Perturbed positively-(p, q)-homogeneous Hamiltonian systems with Frederickson–Lazer conditions

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Abstract

Dealing with a forced system ruled by a positively-(p, q)-homogeneous Hamiltonian function and a friction term, we propose a nonresonance condition in order to generalize a classical result by Frederickson and Lazer. We are thus able to treat both the periodic problem and the boundedness problem. In particular, our results apply to scalar p-Laplacian equations with asymmetric nonlinearities.

1 Introduction and main results

We start considering the scalar equation

$$\frac{d}{dt}\big(|\dot{x}|^{p-2}\dot{x}\big) + h(\dot{x}) + \mu(x^+)^{p-1} - \nu(x^-)^{p-1} = e(t)\,, \tag{1.1}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. Here p, μ and ν are positive constants, with $p > 1$, while h and e are continuous and uniformly bounded functions.

In [9], Frederickson and Lazer have studied the above problem with $p = 2$ and $\mu = \nu$, i.e., dealing with the equation

$$\ddot{x} + h(\dot{x}) + \mu x = e(t)\,.$$

In the case when the function $e(t)$ is 2π-periodic and $\mu = N^2$ for some $N \in \mathbb{N}$, the equation can be seen as a perturbation of a resonant oscillator. Hence, in order to get the existence of a 2π-periodic solution, some additional conditions have to be required. In that paper (see also [11]), the authors provide an existence result under the assumption

$$2\big[h(+\infty) - h(-\infty)\big] > \int_0^{2\pi} e(t)\cos(Nt+\theta)\,dt\,, \qquad \text{for every } \theta \in [0, 2\pi]\,,$$

where $h(\pm\infty)$ denote the limits of the functions h at $\pm\infty$. They also proved that, when h is assumed to be strictly increasing, this condition is indeed *necessary and sufficient* for the existence of a 2π-periodic solution. Note that here and in the sequel we choose $T = 2\pi$ as the value of the period just in order to simplify the notations. Should the period T be different, we can reduce to this case by a simple change of variable.

Our first aim in this paper is to generalize the above result by Frederickson and Lazer to equation (1.1) by introducing a suitable nonresonance condition. We will also deal with the problem of existence of bounded solutions when the function $e(t)$ is not assumed to be periodic. In this case we will need to accordingly modify the Frederickson–Lazer-type nonresonance condition.

In order to explain our results in a more precise way, we remold equation (1.1) to the equivalent planar system

$$\begin{cases} -\dot y = \mu(x^+)^{p-1} - \nu(x^-)^{p-1} + h(|y|^{q-2}y) - e(t)\,, \\ \ \ \dot x = |y|^{q-2}y\,, \end{cases} \tag{1.2}$$

where $(1/p) + (1/q) = 1$. We are thus led to study a more general system,

$$J\dot z = \nabla H(z) + G(t, z)\,, \tag{1.3}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ denotes the standard symplectic matrix. We assume the function $H : \mathbb{R}^2 \to \mathbb{R}$ to be continuously differentiable, and the function $G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ to be continuous. Notice that in (1.2) we have

$$H(x, y) = \frac{1}{q}|y|^q + \frac{1}{p}\left(\mu[x^+]^p + \nu\,[x^-]^p\right)\,.$$

Here are the main hypotheses for our results.

(A1) *The function $H : \mathbb{R}^2 \to \mathbb{R}$ is positively-(p,q)-homogeneous and positive, meaning that, for some $p > 1$ and $q > 1$ with $(1/p) + (1/q) = 1$, we have*

$$H(\gamma^q x, \gamma^p y) = \gamma^{p+q} H(x, y) > 0\,, \text{ for every } (x, y) \in \mathbb{R}^2 \setminus \{0\} \text{ and } \gamma > 0\,.$$

In this setting, the origin $(0,0)$ is an isochronous center for the planar autonomous system

$$J\dot{z} = \nabla H(z)\,, \tag{1.4}$$

i.e., besides the origin, all solutions of system (1.4) are periodic and have the same minimal period, which we denote by $\mathcal{T}$. This fact is a consequence of [4, Lemma 2.1], since one can see that the area $a(E)$ of the set $\{z \in \mathbb{R}^2 : H(z) \le E\}$ is linear in E.

(A2) *One has*

$$\lim_{\gamma\to+\infty} \gamma^{-\frac{p+q}{2}} G(t, \gamma^q x, \gamma^p y) = 0\,,$$

uniformly for $x^2 + y^2 = 1$ *and* $t \in \mathbb{R}$.

It can be easily seen that in the above condition one could equivalently ask that the limit is uniform for (x, y) belonging to compact subsets of $\mathbb{R}^2 \setminus \{0\}$.

Let us first focus our attention on the periodic problem. We then need to introduce the following assumption.

(A3) *Denoting by* $\psi(t) = (\psi_1(t), \psi_2(t))$ *a nontrivial solution of the autonomous system* (1.4)*, we assume that there exist* $d > -1$ *and* $C > 0$ *such that, for every* $\tau \in \mathbb{R}$*,* $\alpha \in [0, \mathcal{T}]$*, and* $\gamma \ge 1$*,*

$$\gamma^{\frac{p+q}{2}d}\big\langle G(\tau, \gamma^q\psi_1(\alpha), \gamma^p\psi_2(\alpha)), (\gamma^{\frac{q-p}{2}}\dot{\psi}_1(\alpha), \gamma^{\frac{p-q}{2}}\dot{\psi}_2(\alpha))\big\rangle \ge -C\,, \tag{1.5}$$

and, for every $\theta \in [0, \mathcal{T}]$*,*

$$\int_0^{2\pi} \liminf_{\substack{\gamma\to+\infty\\ s\to\theta}} \gamma^{\frac{p+q}{2}d}\big\langle G(t, \gamma^q\psi_1(t+s), \gamma^p\psi_2(t+s)), \\ (\gamma^{\frac{q-p}{2}}\dot{\psi}_1(t+s), \gamma^{\frac{p-q}{2}}\dot{\psi}_2(t+s))\big\rangle dt > 0\,. \tag{1.6}$$

Here is our main existence result for the periodic problem.

