case

In the symmetric case, Assumption (A) can be rephrased as follows.

Assumption ($\mathbf{A}_{sym}$). There exist positive constants μ_1, μ_2, for which

$$f(t,x) = \gamma(t,x)x + r(t,x)\,,$$

where $\mu_1 \leq \gamma(t,x) \leq \mu_2$, and $r(t,x)$ is uniformly bounded.

In this case we modify the definition of Property (P) as follows.

Definition 3.1. *We say that the three-number row (c, μ_1, μ_2) satisfies the Property (P_{sym}) if, for every function $\mu \in L^\infty(0,T)$ such that*

$$\mu_1 \leq \mu(t) \leq \mu_2\,, \quad \text{for a.e. } t \in [0,T]\,,$$

the problem

$$\begin{cases} v'' + cv' + \mu(t)v = 0 \\ v(0) = v(T)\,, \quad v'(0) = v'(T) \end{cases} \tag{12}$$

only has the trivial solution.

Notice that if the row $(c, \mu_1, \mu_2, \mu_1, \mu_2)$ satisfies the Property (P) then the row (c, μ_1, μ_2) satisfies the Property (P_{sym}), but the contrary is not guaranteed. Nevertheless, the following theorem still holds.

Theorem 3.2 (Habets–Metzen). *Let Assumption* (A_{sym}) *hold true with either* $\mu_1 > 0$ *or* $\mu_2 < 0$*. Assume* $c \geq 0$ *and that* (c, μ_1, μ_2) *satisfies the Property* (P_{sym})*. Then problem* (1) *has a solution, for every* $e(t)$*.*

We will now provide some sufficient conditions in order to have the Property (P_{sym}) satisfied. The analogue of Theorem 2.3, our main result above, can be stated in the following way.

Theorem 3.3. *Assume* c *and* μ_1^c *to be positive. If there exist a positive integer* N *and* $\gamma > 0$ *such that*

$$\mu_2^c \leq \left(\frac{2\pi(N+1)}{T}\right)^2, \tag{13}$$

and

$$\mathcal{F}_+(\gamma) < e^{\gamma T/2N},$$

then (c, μ_1, μ_2) *satisfies the Property* (P_{sym})*.*

Corollary 2.4, in this case, reads as follows.

Corollary 3.4. *Assume* c *and* μ_1^c *to be positive. If there exist a positive integer* N *and* $\gamma > 0$ *such that* (13) *holds and*

$$\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2} \leq \exp\left(\frac{\gamma T}{N} + \min\left\{c_\gamma \frac{\pi}{\sqrt{\mu_1^c}}\,,\, c_\gamma \frac{\pi}{\sqrt{\mu_2^c}}\right\}\right),$$

then (c, μ_1, μ_2) *satisfies the Property* (P_{sym})*.*

Similarly, Corollaries 2.5–2.8 can be reformulated as follows (see Figure 1 for a visual comparison between them).

Corollary 3.5. *If* c *and* μ_1^c *are positive and*

$$\frac{\mu_2}{\mu_1} < \exp\left(c\,\frac{\pi}{\sqrt{\mu_2^c}}\right),$$

then (c, μ_1, μ_2) *satisfies the Property* (P_{sym})*.*

Corollary 3.6. *If* c *and* μ_1^c *are positive and there exists a positive integer* N *for which*

$$\mu_2^c \leq \left(\frac{2\pi(N+1)}{T}\right)^2, \qquad \frac{\mu_2^c}{\mu_1^c} < \exp\left(c\,\frac{T}{2N}\right), \tag{14}$$

then (c, μ_1, μ_2) *satisfies the Property* (P_{sym})*.*

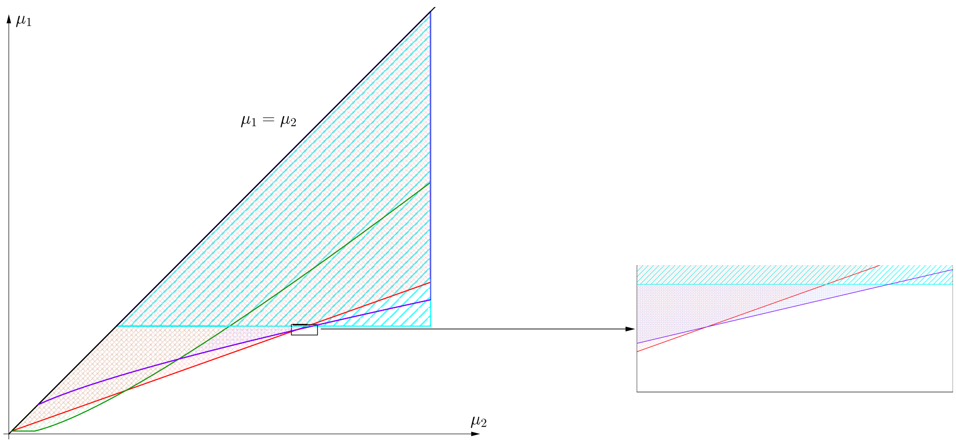

Figure 1: The regions coloured in green, red, violet and light blue represent the couples (μ_2, μ_1) for which Corollaries 3.5, 3.6, 3.7 and 3.8 apply, respectively, when $T = 2\pi$, $c = \frac{1}{3}$, and $N = 1$. We have emphasized the fact that there is also a tiny zone where only Corollary 3.7 applies.

Corollary 3.7. *If c and μ_1^c are positive and there exists a positive integer N for which*

$$\mu_2^c \leq \left(\frac{2\pi(N+1)}{T}\right)^2, \qquad \frac{\mu_2}{\mu_1} < \exp\left(c\left[\frac{T}{N} - \frac{\pi}{\sqrt{\mu_1^c}}\right]\right),$$

then (c, μ_1, μ_2) satisfies the Property (P_{sym}).

Corollary 3.8. *If c and μ_1^c are positive and there exists a positive integer N for which*

$$\left(\frac{2\pi N}{T}\right)^2 \leq \mu_1^c \leq \mu_2^c \leq \left(\frac{2\pi(N+1)}{T}\right)^2, \tag{15}$$

then (c, μ_1, μ_2) satisfies the Property (P_{sym}).

The application of Theorem 2.3 is visualized (in light blue) in Figure 2, where one can appreciate the improvement obtained with respect to the classical result by Drabek and Invernizzi (visualized in green).

We now provide the statements of the dual Theorem 2.10 and Corollaries 2.11, 2.12 in the symmetric case.

Theorem 3.9. *Assume c and μ_1^c to be positive. If there exist a positive integer N and $\gamma > 0$ such that*

$$\mu_1^c \geq \left(\frac{2\pi N}{T}\right)^2, \tag{16}$$

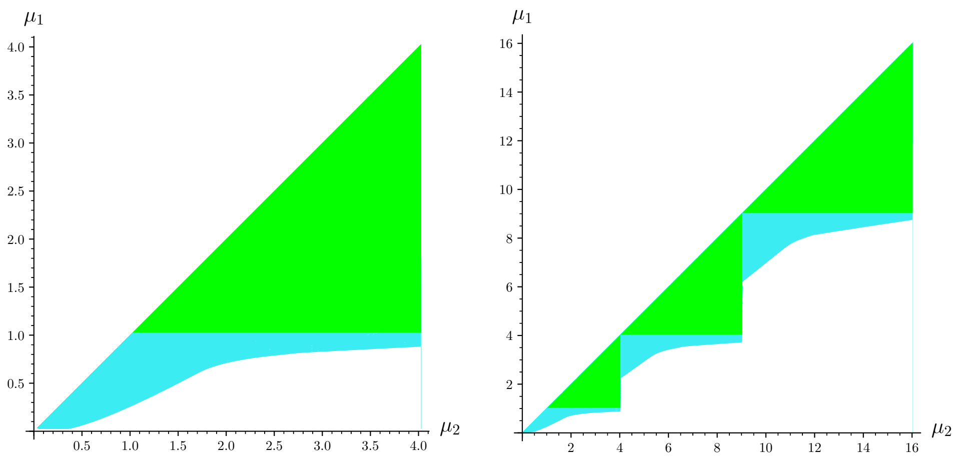

Figure 2: The two figures represent the region where Theorem 3.3 applies. The figure on the left illustrates the case $N = 1$, while the one on the right includes the three choices $N = 1, 2, 3$. Notice that we have highlighted in green the regions where Corollary 3.8 applies.

and

$$\widetilde{\mathcal{F}}_{+}(\gamma) > e^{\gamma T/2(N+1)},$$

then (c, μ_1, μ_2) *satisfies the Property* (P_{sym}).