Theorem 1.1. *Assume the function* G *to be* 2π*-periodic in* t*, and that* (A1)–(A3) *hold true, with*

$$\mathcal{T} = \frac{2\pi}{N}\,, \quad \text{for some } N \in \mathbb{Z} \setminus \{0\}\,.$$

Then system (1.3) *has a* 2π*-periodic solution.*

There is a large literature for the periodic problem associated with scalar second order differential equations approaching resonance (see, e.g., [5, 14] and the references therein). Fewer results are available when dealing with scalar equations ruled by the p-Laplacian. We refer, e.g., to [3, 10], where some kind of Landesman–Lazer conditions have been implemented in order to get existence results.

When no periodicity is assumed on the function $G(t,z)$, we can still look for the existence of *bounded solutions* for (1.3), i.e., solutions $z(t)$ for which

$$\sup\{|z(t)| \,:\, t \in \mathbb{R}\} < +\infty\,.$$

To this aim, instead of $(A3)$, we need the following condition.

(A4) *Denoting by* $\psi(t) = (\psi_1(t), \psi_2(t))$ *a nontrivial solution of the autonomous system* (1.4)*, we assume that there exist* $d > -1$ *and* $C > 0$ *such that* (1.5) *holds, and*

$$\int_0^{\mathcal{T}} \liminf_{\substack{\gamma\to+\infty \\ s\to\theta}} \inf_{\tau\in\mathbb{R}} \gamma^{\frac{p+q}{2}d} \Big\langle G(\tau, \gamma^q\psi_1(t+s), \gamma^p\psi_2(t+s))\,, \\ (\gamma^{\frac{q-p}{2}}\dot{\psi}_1(t+s), \gamma^{\frac{p-q}{2}}\dot{\psi}_2(t+s))\Big\rangle dt > 0\,, \quad (1.7)$$

for every $\theta \in [0, \mathcal{T}]$.

Here is our existence result for bounded solutions.

Theorem 1.2. *Assume that* $(A1)$*,* $(A2)$*, and* $(A4)$ *hold true. Then system* (1.3) *has a bounded solution.*

The search of bounded solutions for scalar equations or systems is an ancient problem dating back to the beginning of the theory of dynamical systems. We just mention Lagrange stability, KAM theory, and Conley–Ważewski theory as classical research line sources. Our approach is somewhat related to the one in [12, § II.8], where some techniques involving the so-called *bound sets* and *guiding functions* are exploited in order to prove the existence of compact invariant sets. Some results more related to Theorem 1.2 can be found in [1, 8, 15, 16, 17, 19].

The paper is organized as follows. In Section 2 we prove our results for a particular class of perturbed linear systems. In Section 3 we introduce a symplectic change of variables which permits us to transform a (p, q)-homogeneous Hamiltonian system into a linear one. The proof of the main results in the general setting is then provided in Section 4. Examples of applications are given in Section 5, and we conclude with some remarks in Section 6.

2 A perturbed linear system

In this section, we provide the proof of Theorems 1.1 and 1.2 in the simpler case when

$$H(z) = \frac{1}{2}N|z|^2 .$$

We are thus considering a Hamiltonian satisfying $(A1)$ with $p = q = 2$.

2.1 Periodic solutions

We are dealing with the 2π-periodic problem associated with

$$J\dot{z} = Nz + G(t, z) , \tag{2.1}$$

where $N \in \mathbb{N} \setminus \{0\}$, and the function $G : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and 2π-periodic in t.

Let us introduce the function

$$\phi(t) = \big(\sin(Nt), \cos(Nt)\big) .$$

Notice that it is a nontrivial solution of the autonomous system $J\dot{z} = Nz$.

Assumptions $(A2)$ and $(A3)$ can be rephrased as follow.

$(A2')$ *One has*

$$\lim_{\lambda \to +\infty} \lambda^{-1} G(t, \lambda z) = 0 ,$$

uniformly for $|z| = 1$ *and* $t \in \mathbb{R}$.

As already remarked above, in condition $(A2')$ one could equivalently ask that the limit is uniform for z belonging to compact subsets of $\mathbb{R}^2 \setminus \{0\}$.

$(A3')$ *There exist* $d > -1$ *and* $C > 0$ *such that, for every* $\tau \in \mathbb{R}$, $\alpha \in [0, 2\pi/N]$, *and* $\lambda \geq 1$,

$$\lambda^d \big\langle G(\tau, \lambda\phi(\alpha)),\, \dot{\phi}(\alpha)\big\rangle \geq -C , \tag{2.2}$$

and, for every $\theta \in [0, 2\pi/N]$,

$$\int_0^{2\pi} \liminf_{\substack{\lambda \to +\infty \\ s \to \theta}} \lambda^d \langle G(t, \lambda\phi(t+s)), \dot{\phi}(t+s) \rangle dt > 0 \,.$$

Now Theorem 1.1, in this setting, can be rephrased as follows.

Theorem 2.1. *Assume that conditions* ($A2'$) *and* ($A3'$) *hold true. Then system* (2.1) *has a* 2π*-periodic solution.*

Proof. We need to prove the following a priori bound for the family of problems

$$\begin{cases} J\dot{z} = \sigma[Nz + G(t,z)] + (1-\sigma)\dfrac{2N+1}{2} z \,, \\ z(0) = z(2\pi) \,, \end{cases} \tag{2.3}$$

parameterized by $\sigma \in [0, 1]$.

Claim. There exists a $\bar{r} > 0$ such that every solution z of (2.3) satisfies $\|z\|_\infty \le \bar{r}$.