Corollary 3.10. *Assume* c *and* μ_1^c *to be positive. If there exist a positive integer* N *and* $\gamma > 0$ *such that* (16) *holds and*

$$\frac{4\mu_1^c + c_\gamma^2}{4\mu_2^c + c_\gamma^2} \geq \exp\left(\frac{\gamma T}{N+1} + \max\left\{c_\gamma \frac{\pi}{\sqrt{\mu_1^c}}\,,\, c_\gamma \frac{\pi}{\sqrt{\mu_2^c}}\right\}\right),$$

then (c, μ_1, μ_2) *satisfies the Property* (P_{sym}).

Corollary 3.11. *Assume* c *and* μ_1^c *to be positive and that there exists a positive integer* N *satisfying* (16) *and, either*

$$\frac{\mu_2^c}{\mu_1^c} > \exp\left(c\,\frac{T}{2(N+1)}\right),$$

or

$$\frac{\mu_2}{\mu_1} > \exp\left(c \cdot \min\left\{\frac{\pi}{\sqrt{\mu_2^c}}\,,\, \frac{T}{N+1} - \frac{\pi}{\sqrt{\mu_1^c}}\right\}\right).$$

then (c, μ_1, μ_2) *satisfies the Property* (P_{sym}).

4 A functional analytic approach

We will now follow a different approach involving some norm estimates for normal operators in Hilbert spaces.

Let $H = L^2(0,T)$. We define the unbounded linear operator $\mathcal{L} : D(\mathcal{L}) \subseteq H \to H$ as follows:

$$D(\mathcal{L}) = \left\{ x \in W^{2,2}(0,T) : x(0) = x(T), x'(0) = x'(T) \right\} ,$$

and

$$\mathcal{L}x = -x'' - cx' .$$

The operator $\mathcal{L}$ is normal (see, e.g., [10, Lemma 5]), and its spectrum is made of isolated eigenvalues, precisely

$$\sigma(\mathcal{L}) = \left\{ \left(\frac{2\pi n}{T}\right)^2 \pm ic\frac{2\pi n}{T} : n \in \mathbb{N} \right\} .$$

The following statement is well known; nevertheless we provide its proof, for the reader's convenience.

Theorem 4.1. *If* $0 < \mu_1 \leq \mu_2$ *are such that*

$$\frac{\mu_2 - \mu_1}{2} < \operatorname{dist}\left(\frac{\mu_1 + \mu_2}{2}, \sigma(\mathcal{L})\right) , \tag{17}$$

then (c, μ_1, μ_2) *satisfies the Property* (P_{sym}).

Proof. If $\lambda \in \mathbb{R} \setminus \sigma(\mathcal{L})$ we can consider the operator $(\mathcal{L} - \lambda I)^{-1} : H \to H$, which is normal, as well. Hence,

$$\|(\mathcal{L} - \lambda I)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(\mathcal{L}))} .$$

Let $\mathcal{N} : H \to H$ be defined as

$$(\mathcal{N}x)(t) = \mu(t)x(t) .$$

Then for every $\lambda \in \mathbb{R} \setminus \sigma(\mathcal{L})$ problem (12) can be rewritten as a fixed point problem, i.e.,

$$\mathcal{L}x = \mathcal{N}x \quad \Leftrightarrow \quad x = \mathcal{T}_\lambda x ,$$

where $\mathcal{T}_\lambda : H \to H$ is defined by

$$\mathcal{T}_\lambda = (\mathcal{L} - \lambda I)^{-1}(\mathcal{N} - \lambda I) .$$

Then,

$$\|\mathcal{T}_\lambda\| \leq \|(\mathcal{L} - \lambda I)^{-1}\| \, \|\mathcal{N} - \lambda I\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(\mathcal{L}))} \|\mathcal{N} - \lambda I\| .$$

Since

$$[(\mathcal{N} - \lambda I)x](t) = (\mu(t) - \lambda)x(t) ,$$

we see that $\mathcal{N} - \lambda I$ is selfadjoint and

$$\|\mathcal{N} - \lambda I\| = \sup_{\|x\|=1} |\langle(\mathcal{N} - \lambda I)x, x\rangle| = \sup_{\|x\|=1} \left| \int_0^T (\mu(t) - \lambda)x(t)^2 \, dt \right| .$$

Let us suppose that $\mu \in L^\infty(0,T)$ satisfies

$$\mu_1 \le \mu(t) \le \mu_2 \,, \qquad \text{for a.e. } t \in [0,T] \,.$$

Then, taking $\lambda = (\mu_1 + \mu_2)/2$, by (17) we have that

$$\begin{aligned} \left\| \mathcal{N} - \frac{\mu_1 + \mu_2}{2} I \right\| &= \sup_{\|x\|=1} \left| \int_0^T \left(\mu(t) - \frac{\mu_1 + \mu_2}{2} \right) x(t)^2 \, dt \right| \\ &\le \frac{\mu_2 - \mu_1}{2} < \text{dist} \left(\frac{\mu_1 + \mu_2}{2}, \sigma(\mathcal{L}) \right) . \end{aligned}$$

Hence, with this choice of λ, the operator $\mathcal{T}_\lambda$ is a contraction, so that problem (12) only has the trivial solution. □

As a first consequence of Theorem 4.1, we have the following.

Corollary 4.2. *If $0 < \mu_1 \le \mu_2$ are such that, either $\mu_2 \le c^2$, or*

$$\mu_2 > c^2 \quad \text{and} \quad \mu_1 + \mu_2 > \frac{1}{2c^2}(\mu_2 - \mu_1)^2 + \frac{c^2}{2} \,, \tag{18}$$

then (c, μ_1, μ_2) satisfies the Property (P_{sym}).

Proof. After noticing that $\sigma(\mathcal{L})$, as a subset of $\mathbb{R}^2$, is contained in the curve

$$C = \left\{ (x,y) : x = \frac{1}{c^2} y^2 \right\} ,$$

it will be useful to compute the distance from C of a given point $(\mu, 0)$, with $\mu > 0$. Since

$$\left[d((\mu,0),(x, c\sqrt{x})) \right]^2 = x^2 + (c^2 - 2\mu)x + \mu^2,$$

taking into account that $x \ge 0$, an elementary argument shows that

$$\text{dist}((\mu,0),C) = \begin{cases} \mu & \text{if } \mu \le c^2/2 \,, \\ c\sqrt{\mu - c^2/4} & \text{if } \mu > c^2/2 \,. \end{cases}$$

Hence, condition (17) will surely hold if either $\mu_1 + \mu_2 \le c^2$, or

$$\mu_1 + \mu_2 > c^2 \quad \text{and} \quad \frac{\mu_2 - \mu_1}{2} < c \sqrt{\frac{\mu_1 + \mu_2}{2} - \frac{c^2}{4}} \,,$$

i.e.,

$$\mu_1+\mu_2>c^2 \quad \text{and} \quad \mu_1+\mu_2>\frac{1}{2c^2}(\mu_2-\mu_1)^2+\frac{c^2}{2}\,.$$

If we replace the last inequality with an equality, we get a parabola which is tangent to the horizontal axis at the point $(c^2,0)$. This is why we can replace the conditions $\mu_1+\mu_2\le c^2$ and $\mu_1+\mu_2>c^2$ by the simpler ones $\mu_2\le c^2$ and $\mu_2>c^2$, respectively, thus obtaining exactly our assumption (18). Theorem 4.1 then applies, to complete the proof. □

Remark 4.3. It could be interesting to observe that the conditions in the above corollary are independent of the period T.

It is rather surprising that, as visualized in Figure 3, Theorem 3.3 and Theorem 4.1 are independent of each other. In Figure 3 we see that the regions in the plane (μ_2,μ_1) where Theorem 3.3 and Theorem 4.1 apply are not contained one in the other. To illustrate this fact, for simplicity let us compare Corollary 3.5 with Theorem 4.1, providing two examples where one applies and the other does not.

Example 1. If $T=2\pi$, $c=1$, $\mu_1=72$ and $\mu_2=98$, then $\operatorname{dist}(85,\sigma(\mathcal{L}))\approx 9.8<13$, so (17) does not hold and we cannot apply Theorem 4.1. However, we can apply Corollary 3.5 since $\mu_2/\mu_1\approx 1.36$ and $\exp\big(c\pi/\sqrt{\mu_2^c}\big)\approx 1.37$.

Example 2. If $T=2\pi$, $c=1$, $\mu_1=0.01$ and $\mu_2=1.1$, then $\mu_2/\mu_1=110>\exp(\pi/\sqrt{0.85})\approx 30.2$, so that the assumption in Corollary 3.5 is not fulfilled. Conversely, Corollary 4.2 applies since $\mu_2>c^2$ and

$$1.11=\mu_1+\mu_2>\frac{1}{2c^2}(\mu_2-\mu_1)^2+\frac{c^2}{2}\approx 1.09\,.$$

Theorem 4.1 will now permit us to improve Corollary 3.8, where the main assumption (15) was that, for some positive integer N,

$$\left(\frac{2\pi N}{T}\right)^2\le \mu_1-\frac{c^2}{4}\le \mu_2-\frac{c^2}{4}\le \left(\frac{2\pi(N+1)}{T}\right)^2.$$

Indeed, as announced in the Introduction, in the following corollary we provide a more general sufficient condition which seems to be new in the literature.

Corollary 4.4. *If $c>0$ and there exist $\chi\in[0,c^2/2]$ and a positive integer N for which*

$$\left(\frac{2\pi N}{T}\right)^2\le \mu_1-\chi\le \mu_2-\chi\le \left(\frac{2\pi(N+1)}{T}\right)^2,$$

then (c,μ_1,μ_2) satisfies the Property (P_{sym}).

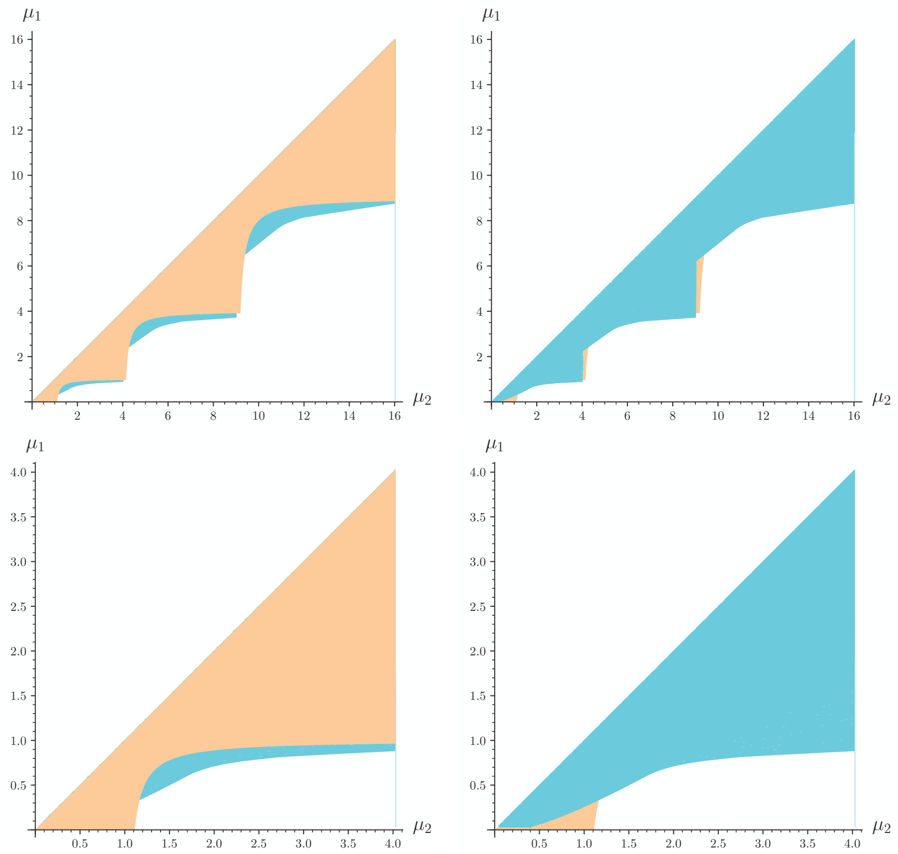

Figure 3: A comparison of the regions where Theorem 3.3 (light blue) and Theorem 4.1 (orange) apply. Here $T = 2\pi$, $c = \frac{1}{3}$ and $N = 1, 2, 3$ for the two first figures, while $N = 1$ for the last ones. The figures on the left show the orange region above the blue one, while the ones on the right show the blue region above the orange one.

Proof. By the change of variable $y(t) = v((2\pi/T)t)$, we can assume $T = 2\pi$ and that there exist $\chi \in [0, c^2/2]$ and a positive integer N for which

$$N^2 \leq \mu_1 - \chi \leq \mu_2 - \chi \leq (N+1)^2 .$$

Let

$$\alpha = \frac{\mu_1 + \mu_2}{2} - \chi .$$

We consider two cases.

First case: $N^2 \leq \alpha \leq N^2 + N + \frac{1}{2}$.

We want to show that, for every integer $\ell \geq 0$,

$$\left|\alpha + \chi - (N+\ell)^2 \pm ic(N+\ell)\right| > \alpha - N^2 , \tag{19}$$

from which we easily deduce that

$$\operatorname{dist}\left(\frac{\mu_1+\mu_2}{2},\sigma(\mathcal{L})\right)=\operatorname{dist}(\alpha+\chi,\sigma(\mathcal{L}))>\alpha-N^2>\frac{\mu_2-\mu_1}{2}\,.$$

The claim (19) is equivalent to

$$\Big(\alpha+\chi-(N+\ell)^2\Big)^2+c^2(N+\ell)^2>(\alpha-N^2)^2,$$

i.e., to

$$\chi^2+2\chi(\alpha-(N+\ell)^2)+((N+\ell)^2-\alpha)^2+c^2(N+\ell)^2>(\alpha-N^2)^2.$$

Since $\chi\in[0,c^2/2]$, we see that

$$\begin{aligned}2\chi(\alpha-(N+\ell)^2)+c^2(N+\ell)^2&=2\chi\alpha+(c^2-2\chi)(N+\ell)^2\\&\geq 2\chi N^2+(c^2-2\chi)N^2=c^2N^2>0\,,\end{aligned}\tag{20}$$

It now suffices to prove that

$$\chi^2+((N+\ell)^2-\alpha)^2\geq(\alpha-N^2)^2.$$

The case $\ell=0$ clearly holds true. If $\ell\geq1$ we have

$$\chi^2+((N+\ell)^2-\alpha)^2\geq((N+1)^2-\alpha)^2\geq(\alpha-N^2)^2\,,$$

and we are done.

Second case: $N^2+N+\frac{1}{2}\leq\alpha\leq(N+1)^2$.