Proof of the Claim. Assume by contradiction that for every positive integer n there exist $\sigma_n \in [0, 1]$ and a solution z_n of (2.3), with $\sigma = \sigma_n$, such that $\|z_n\|_\infty > n$. Passing to a subsequence we can assume that $(\sigma_n)_n$ converges to some $\overline{\sigma} \in [0, 1]$. Set $w_n = \frac{z_n}{\|z_n\|_\infty}$. Then,

$$\begin{cases} J\dot{w}_n = \sigma_n \Big[N w_n + \dfrac{G(t, \|z_n\|_\infty w_n)}{\|z_n\|_\infty} \Big] + (1-\sigma_n)\dfrac{2N+1}{2} w_n \,, \\ w_n(0) = w_n(2\pi) \,. \end{cases} \tag{2.4}$$

Since $(w_n)_n$ is uniformly bounded, the differential equation in (2.4) implies that $(w_n)_n$ is bounded in $H^1(0, 2\pi)$, and so there exists a 2π-periodic function $w \in H^1(0, 2\pi)$ such that (up to a subsequence) $w_n \to w$ uniformly and $w_n \rightharpoonup w$ weakly in $H^1(0, 2\pi)$. Therefore, $\|w\|_\infty = 1$ and passing to the weak limit in (2.4), by using ($A2'$) we see that w solves

$$\begin{cases} J\dot{w} = \overline{\sigma} N w + (1-\overline{\sigma})\dfrac{2N+1}{2} w \,, \\ w(0) = w(2\pi) \,. \end{cases}$$

Hence, it has to be $\overline{\sigma} = 1$, and $J\dot{w} = Nw$. In particular, $w(t) \neq 0$ for every $t \in [0, 2\pi]$, and we can write $w(t) = \phi(t + \theta)$ for some $\theta \in [0, 2\pi/N]$. Let us also write

$$z_n(t) = r_n(t)\, \phi(t + \chi_n(t)) \,,$$

where $r_n : [0, 2\pi] \to \mathbb{R}$ and $\chi_n : [0, 2\pi] \to \mathbb{R}$ are continuous functions. Then, recalling that z_n is a solution of (2.3),

$$\begin{aligned}\dot r_n(t) &= \frac{\langle \dot z_n(t), z_n(t)\rangle}{r_n(t)} \\ &= -\sigma_n \Big\langle JG(t, r_n(t)\phi(t + \chi_n(t)))\,, \phi(t + \chi_n(t))\Big\rangle \\ &= -\frac{\sigma_n}{N}\Big\langle G(t, r_n(t)\phi(t + \chi_n(t))), \dot\phi(t + \chi_n(t))\Big\rangle\,.\end{aligned}$$

Multiplying both sides by $[r_n(t)]^d$ and integrating the above equation between 0 and 2π, since

$$\int_0^{2\pi} [r_n(t)]^d \dot r_n(t)\, dt = 0\,,$$

recalling that $\sigma_n \neq 0$ for n large enough, we have that

$$\int_0^{2\pi} [r_n(t)]^d \Big\langle G(t, r_n(t)\phi(t + \chi_n(t))), \dot\phi(t + \chi_n(t))\Big\rangle\, dt = 0\,.$$

Now, thanks to (2.2), we can apply Fatou's Lemma to obtain

$$\int_0^{2\pi} \liminf_n\, [r_n(t)]^d \Big\langle G(t, r_n(t)\phi(t + \chi_n(t))), \dot\phi(t + \chi_n(t))\Big\rangle\, dt \le 0\,.$$

Since $w_n \to w$ uniformly, we have that $r_n(t) \to +\infty$ and $\chi_n(t) \to \theta$ for some $\theta \in \mathbb{R}$, both limits being uniform in t. Without loss of generality we can assume that $\theta \in [0, 2\pi/N]$. Hence,

$$\begin{aligned}&\liminf_n\, [r_n(t)]^d \Big\langle G(t, r_n(t)\phi(t + \chi_n(t)), \dot\phi(t + \chi_n(t))\Big\rangle \\ &\quad \ge \liminf_{\substack{\lambda\to+\infty \\ s\to\theta}} \lambda^d \big\langle G(t, \lambda\phi(t + s)),\, \dot\phi(t + s)\big\rangle\,,\end{aligned}$$

and integrating we get a contradiction with $(A3')$, thus ending the proof of the claim.

The proof of the theorem can be now completed by a standard application of the Leray–Schauder topological degree theory. □

2.2 Bounded solutions

We now consider system (2.1) without assuming $G(t, z)$ to be periodic in t. Instead of $(A3')$, we consider the following assumption.

(A4′) *There exist $d > -1$ and $C > 0$ such that* (2.2) *holds, and*

$$\int_0^{2\pi/N} \liminf_{\substack{\lambda\to+\infty \\ s\to\theta}} \inf_{\tau\in\mathbb{R}} \lambda^d \big\langle G(\tau, \lambda\phi(t+s)),\, \dot\phi(t+s)\big\rangle dt > 0\,,$$

for every $\theta \in [0, 2\pi/N]$.

Theorem 2.2. *Assume that conditions* ($A2'$) *and* ($A4'$) *hold true. Then system* (2.1) *has a bounded solution.*

Proof. We first need to prove the following a priori bound.

Claim. There exists $R > 0$ such that every solution z of system (2.1) satisfying $z(t_0) = 0$ for some t_0 is such that $|z(t)| \le R$, for every $t \ge t_0$.

Proof of the Claim. Assume by contradiction that for every positive integer n there is a solution z_n of system (2.1) satisfying $z_n(t_n^0) = 0$ for a certain $t_n^0 \in \mathbb{R}$ and there is $t_n > t_n^0$ such that $|z_n(t_n)| = n$ and $|z_n(t)| < n$, for all $t \in [t_n^0, t_n[$. Let $\bar t_n^0 \ge t_n^0$ be such that $z_n(\bar t_n^0) = 0$ and $z_n(t) \ne 0$ for every $t \in]\bar t_n^0, t_n]$. For those values of t we introduce the polar coordinates $z_n(t) = \rho_n(t)\phi(\theta_n(t))$, where ρ and θ are continuous functions. Then, since z_n solves (2.1), we have

$$\dot\theta_n(t) = 1 + \frac{1}{N\rho_n(t)} \big\langle G\big(t, \rho_n(t)\phi(\theta_n(t))\big)\,,\, \phi(\theta_n(t))\big\rangle\,. \tag{2.5}$$

By ($A2'$), for every $\varepsilon \in]0,1[$, we can find a sufficiently large $R(\varepsilon) > 1$ such that, if $|z_n(t)| \ge R(\varepsilon)$, then

$$1 - \varepsilon \le \dot\theta_n(t) \le 1 + \varepsilon\,.$$

In particular, we can find $R_1 > 0$ such that, if $|z_n(t)| \ge R_1$, then $\dot\theta_n(t) > 1/2$. So, z_n rotates clockwise around the origin when $|z_n(t)| \ge R_1$. Moreover, since $J\dot\phi = N\phi$, for the radial speed we have

$$\dot\rho_n(t) = -\frac{1}{N} \big\langle G\big(t, \rho_n(t)\phi(\theta_n(t))\big)\,,\, \dot\phi(\theta_n(t))\big\rangle\,. \tag{2.6}$$

For $n > R_1$, we can select a time t_n^1 with the following property: $\rho_n(t_n^1) = R_1$ and $R_1 < \rho_n(t) < n$ for every $t \in]t_n^1, t_n[$.