We want to show that, for every integer $\ell\geq0$,

$$\left|\alpha+\chi-(N+\ell)^2\pm ic(N+\ell)\right|>(N+1)^2-\alpha\,,\tag{21}$$

from which we deduce easily

$$\operatorname{dist}(\alpha+\chi,\sigma(\mathcal{L}))>(N+1)^2-\alpha>\frac{\mu_2-\mu_1}{2}\,.$$

If $\ell=0$, since

$$(\alpha+\chi-N^2)^2+c^2N^2>(\alpha+\chi-N^2)^2\geq(\alpha-N^2)^2\geq((N+1)^2-\alpha)^2\,,$$

we easily conclude.

Let now $\ell\geq1$. The claim (21) is equivalent to

$$\chi^2+2\chi(\alpha-(N+\ell)^2)+((N+\ell)^2-\alpha)^2+c^2(N+\ell)^2>((N+1)^2-\alpha)^2.$$

Using (20), it suffices to prove that

$$\chi^2+((N+\ell)^2-\alpha)^2\geq((N+1)^2-\alpha)^2\,,$$

which holds true since

$$(N+\ell)^2-\alpha\geq(N+1)^2-\alpha>0\,.$$

The proof is completed. □

5 A counterexample

We will now show that the condition (14) in Corollary 3.6 is, in some cases, the best possible.

Theorem 5.1. *For every $c > 0$ there exist $\mu_1 < \mu_2$ such that $\mu_1^c > 0$ and (c, μ_1, μ_2) does not satisfy the Property (P_{sym}). Moreover, if there exists a positive integer N for which*

$$0 < c \leq \frac{4N}{T} \ln\left(1 + \frac{2}{N}\right), \tag{22}$$

then μ_1, μ_2 can be chosen in such a way that

$$\mu_2^c \leq \left(\frac{2\pi(N+1)}{T}\right)^2, \qquad \frac{\mu_2^c}{\mu_1^c} = \exp\left(c\,\frac{T}{2N}\right). \tag{23}$$

Proof. Fix any positive integer N and define the positive constants

$$\mu_1^c = \left[\frac{\pi}{\omega}\left(1 + e^{-\omega c/4}\right)\right]^2, \qquad \mu_2^c = \left[\frac{\pi}{\omega}\left(1 + e^{\omega c/4}\right)\right]^2,$$

where $\omega = T/N$. Then,

$$\frac{1}{\sqrt{\mu_1^c}} + \frac{1}{\sqrt{\mu_2^c}} = \frac{\omega}{\pi},$$

and

$$\frac{\mu_2^c}{\mu_1^c} = e^{\omega c/2}.$$

Now, set

$$S_1 = \frac{\pi}{2\sqrt{\mu_1^c}}, \quad S_2 = \frac{\pi}{2\sqrt{\mu_1^c}} + \frac{\pi}{2\sqrt{\mu_2^c}} = \frac{\omega}{2}, \quad S_3 = \frac{\pi}{\sqrt{\mu_1^c}} + \frac{\pi}{2\sqrt{\mu_2^c}},$$

and

$$\mu^c(t) = \begin{cases} \mu_1^c & \text{if } t \in [0, S_1[\, \cup [S_2, S_3[\,, \\ \mu_2^c & \text{if } t \in [S_1, S_2[\, \cup [S_3, \omega[\,. \end{cases}$$

Let $v_o : [0, \omega] \to \mathbb{R}$ be the function defined as

$$v_o(t) = \begin{cases} \dfrac{1}{\sqrt{\mu_1^c}} \sin(\sqrt{\mu_1^c}\,t) & \text{if } t \in [0, S_1[\,, \\ \dfrac{1}{\sqrt{\mu_1^c}} \cos(\sqrt{\mu_2^c}\,(t - S_1)) & \text{if } t \in [S_1, S_2[\,, \\ -\dfrac{\sqrt{\mu_2^c}}{\mu_1^c} \sin(\sqrt{\mu_1^c}\,(t - S_2)) & \text{if } t \in [S_2, S_3[\,, \\ -\dfrac{\sqrt{\mu_2^c}}{\mu_1^c} \cos(\sqrt{\mu_2^c}\,(t - S_3)) & \text{if } t \in [S_3, \omega]\,, \end{cases}$$

It is a solution of

$$\begin{cases} v'' + \mu^c(t)v = 0\,, \\ v(0) = v(\omega) = 0, \quad v'(0) = 1, \quad v'(\omega) = e^{\omega c/2}\,. \end{cases}$$

Finally, let $x_o : [0, \omega] \to \mathbb{R}$ be defined as

$$x_o(t) = e^{-ct/2} v_o(t)\,.$$

Then x_o shares the same regularity of v_o and the same sign, as well, and

$$x_o(0) = x_o(\omega) = 0\,, \quad x_o{}'(0) = x_o{}'(\omega) = 1\,.$$

Now, if we choose $\mu_1 = \mu_1^c + c^2/4$, $\mu_2 = \mu_2^c + c^2/4$, and

$$\mu(t) = \mu^c(t) + \frac{c^2}{4}\,,$$

extending both $\mu(t)$ and $x_o(t)$ by $\frac{T}{N}$-periodicity over the interval $[0, T]$ we have that $\mu \in L^\infty(0, T)$ and x_o is a nontrivial solution of

$$\begin{cases} x'' + cx' + \mu(t)x = 0 \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,. \end{cases}$$

We have thus proved that (c, μ_1, μ_2) does not satisfy the Property (P_{sym}). To conclude the proof observe that, if (22) holds, then the inequality in (23) also holds. □

6 A generalization of Theorem 1.1

In this section we will use the notation $\bar{e} = \frac{1}{T}\int_0^T e(t)\,dt$ and $E(t) = \int_0^t e(s)\,ds$.

The following result generalizes Theorem 1.1 stated in the Introduction.

Theorem 6.1. *Assume that there exist three positive constants d, a_1, a_2, with $a_1 < a_2$, and a function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$\begin{aligned} 0 < a_1 g(x) \le f(t, x) \le a_2 g(x) \qquad & \text{if } x \ge d\,, \\ a_2 g(x) \le f(t, x) \le a_1 g(x) < 0 \quad & \text{if } x \le -d\,. \end{aligned} \tag{24}$$

Let

$$M_d = \max\{|f(t, x)| \,:\, t \in [0, T]\,, |x| \le d\}\,,$$

and

$$K = \left(\sqrt{\frac{a_2}{a_1}} - 1\right)^{-1}. \tag{25}$$

If $\bar{e} = 0$ and

$$cd + \|E\|_\infty + \sqrt{(cd + \|E\|_\infty)^2 + 8dM_d K} < K(cd - \|E\|_\infty)\,, \tag{26}$$

then problem (1) *has a solution.*

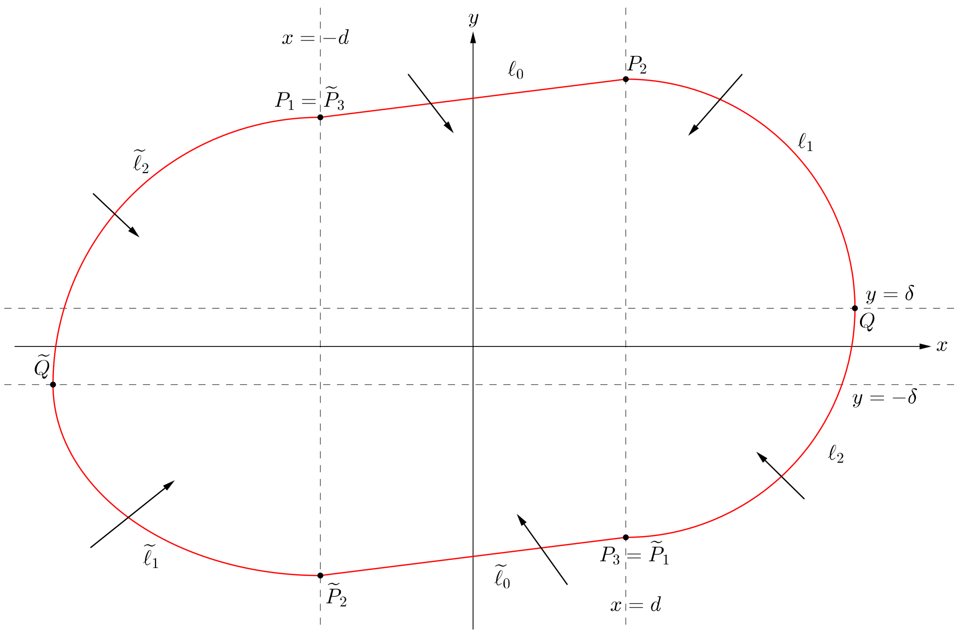

Figure 4: The invariant set.

Proof. We write the equivalent system

$$x' = y - cx + E(t)\,, \qquad y' = -f(t,x)\,, \tag{27}$$

and follow the lines of [7], constructing a positively invariant compact set containing the origin. This set, depicited in Figure 4, is delimited by some piecewise smooth simple closed curves, as described below.

Let δ be the positive solution of the equation

$$K\delta^2 - (cd + \|E\|_\infty)\delta - 2dM_d = 0\,,$$

i.e.,

$$\delta = \frac{cd + \|E\|_\infty + \sqrt{(cd + \|E\|_\infty)^2 + 8dM_dK}}{2K}\,.$$

We set

$$y_1 = K\delta = cd + \|E\|_\infty + 2dM_d/\delta\,. \tag{28}$$

By (26), we have

$$2\delta < cd - \|E\|_\infty\,. \tag{29}$$

We start from a point $P_1 = (-d, y_1)$. Define the point $P_2 = (d, y_1 + \delta)$ and let ℓ_0 be the segment joining P_1 and P_2. Now let $G(x) = \int_d^x g(\xi)\,d\xi$ and consider the two functions

$$V_1(x,y) = \frac{1}{2}(y-\delta)^2 + a_1 G(x)\,,$$
$$V_2(x,y) = \frac{1}{2}(y-\delta)^2 + a_2 G(x)\,,$$

We follow the curve $V_1(x,y) = V_1(P_2)$ starting from P_2 until we reach the line $y = \delta$ at a point $Q = (x_Q, \delta)$ and then the curve $V_2(x,y) = V_2(Q)$ until we reach $x = d$ at some point $P_3 = (d, y_3)$ with $y_3 < \delta$. Denote by ℓ_1 and ℓ_2 these pieces of curves. It has to be

$$V_1(P_2) = V_1(Q) = \frac{a_1}{a_2} V_2(Q) = \frac{a_1}{a_2} V_2(P_3)\,,$$

hence

$$(y_3 - \delta)^2 = \frac{a_2}{a_1} y_1^2 = \frac{a_2}{a_1} K^2 \delta^2 > \delta^2\,,$$

where the last inequality follows by (25). Hence $y_3 < 0$, and we deduce that

$$\sqrt{\frac{a_2}{a_1}}\; y_1 = |y_3 - \delta| = \delta - y_3\,,$$

and so, by (28),

$$|y_3| = \sqrt{\frac{a_2}{a_1}} y_1 - \delta = y_1 \left(\sqrt{\frac{a_2}{a_1}} - K^{-1} \right) = y_1\,.$$

We can now proceed symmetrically and define $\widetilde{P}_2 = (-d, y_3 - \delta)$ and let $\widetilde{\ell}_0$ be the segment joining $\widetilde{P}_1 = P_3$ and $\widetilde{P}_2$. Now let $\widetilde{G}(x) = \int_{-d}^{x} g(\xi)\, d\xi$ and consider the two functions

$$\widetilde{V}_1(x,y) = \frac{1}{2}(y+\delta)^2 + a_1 \widetilde{G}(x)\,,$$
$$\widetilde{V}_2(x,y) = \frac{1}{2}(y+\delta)^2 + a_2 \widetilde{G}(x)\,,$$

We follow the curve $\widetilde{V}_1(x,y) = \widetilde{V}_1(\widetilde{P}_2)$ starting from $\widetilde{P}_2$ until we reach the line $y = -\delta$ at a point $\widetilde{Q} = (x_{\widetilde{Q}}, -\delta)$ and then the curve $\widetilde{V}_2(x,y) = \widetilde{V}_2(\widetilde{Q})$ until we reach $x = -d$ at some point $\widetilde{P}_3 = (-d, \widetilde{y}_3)$. Denote by $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ these pieces of curves. Arguing as above we can prove that $\widetilde{y}_3 = |y_3|$, and so $\widetilde{y}_3 = y_1$. Hence $\widetilde{P}_3$ coincides with P_1.

We want to prove now that the compact region Ω bounded by the lines

$$\ell_0\,,\, \ell_1\,,\, \ell_2\,,\, \tilde{\ell}_0\,,\, \tilde{\ell}_1\,,\, \tilde{\ell}_2$$

is strictly positively invariant. Let

$$\Phi(t,x,y) = (y - cx + E(t), -f(t,x))$$

be the field associated with (27).

- On ℓ_0, if we denote with $\nu_1 = (\delta, -2d)$ the *inner* normal to ℓ_1, and recalling that $-d \le x \le d$, we obtain

  $$\begin{aligned}\Phi(t,x,y) \cdot \nu_1 &= \delta y - c\delta x + \delta E(t) + 2df(t,x) \\ &> \delta\,(y_1 - cd - \|E\|_\infty - 2dM_d/\delta) = 0\,.\end{aligned}$$

  Notice that in the above computation the strict inequality $y - cx > y_1 - cd$ holds along ℓ_0.

- On ℓ_1, from (29) and recalling that $x \geq d$ and $y \geq \delta$ here, we have, along the trajectories of (27),

$$\begin{aligned}\frac{d}{dt}V_1(x(t),y(t)) &= \big(a_1 g(x(t)), y(t)-\delta\big)\cdot \Phi\big(t,x(t),y(t)\big)\\ &= \big(y(t)-\delta\big)\big(a_1 g(x(t)) - f(t,x(t))\big)\\ &\qquad + a_1 g(x(t))\big(E(t) - cx(t) + \delta\big)\\ &\leq a_1 g(x(t))\big(\|E\|_\infty - cd + \delta\big) < 0\,.\end{aligned}$$

- On ℓ_2 the same reasoning holds considering that now we are dealing with V_2.
- A similar argument can be performed for $\tilde{\ell}_0$, $\tilde{\ell}_1$, and $\tilde{\ell}_2$ considering that now we are dealing with $\widetilde{V}_1$ and $\widetilde{V}_2$, respectively.

The above construction shows that the compact set Ω is strictly positively invariant.

If we assume the uniqueness of the solutions to initial value problem associated with (27), then the conclusion directly follows from Brouwer's fixed point theorem. Otherwise we can uniformly approximate the function $f(t,x)$ on $[0,T]\times[x_{\widetilde{Q}}, x_Q]$ by a sequence of continuous functions $f_n(t,x)$ which are locally Lipschitz continuous in x. The *strict* positive invariance of Ω persists when n is large enough if f is replaced by f_n. A standard compactness argument allows us to complete the proof. □

Remark 6.2. The result of Theorem 1.1 can be recovered as a consequence of Theorem 6.1. Indeed, there is no loss of generality in assuming $\bar{e} = 0$, simply taking $g(x) - \bar{e}$ instead of $g(x)$. Then, choosing $f(t,x) = g(x)$, $d > \|E\|_\infty / c$, $a_1 = 1-\varepsilon$ and $a_2 = 1+\varepsilon$, with $\varepsilon > 0$ sufficiently small, all the assumptions of Theorem 6.1 are easily verified.

Here is a consequence in the case when $f(t,x)$ has a linear growth in x.