We now show that $t_n - t_n^1 \to +\infty$. Indeed, from (2.6), using ($A2'$), we can find a positive constant $\bar c$ such that $|\dot\rho_n(t)| \le \bar c \rho_n(t)$ for every $t \in [t_n^1, t_n]$, and so

$$\rho_n(t) \ge \rho_n(t_n) \mathrm{e}^{-\bar c(t_n - t)} = n \mathrm{e}^{-\bar c(t_n - t)}\,. \tag{2.7}$$

In particular, we get $\mathrm{e}^{\bar{c}(t_n - t_n^1)} \geq n/R_1$, whence $t_n - t_n^1 \to +\infty$.

As a consequence, for n large, $t_n - t_n^1 > 4\pi/N$ and since $\dot\theta_n > 1/2$ in the interval $[t_n^1, t_n]$, the solution z_n performs more than one complete rotation there. So, there exists $s_n \in]t_n - 4\pi/N, t_n[$ such that z_n performs exactly one rotation around the origin in the time interval $[s_n, t_n]$. Let $\alpha_n \in [0, 2\pi/N]$ be such that $\theta_n(s_n) = \alpha_n$. So, $z_n(s_n) = \rho_n(s_n)\phi(\alpha_n)$ and consequently we have $z_n(t_n) = \rho_n(t_n)\phi(\alpha_n + 2\pi/N) = n\phi(\alpha_n + 2\pi/N)$.

By (2.7), we have

$$\lim_n \min\{\rho_n(t) \,:\, t \in [s_n, t_n]\} = +\infty\,,$$

hence, from (2.5) and $(A2')$, we deduce that $\dot\theta_n \to 1$ uniformly.

By letting $\eta = \theta_n(t)$, for n large enough, we get

$$\begin{aligned}
0 &\geq \frac{[\rho_n(s_n)]^{d+1}}{d+1} - \frac{[\rho_n(t_n)]^{d+1}}{d+1} \\
&= -\int_{s_n}^{t_n} [\rho_n(t)]^d \dot\rho_n(t)\, dt \\
&= \frac{1}{N}\int_{\alpha_n}^{\alpha_n + 2\pi/N} \frac{[\rho_n(\theta_n^{-1}(\eta))]^d}{\dot\theta_n(\theta_n^{-1}(\eta))} \Big\langle G\big(\theta_n^{-1}(\eta), \rho_n(\theta_n^{-1}(\eta))\phi(\eta)\big), \dot\phi(\eta)\Big\rangle\, d\eta \\
&\geq \frac{1}{N}\int_{\alpha_n}^{\alpha_n + 2\pi/N} \frac{[\rho_n(\theta_n^{-1}(\eta))]^d}{\dot\theta_n(\theta_n^{-1}(\eta))} \inf_{\tau\in\mathbb{R}} \Big\langle G\big(\tau, \rho_n(\theta_n^{-1}(\eta))\phi(\eta)\big), \dot\phi(\eta)\Big\rangle\, d\eta\,.
\end{aligned}$$

Hence, by the change of variable $\omega = \eta - \alpha_n$, setting $\lambda_n(\omega) = \rho_n(\theta_n^{-1}(\omega + \alpha_n))$ and $b_n(\omega) = \dot\theta_n(\theta_n^{-1}(\omega + \alpha_n))$,

$$0 \geq \int_0^{2\pi/N} \frac{[\lambda_n(\omega)]^d}{b_n(\omega)} \inf_{\tau\in\mathbb{R}} \Big\langle G\big(\tau, \lambda_n(\omega)\phi(\omega + \alpha_n)\big), \dot\phi(\omega + \alpha_n)\Big\rangle\, d\omega\,.$$

Since, by $(A4')$, assumption (2.2) holds, we can apply Fatou's Lemma so to get

$$\int_0^{2\pi/N} \liminf_n \frac{[\lambda_n(\omega)]^d}{b_n(\omega)} \inf_{\tau\in\mathbb{R}} \Big\langle G\big(\tau, \lambda_n(\omega)\phi(\omega + \alpha_n)\big), \dot\phi(\omega + \alpha_n)\Big\rangle\, d\omega \,\leq 0\,.$$

Being $(\alpha_n)_n$ in $[0, 2\pi/N]$, we can assume that, up to a subsequence, $\alpha_n \to \alpha \in [0, 2\pi/N]$. Recalling that $\dot\theta_n(t) \to 1$ uniformly in t, we see that $b_n(\omega) \to 1$ uniformly in ω, hence

$$\int_0^{2\pi/N} \liminf_{\substack{\lambda\to+\infty \\ s\to\alpha}} \lambda^d \inf_{\tau\in\mathbb{R}} \Big\langle G\big(\tau, \lambda\phi(\omega + s)\big), \dot\phi(\omega + s)\Big\rangle\, d\omega \,\leq 0\,,$$

thus contradicting $(A4')$. The claim is thus proved.

Now, let us prove the existence of the bounded solution we are looking for. To this purpose, let z_n be a solution of (2.1) such that $z_n(-n) = 0$. By the above argument, we have that $|z_n(t)| \leq R$ for all $t \geq -n$, and from (2.1) we see that the sequence $(z_n)_n$ is equibounded and equicontinuous. So by the Ascoli–Arzelà Theorem there exists a subsequence $(z_n^{(1)})_{n\geq 1}$ which converges uniformly on $[-1, 1]$ to some function z, which is a solution of (2.1) on that interval. Consider now the sequence $(z_n^{(1)})_{n\geq 2}$. Again, there exists a subsequence $(z_n^{(2)})_{n\geq 2}$ converging uniformly on $[-2, 2]$ to some solution of (2.1), which we still denote by z. Indeed, by the uniqueness of the limit, it is the extension of the previously found function z. In the similar way, we define on each interval $[-j, j]$ a subsequence $(z_n^{(j)})_{n\geq j}$ which converges uniformly to a solution of (2.1) on $[-j, j]$, which we still denote by z since it coincides with the previously found functions on the domains $[-k, k]$, with $k < j$. Hence, the diagonal sequence $(z_j^{(j)})_{j\geq 1}$ converges to a solution z of (2.1), uniformly on every compact subset of $\mathbb{R}$. Clearly enough, we have that $|z(t)| \leq R$ for all $t \in \mathbb{R}$, thus completing the proof. □

3 A symplectic change of variables

For the reader's convenience, we report in this section the main ideas discussed in [7].

By using $(A1)$, we have that $H(0, 0) = 0$ and the generalized Euler Identity holds true, i.e.,

$$\left\langle \nabla H(x, y), \left(\frac{x}{p}, \frac{y}{q}\right) \right\rangle = H(x, y) \,. \tag{3.1}$$

Choose the positive constant

$$\Upsilon = \min \left\{ \frac{1}{|z|^2} H(z) : 1 \leq |z| \leq 2 \right\} , \tag{3.2}$$

and let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $\eta'(s) \leq 0$ for all $s \in \mathbb{R}$ and

$$\eta(s) = \begin{cases} 1\,, & \text{if } \; s \leq 1\,, \\ 0\,, & \text{if } \; s \geq 2\,. \end{cases}$$

For $z = (x, y)$, set

$$\widehat{H}(z) = \eta(|z|)\Upsilon|z|^2 + (1 - \eta(|z|))H(z)\,,$$

and consider the new system

$$J\dot{z} = \nabla \widehat{H}(z)\,. \tag{3.3}$$

Notice that $\widehat{H}(0) = 0$, and $\widehat{H}(z) \neq 0$ for every $z \neq 0$. This implies that every non-zero solution of system (3.3) does not pass through the origin. Moreover, for every $z \neq 0$, we have

$$\nabla \widehat{H}(z) = \Big(\Upsilon\eta'(|z|)|z| + 2\Upsilon\eta(|z|) - \frac{\eta'(|z|)}{|z|}H(z)\Big)z + (1-\eta(|z|))\nabla H(z)\,.$$

Then, using (3.1) and (3.2), if $z = (x, y)$ is such that $1 \leq |z| \leq 2$, we have

$$\begin{aligned}\Big\langle \nabla \widehat{H}(z), \Big(\frac{x}{p}, \frac{y}{q}\Big)\Big\rangle &= \eta'(|z|)|z|\Big(\frac{x^2}{p} + \frac{y^2}{q}\Big)\big(\Upsilon - \frac{1}{|z|^2}H(z)\big) \\ &\quad + 2\eta(|z|)\,\Upsilon\Big(\frac{x^2}{p} + \frac{y^2}{q}\Big) + (1-\eta(|z|))H(z) > 0\,. \end{aligned} \tag{3.4}$$

This implies that $\nabla \widehat{H}(z) \neq 0$, for $1 \leq |z| \leq 2$. On the other hand, for $0 < |z| \leq 1$ the Hamiltonian function $\widehat{H}$ is quadratic, so that $\nabla \widehat{H}(z) \neq 0$. Lastly, for $|z| \geq 2$, we have $\nabla \widehat{H}(z) = \nabla H(z)$, and it is clear from (3.1) that $\nabla H(z) \neq 0$. Hence $\nabla \widehat{H}(z) \neq 0$ for every $z \neq 0$, and the Poincaré–Bendixson theory guarantees that all the solutions of system (3.3) are periodic. Thus, the origin is still a global center for system (3.3).

Now for any $z_0 \in \mathbb{R}^2 \setminus \{0\}$, we denote by $\widehat{T}(z_0)$ the minimal period of the solution of (3.3) passing through z_0. We notice here that this solution is unique, even if we are not assuming ∇H to be locally Lipschitz continuous, cf. [18]. The function $\widehat{T} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ thus defined is continuously differentiable (see [2]).

Define

$$\delta^\star = [0, +\infty[\times\{0\}\,,$$

and a function $\xi :]0, +\infty[\to]0, +\infty[$ as follows: for every $E > 0$, the level line $\{z \in \mathbb{R}^2 : \widehat{H}(z) = E\}$ intersects $\delta^\star$ at the point $(\xi(E), 0)$. Such a point is unique, since by (3.4) we have that $\frac{\partial \widehat{H}}{\partial x}(\xi, 0) > 0$ when $\xi > 0$.

Now, choose $E_0 > \max\{H(z) : |z| \leq 2\}$ and define $\widehat{K} : \mathbb{R}^2 \to \mathbb{R}$ as

$$\widehat{K}(z) = \frac{1}{\mathcal{T}}\int_{E_0}^{\widehat{H}(z)} \widehat{T}(\xi(E), 0)\, dE + E_0\,.$$

In particular, setting

$$\mathcal{E}_0 = \{z \in \mathbb{R}^2 \,:\, H(z) \geq E_0\}\,, \tag{3.5}$$

we have

$$\widehat{K}(z) = \widehat{H}(z) = H(z)\,, \quad \text{for every } z \in \mathcal{E}_0\,. \tag{3.6}$$

The function $\widehat{K}$ is continuously differentiable, and

$$\nabla \widehat{K}(z) = \frac{\widehat{T}(z)}{\mathcal{T}} \nabla \widehat{H}(z)\,.$$

Hence, the origin is an *isochronous* center for the system

$$J\dot{z} = \nabla \widehat{K}(z)\,, \tag{3.7}$$

since all solutions except the equilibrium 0 are periodic with minimal period $\mathcal{T}$. Moreover,

$$\widehat{K}(z) = \frac{\pi}{\mathcal{T}}|z|^2, \quad \text{if } |z| \leq 1\,.$$

Now, for every $z_0 \in \mathbb{R}^2 \setminus \{0\}$, let $\zeta(t\,;z_0)$ be the solution of system (3.7) satisfying $\zeta(0\,;z_0) = z_0$, and define $\theta(z_0) \in [0, 2\pi[$ as the minimum time for which

$$\zeta\Big(-\frac{\mathcal{T}}{2\pi}\theta(z_0)\,;z_0\Big) \in \delta^\star.$$

As shown in [2], the restricted function $\theta : \mathbb{R}^2 \setminus \delta^\star \to]0, 2\pi[$ is continuously differentiable, and its gradient $\nabla\theta$ can be continuously extended to $\mathbb{R}^2 \setminus \{0\}$. We will still denote this extension by $\nabla\theta : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$.