Corollary 6.3. *Assume that there exist two positive constants μ_1, μ_2 such that*

$$\mu_1 \leq \liminf_{|x|\to\infty} \frac{f(t,x)}{x} \leq \limsup_{|x|\to\infty} \frac{f(t,x)}{x} \leq \mu_2\,.$$

If

$$\mu_1 > \mu_2 \left(\frac{2c^2 + 8\mu_2}{3c^2 + 8\mu_2}\right)^2,$$

then problem (1) *has a solution.*

Proof. Without loss of generality we can assume that $\bar{e} = 0$, simply replacing $f(t,x)$ by $f(t,x) - \bar{e}$. Moreover, we can find $\varepsilon \in]0, a_1[$ such that, setting

$$a_1 = \mu_1 - \varepsilon\,, \qquad a_2 = \mu_2 + \varepsilon\,,$$

one has

$$a_1 > a_2 \left(\frac{2c^2 + 8a_2}{3c^2 + 8a_2} \right)^2 , \tag{30}$$

and, for $d > 0$ sufficiently large, (24) holds with $g(x) = x$ and

$$M_d = \max\{ |f(t,x)| \,:\, t \in [0,T]\,, |x| \le d\} \le a_2 d\,. \tag{31}$$

Recalling (25), after some rearrangements we can show that (30) is equivalent to

$$1 + \sqrt{1 + 8a_2 K/c^2} < K\,.$$

Then, there is $\eta_0 > 0$ such that if $|\eta| < \eta_0$ we get

$$1 + \eta + \sqrt{(1+\eta)^2 + 8a_2 K/c^2} < K(1-\eta)\,,$$

So, for d large enough we have

$$1 + \frac{\|E\|_\infty}{cd} + \sqrt{\left(1 + \frac{\|E\|_\infty}{cd}\right)^2 + 8a_2 K/c^2} < K\left(1 - \frac{\|E\|_\infty}{cd}\right),$$

leading to

$$cd + \|E\|_\infty + \sqrt{(cd + \|E\|_\infty)^2 + 8a_2 d^2 K} < K(cd - \|E\|_\infty)\,,$$

which implies (26), in view of (31). Theorem 6.1 then applies to complete the proof. □

We now consider the case $f(t,x) = \mu(t)x$, and obtain a new sufficient condition in order to have the Property (P_{sym}).

Corollary 6.4. *Assume* $0 < \mu_1 \le \mu_2$ *and* $c > 0$ *are such that*

$$\mu_1 > \mu_2 \left(\frac{2c^2 + 8\mu_2}{3c^2 + 8\mu_2} \right)^2 . \tag{32}$$

Then (c, μ_1, μ_2) *satisfies the Property* (P_{sym}).

Proof. By Corollary 6.3, the problem

$$\begin{cases} x'' + cx' + \mu(t)x = e(t)\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,. \end{cases}$$

has a solution for every $e \in L^2(0,T)$. The conclusion then follows from the Fredholm alternative. □

Remark 6.5. It could be surprising that in condition (32) there is no dependence on the period T. However, in the proof of Theorem 6.1 we have discovered the existence of a positively invariant set, and this fact guarantees that the condition will apply to any possible period T.

Remark 6.6. Corollary 6.4 could be deduced from Corollary 4.2. Indeed the function $p : [0, +\infty[\to \mathbb{R}$ defined as

$$p(x) = x \left(\frac{2c^2 + 8x}{3c^2 + 8x} \right)^2 ,$$

appearing in (32), is strictly convex and $\lim_{x\to+\infty}(p(x) - x) = -c^2/4$. Hence $p(x) > x - c^2/4$ for every $x \geq 0$. Since the parabola $x + y = \frac{1}{2c^2}(y - x)^2 + c^2/2$, appearing in (18), is symmetric with respect to the line $y = x$ and contains the point $(c^2, 0)$ one can deduce that the region where Corollary 4.2 can be applied contains the region where the hypotheses of Corollary 6.4 are fulfilled.

7 Proof of Theorems 2.3 and 2.10

Let us first prove Theorem 2.3. To this aim, we start providing some estimates on the number of clockwise rotations around the origin of a solution of (7) in the phase plane.

Lemma 7.1. *Let N be a nonnegative integer satisfying* (8)*. Then, for any nontrivial solution v of* (7)*, the curve $t \mapsto (v(t), v'(t))$ makes at most N clockwise rotations around the origin in the time interval* $[0, T]$.

Proof. Let $\alpha \geq 0$ and $\beta \geq 0$ be such that $\alpha + \beta = c$. We write the differential equation in (7) as a system

$$v' = w - \alpha v\,, \qquad -w' = \beta w + (\mu(t) - \alpha\beta)v^+ - (\nu(t) - \alpha\beta)v^- .$$

and fix any nontrivial solution (v, w). By uniqueness of the solutions of Cauchy problems, it has to be $(v(t), w(t)) \neq (0, 0)$ for every $t \in [0, T]$. Let $\mathscr{N}$ be the integer number of clockwise rotations of (v, w) around the origin in the time interval $[0, T]$. We introduce the modified polar coordinates

$$v(t) = \begin{cases} \delta_+ \rho(t) \cos\theta(t) & \text{if } \theta(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right[\,, \\ \delta_- \rho(t) \cos\theta(t) & \text{if } \theta(t) \in \left]-\frac{3\pi}{2}, -\frac{\pi}{2}\right[\,, \end{cases} \qquad w(t) = \rho(t) \sin\theta(t)\,,$$

for some $\delta_+, \delta_- > 0$.

We now focus our attention on the case when $\theta(t) \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. A simple computation provides

$$\begin{cases} \delta_+ \rho' \cos\theta - \delta_+ \rho\theta' \sin\theta = \rho \sin\theta - \delta_+ \rho\alpha \cos\theta \\ -\rho' \sin\theta - \rho\theta' \cos\theta = \rho\beta \sin\theta + \delta_+ \rho(\mu(t) - \alpha\beta) \cos\theta\,. \end{cases}$$

Hence,

$$-\theta' = \frac{1}{\delta_+} \sin^2\theta + \frac{\beta - \alpha}{2} \sin(2\theta) + \delta_+(\mu(t) - \alpha\beta) \cos^2\theta\,.$$

The particular choice $\alpha = \beta$ leads us to

$$-\theta'(t) = \frac{1}{\delta_+} \sin^2 \theta(t) + \delta_+ \left(\mu(t) - \frac{c^2}{4} \right) \cos^2 \theta(t) \,.$$

Then, using the notation in (6),

$$\sin^2 \theta(t) + \delta_+^2 \mu_1^c \cos^2 \theta(t) \leq -\delta_+ \theta'(t) \leq \sin^2 \theta(t) + \delta_+^2 \mu_2^c \cos^2 \theta(t) \,. \qquad (33)$$

Analogously, when $\theta(t) \in \left]-\frac{3\pi}{2}, -\frac{\pi}{2}\right[$ we get

$$\sin^2 \theta(t) + \delta_-^2 \nu_1^c \cos^2 \theta(t) \leq -\delta_- \theta'(t) \leq \sin^2 \theta(t) + \delta_-^2 \nu_2^c \cos^2 \theta(t) \,. \qquad (34)$$

In particular we have $-\theta'(t) > 0$ for every t. We now provide an estimate of the time needed by the solution in order to complete a rotation around the origin. Let $t_1 < t_2 < t_3$ be such that

$$\theta(t_1) = \frac{\pi}{2} \,, \quad \theta(t_2) = -\frac{\pi}{2} \,, \quad \theta(t_3) = -\frac{3\pi}{2} \,,$$

and

$$\begin{aligned} &\theta(t) \in \left]-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right[\quad \text{for every } t \in]t_1, t_2[\,, \\ &\theta(t) \in \left]-\tfrac{3\pi}{2}, -\tfrac{\pi}{2}\right[\quad \text{for every } t \in]t_2, t_3[\,. \end{aligned}$$

Taking $\delta_+ = 1/\sqrt{\mu_2^*}$, integrating in (33) on the interval $[t_1, t_2]$, we deduce

$$t_2 - t_1 \geq \frac{\pi}{\sqrt{\mu_2^c}} \,. \qquad (35)$$

On the other hand, taking $\delta_- = 1/\sqrt{\nu_2^*}$, from (34) we deduce

$$t_3 - t_2 \geq \frac{\pi}{\sqrt{\nu_2^c}} \,. \qquad (36)$$

Hence, summing up, the time $t_3 - t_1$ needed by the solution to perform a complete rotation in the plane satisfies

$$t_3 - t_1 \geq \frac{\pi}{\sqrt{\mu_2^c}} + \frac{\pi}{\sqrt{\nu_2^c}} \,.$$

We have thus proved that $\mathscr{N} \leq N + 1$.