Hence, by [2, Proposition 2.2], there exists a symplectic diffeomorphism $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Lambda(z) = \begin{cases} \sqrt{\dfrac{\mathcal{T}}{\pi}\widehat{K}(z)}\big(\cos\theta(z),\, -\sin\theta(z)\big), & \text{if } z \neq 0\,, \\ 0\,, & \text{if } z = 0\,, \end{cases} \tag{3.8}$$

such that, by the change of variable $w = \Lambda(z)$, system (3.7) is changed to

$$J\dot{w} = \frac{2\pi}{\mathcal{T}}w\,,$$

i.e., to $J\dot{w} = Nw$. The function Λ satisfies the following relation

$$[\Lambda'(z)]^T J \Lambda'(z) = J\,, \tag{3.9}$$

for every $z \in \mathbb{R}^2$, where $\Lambda'(z)$ represents the Jacobian matrix of Λ. Thus, we see that

$$\nabla \widehat{K}(z) = N[\Lambda'(z)]^T \Lambda(z) \tag{3.10}$$

for every $z \in \mathbb{R}^2$.

4 Proofs of the main results

Let $\Lambda^{-1} = (\Psi_1, \Psi_2)$ and set $\mathcal{F}_0 = \Lambda(\mathcal{E}_0)$, where $\mathcal{E}_0$ was introduced in (3.5).

Proposition 4.1. *If $(u, v) \in \mathcal{F}_0$, then*

$$\Psi_1(\lambda u, \lambda v) = \lambda^{\frac{2}{p}} \Psi_1(u, v), \quad \Psi_2(\lambda u, \lambda v) = \lambda^{\frac{2}{q}} \Psi_2(u, v)\,,$$

for every $\lambda \geq 1$.

Proof. Since H is (p, q)-homogeneous, for $\gamma = \lambda^{\frac{2}{pq}}$, we have

$$H(\lambda^{\frac{2}{p}} x, \lambda^{\frac{2}{q}} y) = H(\gamma^q x, \gamma^p y) = \gamma^{pq} H(x, y) = \lambda^2 H(x, y)\,.$$

Recalling (3.6) for $\lambda \geq 1$, we have

$$\widehat{K}(\lambda^{\frac{2}{p}} x, \lambda^{\frac{2}{q}} y) = \lambda^2 \widehat{K}(x, y)\,,$$

when $z = (x, y) \in \mathcal{E}_0$. It has been proved in [7, Section 3.2] that the function θ satisfies

$$\theta(\gamma^q x, \gamma^p y) = \theta(x, y)\,. \tag{4.1}$$

Thus (3.8) implies that

$$\Lambda(\lambda^{\frac{2}{p}} x, \lambda^{\frac{2}{q}} y) = \sqrt{\frac{\mathcal{T}}{\pi} \lambda^2 \widehat{K}(x, y)} \Big(\cos(\theta(x, y)), -\sin(\theta(x, y)) \Big) = \lambda \Lambda(x, y)\,.$$

Hence, writing $\Lambda = (\Lambda_1, \Lambda_2)$, for every $\lambda \geq 1$ we have

$$\Lambda_1(\lambda^{\frac{2}{p}} x, \lambda^{\frac{2}{q}} y) = \lambda \Lambda_1(x, y), \quad \Lambda_2(\lambda^{\frac{2}{p}} x, \lambda^{\frac{2}{q}} y) = \lambda \Lambda_2(x, y)\,.$$

Setting $u = \Lambda_1(x, y)$ and $v = \Lambda_2(x, y)$, this implies $(u, v) \in \mathcal{F}_0$ and

$$(\lambda^{\frac{2}{p}} x, \lambda^{\frac{2}{q}} y) = \Lambda^{-1}(\lambda u, \lambda v) = \big(\Psi_1(\lambda u, \lambda v)\,,\, \Psi_2(\lambda u, \lambda v)\big)\,,$$

and the proof is easily completed. □

Let us consider a solution z of (1.3) and define $w(t) = \Lambda(z(t))$. If $z(t) \in \mathcal{E}_0$ for t in a certain interval I, then $w(t) \in \mathcal{F}_0$, so using (3.9) and recalling (3.6) and (3.10), in that interval we have

$$\begin{aligned}
J\dot{w} &= J\Lambda'(z)\dot{z} \\
&= [(\Lambda'(z))^T]^{-1} J\dot{z} \\
&= [(\Lambda'(z))^T]^{-1} [\nabla H(z) + G(t, z)] \\
&= Nw + [(\Lambda'(z))^T]^{-1} G(t, z) \\
&= Nw + [(\Lambda^{-1})'(w)]^T G(t, \Lambda^{-1}(w))\,.
\end{aligned}$$

Define

$$\widetilde{G}(t,w) = [(\Lambda^{-1})'(w)]^T G(t,\Lambda^{-1}(w))\,,$$

and consider the following modified system

$$J\dot{w} = Nw + \widetilde{G}(t,w)\,. \tag{4.2}$$

Given a nontrivial solution ψ of the autonomous system (1.4) satisfying $\psi(t) \in \mathcal{E}_0$, cf. (3.5), let $\phi = \Lambda(\psi)$. Then $\phi \neq 0$ and $J\dot{\phi} = N\phi$. Now we need the following result concerning G and $\widetilde{G}$.