By contradiction, assume that $\mathscr{N} = N + 1$. Then equalities must hold in (35) and (36). Moreover, one has $\mu(t) = \mu_2$ for almost every $t \in [t_1, t_2]$ while $\nu(t) = \nu_2$ for almost every $t \in [t_2, t_3]$. Consequently, v solves

$$\begin{cases} v'' + cv' + \mu_2 v^+ - \nu_2 v^- = 0 \\ v(0) = v(T) \,, \quad v'(0) = v'(T) \,. \end{cases}$$

Since $c > 0$, multiplying in the equation by v' and integrating over $[0, T]$, one easily proves that $v \equiv 0$, a contradiction.

Hence $\mathscr{N} \leq N$, thus concluding the proof. $\square$

Let us now introduce a notion related to the Property (P).

Definition 7.2. *Given $\gamma \in \mathbb{R}$, we say that the row $(\hat{c}, \hat{\mu}_1, \hat{\mu}_2, \hat{\nu}_1, \hat{\nu}_2)$ satisfies the Property (P_γ) if, for every pair of functions $\hat{\mu}, \hat{\nu}$ in $L^\infty(0,T)$ such that*

$$\hat{\mu}_1 \leq \hat{\mu}(t) \leq \hat{\mu}_2 \,, \quad \hat{\nu}_1 \leq \hat{\nu}(t) \leq \hat{\nu}_2 \,, \quad \text{for a.e. } t \in [0,T] \,, \tag{37}$$

the problem

$$\begin{cases} z'' + \hat{c}z' + \hat{\mu}(t)z^+ - \hat{\nu}(t)z^- = 0 \\ z(0) = z(T) = 0 \,, \quad z'(0) = 1 \,, \quad z'(T) = e^{\gamma T} \end{cases} \tag{38}$$

has no solution.

The following lemma relating the Properties (P) and (P_γ) will be useful.

Lemma 7.3. *Given the five-number row $(c, \mu_1, \mu_2, \nu_1, \nu_2)$, recall the notation $c_\gamma = c - 2\gamma$ and set*

$$\mu_{j,\gamma} = \mu_j - c\gamma + \gamma^2 \,, \quad \nu_{j,\gamma} = \nu_j - c\gamma + \gamma^2 \,, \quad j = 1, 2 \,.$$

Then, the following statements are equivalent.

(i) The row $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ satisfies the Property (P).

(ii) For every $\gamma \in \mathbb{R}$, the row $(c_\gamma, \mu_{1,\gamma}, \mu_{2,\gamma}, \nu_{1,\gamma}, \nu_{2,\gamma})$ satisfies the Property (P_γ).

(iii) There exists $\gamma \in \mathbb{R}$ for which $(c_\gamma, \mu_{1,\gamma}, \mu_{2,\gamma}, \nu_{1,\gamma}, \nu_{2,\gamma})$ satisfies the Property (P_γ).

Proof. $(i) \Rightarrow (ii)$ By contradiction, let $z(t)$ be a solution of (38), for some $\gamma \in \mathbb{R}$, $\hat{c} = c_\gamma$, and $\hat{\mu}(t)$, $\hat{\nu}(t)$ satisfying (37), with $\hat{\mu}_j = \mu_{j,\gamma}$, $\hat{\nu}_j = \nu_{j,\gamma}$, $j = 1, 2$. Setting $v(t) = e^{-\gamma t} z(t)$ we get a solution of (7) with

$$c = \hat{c} + 2\gamma \,, \quad \mu(t) = \hat{\mu}(t) + c\gamma - \gamma^2 \,, \quad \nu(t) = \hat{\nu}(t) + c\gamma - \gamma^2 \,,$$

and (5) holds, with $\mu_j = \hat{\mu}_j + c\gamma - \gamma^2$ and $\nu_j = \hat{\nu}_j + c\gamma - \gamma^2$, $j = 1, 2$.

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$ By contradiction, let $v(t)$ be a nontrivial solution of (7), for some $\mu(t)$, $\nu(t)$ satisfying (5). After extending it on $\mathbb{R}$ by T-periodicity, we can find a positive constant C^* and a $t^* \in [0,T]$ for which the function $w(t) = C^* v(t + t^*)$ solves

$$\begin{cases} w'' + cw' + \mu(t + t^*)w^+ - \nu(t + t^*)w^- = 0 \\ w(0) = w(T) = 0 \,, \quad w'(0) = w'(T) = 1 \,. \end{cases}$$

Then, $z(t) = e^{\gamma t} w(t)$ provides a solution of (38) with

$$\hat{c} = c - 2\gamma\,, \quad \hat{\mu}(t) = \mu(t+t^*) - c\gamma + \gamma^2\,, \quad \hat{\nu}(t) = \nu(t+t^*) - c\gamma + \gamma^2\,,$$

and (37) holds, with $\hat{\mu}_j = \mu_{j,\gamma}$ and $\hat{\nu}_j = \nu_{j,\gamma}$, $j = 1, 2$. □

We are now ready to carry on the proof of our main result.

Proof of Theorem 2.3. Assume by contradiction that $(c, \mu_1, \mu_2, \nu_1, \nu_2)$ does not satisfy the Property (P). Fix $\gamma > 0$ satisfying (9). Then by Lemma 7.3, there exist some functions $\hat{\mu}, \hat{\nu}$ in $L^\infty(0,T)$ such that

$$\mu_{1,\gamma} \le \hat{\mu}(t) \le \mu_{2,\gamma}\,, \quad \nu_{1,\gamma} \le \hat{\nu}(t) \le \nu_{2,\gamma}\,, \qquad \text{for a.e. } t \in [0,T]\,,$$

and the problem

$$\begin{cases} v'' + c_\gamma v' + \hat{\mu}(t) v^+ - \hat{\nu}(t) v^- = 0 \\ v(0) = v(T) = 0\,, \quad v'(0) = 1\,, \quad v'(T) = e^{\gamma T} \end{cases}$$

has a solution. We define

$$\mu(t) = \hat{\mu}(t) + c\gamma - \gamma^2\,, \qquad \nu(t) = \hat{\nu}(t) + c\gamma - \gamma^2\,,$$

and notice that

$$\mu_1 \le \mu(t) \le \mu_2\,, \quad \nu_1 \le \nu(t) \le \nu_2\,, \qquad \text{for a.e. } t \in [0,T]\,,$$

and

$$\hat{\mu}(t) - c_\gamma^2/4 = \mu(t) - c^2/4, \qquad \hat{\nu}(t) - c_\gamma^2/4 = \nu(t) - c^2/4.$$

We then write the equivalent system

$$\begin{cases} v' = w - (c_\gamma/2) v\,, \\ w' = -(c_\gamma/2) w - (\mu(t) - c^2/4) v^+ + (\nu(t) - c^2/4) v^-\,. \end{cases} \tag{39}$$

Now we introduce the modified polar coordinates

$$v = \delta \rho \cos\theta\,, \qquad w = \rho \sin\theta\,, \tag{40}$$

for some $\delta > 0$, and we first concentrate on the half-plane $v \ge 0$. Let

$$A_\gamma = \{(v,w) : v \ge 0\,,\, w \ge (c_\gamma/2) v\}\,, \quad B_\gamma = \{(v,w) : v \ge 0\,,\, w \le (c_\gamma/2) v\}\,.$$

We consider different polar coordinates (40) for the sets A_γ and B_γ choosing different values of δ as follows: we take $\delta = 1/\sqrt{\mu_1^c}$ on A_γ and $\delta = 1/\sqrt{\mu_2^c}$ on B_γ.