Lemma 4.2. *If $\psi(\alpha) \in \mathcal{E}_0$ for some α, then, for every $\gamma > 1$ and $\tau \in \mathbb{R}$,*

$$\begin{aligned}&\langle G(\tau,\gamma^q\psi_1(\alpha),\gamma^p\psi_2(\alpha)),\,(\gamma^{\frac{q-p}{2}}\dot{\psi}_1(\alpha),\gamma^{\frac{p-q}{2}}\dot{\psi}_2(\alpha))\rangle\\ &= \langle \widetilde{G}(\tau,\gamma^{\frac{p+q}{2}}\phi(\alpha)),\dot{\phi}(\alpha)\rangle\,.\end{aligned}$$

Proof. Recalling (3.6), the computations in [7, Section 3.2] show that for every $z = (x,y) \in \mathcal{E}_0$ we have

$$\Lambda'(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z)\end{bmatrix}\,,$$

where, by denoting $c(z) = \cos\theta(z)$ and $s(z) = \sin\theta(z)$,

$$a_{11}(z) = \sqrt{\frac{\mathcal{T}}{\pi}}\left(\frac{\partial_x H(z)}{2\sqrt{H(z)}}\,c(z) - \sqrt{H(z)}\partial_x\theta(z)s(z)\right),$$

$$a_{12}(z) = \sqrt{\frac{\mathcal{T}}{\pi}}\left(\frac{\partial_y H(z)}{2\sqrt{H(z)}}\,c(z) - \sqrt{H(z)}\partial_y\theta(z)s(z)\right),$$

$$a_{21}(z) = \sqrt{\frac{\mathcal{T}}{\pi}}\left(-\frac{\partial_x H(z)}{2\sqrt{H(z)}}\,s(z) - \sqrt{H(z)}\partial_x\theta(z)c(z)\right),$$

$$a_{22}(z) = \sqrt{\frac{\mathcal{T}}{\pi}}\left(-\frac{\partial_y H(z)}{2\sqrt{H(z)}}\,s(z) - \sqrt{H(z)}\partial_y\theta(z)c(z)\right).$$

Recalling that Λ is symplectic, so $\det\Lambda'(z) = 1$, the inverse matrix is

$$(\Lambda'(z))^{-1} = \begin{bmatrix} a_{22}(z) & -a_{12}(z) \\ -a_{21}(z) & a_{11}(z)\end{bmatrix}\,.$$

The following identities have been proved in [6]:

$$\frac{\partial H}{\partial x}(\gamma^q x,\gamma^p y) = \gamma^p\frac{\partial H}{\partial x}(x,y)\,, \qquad \frac{\partial H}{\partial y}(\gamma^q x,\gamma^p y) = \gamma^q\frac{\partial H}{\partial y}(x,y)\,.$$

In addition, it has been proved in [7, Section 3.2] that the function θ satisfies

$$\partial_x\theta(\gamma^q x, \gamma^p y)\gamma^q = \partial_x\theta(x,y)\,, \qquad \partial_y\theta(\gamma^q x, \gamma^p y)\gamma^p = \partial_y\theta(x,y)\,.$$

Thus, by using all the above together with (4.1), we have

$$a_{11}(\gamma^q x, \gamma^p y) = \gamma^{\frac{p-q}{2}} a_{11}(x,y)\,, \qquad a_{12}(\gamma^q x, \gamma^p y) = \gamma^{\frac{q-p}{2}} a_{12}(x,y)\,,$$

$$a_{21}(\gamma^q x, \gamma^p y) = \gamma^{\frac{p-q}{2}} a_{21}(x,y)\,, \qquad a_{22}(\gamma^q x, \gamma^p y) = \gamma^{\frac{q-p}{2}} a_{22}(x,y)\,.$$

Now, for every $\lambda > 1$,

$$\begin{aligned}
&\langle \widetilde{G}(\tau, \lambda\phi(\alpha)), \dot{\phi}(\alpha)\rangle \\
&\quad = \Big\langle [(\Lambda^{-1})'(\lambda\phi(\alpha))]^T G(\tau, \Lambda^{-1}(\lambda\phi(\alpha))), \dot{\phi}(\alpha)\Big\rangle \\
&\quad = \Big\langle G(\tau, \Lambda^{-1}(\lambda\phi(\alpha))),\ [(\Lambda^{-1})'(\lambda\phi(\alpha))]\dot{\phi}(\alpha)\Big\rangle \\
&\quad = \Big\langle G(\tau, \Lambda^{-1}(\lambda\phi(\alpha))),\ [\Lambda'(\Lambda^{-1}(\lambda\phi(\alpha)))]^{-1}\dot{\phi}(\alpha)\Big\rangle\,.
\end{aligned} \tag{4.3}$$

Setting $\gamma = \lambda^{\frac{2}{pq}} > 1$, by Proposition 4.1, we have

$$\begin{aligned}
\Lambda^{-1}\Big(\lambda\phi(\alpha)\Big) &= \Big(\Psi_1(\lambda\phi(\alpha)), \Psi_2(\lambda\phi(\alpha))\Big) \\
&= \Big(\lambda^{\frac{2}{p}}\psi_1(\alpha), \lambda^{\frac{2}{q}}\psi_2(\alpha)\Big) \\
&= \Big(\gamma^q\psi_1(\alpha), \gamma^p\psi_2(\alpha)\Big)\,.
\end{aligned}$$

This implies that

$$\begin{aligned}
\Big[\Lambda'\Big(\Lambda^{-1}(\lambda\phi(\alpha))\Big)\Big]^{-1} &= \Big[\Lambda'\Big(\gamma^q\psi_1(\alpha), \gamma^p\psi_2(\alpha)\Big)\Big]^{-1} \\
&= \begin{bmatrix} \gamma^{\frac{q-p}{2}} a_{22}(\psi(\alpha)) & -\gamma^{\frac{q-p}{2}} a_{12}(\psi(\alpha)) \\ -\gamma^{\frac{p-q}{2}} a_{21}(\psi(\alpha)) & \gamma^{\frac{p-q}{2}} a_{11}(\psi(\alpha)) \end{bmatrix}\,,
\end{aligned}$$

hence

$$\Big[\Lambda'\Big(\Lambda^{-1}(\lambda\phi(\alpha))\Big)\Big]^{-1}\dot{\phi}(\alpha) = \Big(\gamma^{\frac{q-p}{2}}\dot{\psi}_1(\alpha), \gamma^{\frac{p-q}{2}}\dot{\psi}_2(\alpha)\Big)^T\,,$$

where we use the fact that $\dot{\psi} = [\Lambda'(\psi)]^{-1}\dot{\phi}$. This, together with (4.3), implies that

$$\big\langle \widetilde{G}\big(\tau, \lambda\phi(\alpha)\big), \dot{\phi}(\alpha)\big\rangle = \Big\langle G\big(\tau, \gamma^q\psi_1(\alpha), \gamma^p\psi_2(\alpha)\big),\ \big(\gamma^{\frac{q-p}{2}}\dot{\psi}_1(\alpha), \gamma^{\frac{p-q}{2}}\dot{\psi}_2(\alpha)\big)\Big\rangle\,.$$