Now, let us consider a point on the half-line $w = \frac{c_\gamma}{2} v$ with $v \geq 0$. We denote by $(\rho[A_\gamma], \theta[A_\gamma])$ its polar coordinates as a point of the set A_γ. Similarly we denote by $(\rho[B_\gamma], \theta[B_\gamma])$ its polar coordinates as a point of the set B_γ. We can compute

$$\theta[A_\gamma] = \arctan \frac{c_\gamma}{2\sqrt{\mu_1^c}}\,, \quad \theta[B_\gamma] = \arctan \frac{c_\gamma}{2\sqrt{\mu_2^c}}\,, \tag{41}$$

while

$$\rho[B_\gamma] = \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2}}\, \rho[A_\gamma]\,. \tag{42}$$

We have the following estimates:

$$\begin{cases} \dfrac{\delta\rho'}{\rho} = [1 - \delta^2(\mu(t) - c^2/4)] \cos\theta \sin\theta - \delta c_\gamma/2\,, \\ -\delta\theta' = 1 - [1 - \delta^2(\mu(t) - c^2/4)] \cos^2\theta\,. \end{cases}$$

Hence,

$$\frac{1}{\rho}\frac{d\rho}{d\theta} = \frac{\delta c_\gamma/2 - [1 - \delta^2(\mu(t) - c^2/4)] \cos\theta \sin\theta}{1 - [1 - \delta^2(\mu(t) - c^2/4)] \cos^2\theta}\,. \tag{43}$$

It can be seen that the function

$$\phi_\theta(s) = \frac{\delta c_\gamma/2 - [1 - \delta^2 s] \cos\theta \sin\theta}{1 - [1 - \delta^2 s] \cos^2\theta}\,, \qquad s > 0\,,$$

has derivative of the same sign of $\sin\theta - (c_\gamma/2)\delta\cos\theta = \frac{1}{\rho}(w - (c_\gamma/2)v)$, and so, roughly speaking, it is increasing on A_γ and decreasing on B_γ. This fact permits us to deduce from (43) the following estimate

$$\frac{1}{\rho}\frac{d\rho}{d\theta} \geq \begin{cases} \dfrac{c_\gamma}{2\sqrt{\mu_1^c}} & \text{on } A_\gamma\,, \\ \dfrac{c_\gamma}{2\sqrt{\mu_2^c}} & \text{on } B_\gamma\,. \end{cases} \tag{44}$$

Let us now consider a solution performing a clockwise half-rotation around the origin in the half-plane $v \geq 0$ in a certain time interval. Assume that it starts from a point $Q_0 = (0, \rho(\pi/2))$, crosses the half-line $w = (c_\gamma/2)v$ at a point Q_1 and arrives at a point $Q_2 = (0, -\rho(-\pi/2))$. We denote as above with $(\rho[A_\gamma], \theta[A_\gamma])$ and $(\rho[B_\gamma], \theta[B_\gamma])$ the two possible variants of polar coordinates of the point Q_1.

Integrating in (44), we get

$$\rho[A_\gamma] \leq \rho\Big(\frac{\pi}{2}\Big) \exp\Big(- \frac{c_\gamma}{2\sqrt{\mu_1^c}}\Big(\frac{\pi}{2} - \theta[A_\gamma]\Big)\Big)\,,$$

$$\rho\Big(-\frac{\pi}{2}\Big) \le \rho[B_\gamma]\exp\bigg(-\frac{c_\gamma}{2\sqrt{\mu_2^c}}\Big(\frac{\pi}{2}+\theta[B_\gamma]\Big)\bigg),$$

hence, recalling (41) and (42),

$$\rho\Big(-\frac{\pi}{2}\Big) \le \rho\Big(\frac{\pi}{2}\Big)\mathcal{F}_+(\gamma),$$

where

$$\mathcal{F}_+(\gamma) = \sqrt{\frac{4\mu_2^c + c_\gamma^2}{4\mu_1^c + c_\gamma^2}}\,\exp\bigg(-\frac{c_\gamma}{2\sqrt{\mu_1^c}}\Big(\frac{\pi}{2}-\theta[A_\gamma]\Big) - \frac{c_\gamma}{2\sqrt{\mu_2^c}}\Big(\frac{\pi}{2}+\theta[B_\gamma]\Big)\bigg).$$

Analogously, we can consider a solution performing a half-rotation in the half-plane $v \le 0$. Setting

$$A_\gamma^\dagger = \{(v,w) : v \le 0\,,\, w \le (c_\gamma/2)v\}\,,\quad B_\gamma^\dagger = \{(v,w) : v \le 0\,,\, w \ge (c_\gamma/2)v\}\,.$$

and

$$\theta[A_\gamma^\dagger] = \arctan\frac{c_\gamma}{2\sqrt{\nu_1^c}}\,,\quad \theta[B_\gamma^\dagger] = \arctan\frac{c_\gamma}{2\sqrt{\nu_2^c}}\,,$$

we find that

$$\rho\Big(-\frac{3\pi}{2}\Big) \le \rho\Big(-\frac{\pi}{2}\Big)\mathcal{F}_-(\gamma),$$

with

$$\mathcal{F}_-(\gamma) = \sqrt{\frac{4\nu_1^c + c_\gamma^2}{4\nu_2^c + c_\gamma^2}}\,\exp\bigg(-\frac{c_\gamma}{2\sqrt{\nu_2^c}}\Big(\frac{\pi}{2}-\theta[A_\gamma^\dagger]\Big) - \frac{c_\gamma}{2\sqrt{\nu_1^c}}\Big(\frac{\pi}{2}+\theta[B_\gamma^\dagger]\Big)\bigg).$$

So,

$$\rho\Big(-\frac{3\pi}{2}\Big) \le \rho\Big(\frac{\pi}{2}\Big)\mathcal{F}_+(\gamma)\mathcal{F}_-(\gamma)\,. \tag{45}$$

We now claim that the solution (v, w) of (39) performs at most N clockwise rotations around the origin in the time interval $[0, T]$. Indeed, by the change of variable $u(t) = e^{\gamma t}v(t)$ we recover a solution of (7) and Lemma 7.1 can be applied, to confirm our claim.

To conclude, since the estimate (45) holds for any rotation around the origin and there are $\mathscr{N} \le N$ of them, we can compute, using assumption (9), since $\rho(\pi/2) = 1$ and $\gamma > 0$,

$$e^{\gamma T} = \rho\left(\frac{\pi}{2} - 2\pi\mathscr{N}\right) \le \rho\left(\frac{\pi}{2}\right)[\mathcal{F}_+(\gamma)\mathcal{F}_-(\gamma)]^{\mathscr{N}} < 1 \cdot e^{\gamma\frac{T}{N}\mathscr{N}} \le e^{\gamma T},$$

a contradiction. The proof of Theorem 2.3 is thus completed. □

Proof of Theorem 2.10. We will be very sketchy. First of all, in the spirit of Lemma 7.1, we can prove that in this case the solution makes *at least* $N+1$ clockwise rotations around the origin in the time interval $[0, T]$. Then, we need to switch the choice of δ in the polar coordinates (40) so to obtain (44) with the reversed inequality, where the values μ_1^c and μ_2^c are swapped in the two lines. Finally, we similarly obtain the analogue of estimate (45), with reversed inequality, thus reaching the final contradiction. □

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