The observation

$$\gamma = \lambda^{\frac{2}{pq}} \quad \Leftrightarrow \quad \lambda = \gamma^{\frac{pq}{2}} = \gamma^{\frac{p+q}{2}}$$

completes the proof of the lemma. $\square$

After these preliminary considerations, we now complete the proof of Theorems 1.1 and 1.2. Let us focus on the first one.

We prove that $(A2)$ implies $(A2')$ with G replaced by $\widetilde{G}$. It has been proved in [7, Section 3.2] that $(\Lambda^{-1})'(w)$ is bounded for every $w \in \mathbb{R}^2$, i.e., there exists $C_1 > 0$ such that for every $w \in \mathbb{R}^2$, we have

$$\|(\Lambda^{-1})'(w)\| \leq C_1 \,.$$

Let w belong to a compact subset $\mathcal{K}$ of $\mathbb{R}^2 \setminus \{0\}$ and let $(x, y) = \Lambda^{-1}(w)$. Since $\Lambda(0) = 0$, also the image $\Lambda(\mathcal{K})$ is a compact subset of $\mathbb{R}^2 \setminus \{0\}$. Recalling Proposition 4.1, we get $\Lambda^{-1}(\lambda w) = (\lambda^{\frac{2}{p}} x, \lambda^{\frac{2}{q}} y)$, when λ is sufficiently large. So, by using the definition of $\widetilde{G}$, for $\lambda = \gamma^{\frac{p+q}{2}} \geq 1$ we have

$$\begin{aligned}\lambda^{-1}|\widetilde{G}(t, \lambda w)| &\leq \gamma^{-\frac{p+q}{2}} \left\| [(\Lambda^{-1})'(\lambda w)]^T \right\| |G(t, \Lambda^{-1}(\lambda w))| \\ &\leq C_1 \gamma^{-\frac{p+q}{2}} |G(t, \gamma^q x, \gamma^p y)| \,.\end{aligned}$$

Hence, $(A2')$ follows from $(A2)$.

Now, we show that $(A3)$ implies $(A3')$ with G replaced by $\widetilde{G}$. Indeed, taking $\lambda = \gamma^{\frac{p+q}{2}}$ large enough, by Lemma 4.2, we have

$$\begin{aligned}&\gamma^{\frac{p+q}{2}d} \langle G(\tau, \gamma^q \psi_1(t+s), \gamma^p \psi_2(t+s)), \, (\gamma^{\frac{q-p}{2}} \dot{\psi}_1(t+s), \gamma^{\frac{p-q}{2}} \dot{\psi}_2(t+s)) \rangle \\ &\quad = \lambda^d \langle \widetilde{G}(\tau, \lambda \phi(t+s)), \dot{\phi}(t+s) \rangle \,.\end{aligned}$$

Hence, we can apply Theorem 2.1, so problem (4.2) has a 2π-periodic solution w. Since Λ is a diffeomorphism, by the inverse change of variables $z = \Lambda^{-1}(w)$ we obtain a 2π-periodic solution of system (1.3), as desired. The proof of Theorem 1.1 is thus completed.

Concerning Theorem 1.2, proceeding as above we see that $(A4)$ implies $(A4')$ with G replaced by $\widetilde{G}$. We can then apply Theorem 2.2, so to obtain a bounded solution w of system (4.2). By the inverse change of variables $z = \Lambda^{-1}(w)$, we obtain a bounded solution z of system (1.3). The proof of Theorem 1.2 is thus completed, as well. □

5 Some possible examples

Concerning an application of Theorem 1.1, we propose a periodic problem associated with the following asymmetric scalar equation

$$\frac{d}{dt}\big(|\dot{x}|^{p-2}\dot{x}\big) + h(\dot{x}) + \mu(x^+)^{p-1} - \nu(x^-)^{p-1} = e(t) \,, \tag{5.1}$$

with $p > 1$, where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, $h : \mathbb{R} \to \mathbb{R}$ and $e : \mathbb{R} \to \mathbb{R}$ are continuous. We assume that the positive constants μ, ν satisfy

$$\pi_p(\mu^{-1/p} + \nu^{-1/p}) = \frac{2\pi}{N}\,, \tag{5.2}$$

for a certain positive integer N, where

$$\pi_p = \frac{2(p-1)^{\frac{1}{p}}}{p\sin(\frac{\pi}{p})}\pi\,.$$

The scalar equation (5.1) corresponds to the system (1.3), where

$$H(z) = \frac{|y|^q}{q} + \frac{1}{p}\big(\mu[x^+]^p + \nu[x^-]^p\big)\,, \qquad G(t,z) = \begin{pmatrix} h(|y|^{q-2}y) - e(t) \\ 0 \end{pmatrix},$$

with $(1/p)+(1/q) = 1$. The function H is positively-(p,q)-homogeneous and positive. The nontrivial solutions of the equation

$$\frac{d}{dt}\big(|\dot{x}|^{p-2}\dot{x}\big) + \mu(x^+)^{p-1} - \nu(x^-)^{p-1} = 0$$

are of the form $x(t) = c\psi_1(t-\theta)$ with $c > 0$, $\theta \in [0, 2\pi/N]$, and ψ_1 is the function, with minimal period $2\pi/N$, defined on the interval $[0, 2\pi/N]$ as

$$\psi_1(t) = \begin{cases} \mu^{-1/p}\sin_p(\mu^{-1/p}t) & \text{if } t \in \big[0, \pi_p\mu^{-1/p}\big]\,, \\ \nu^{-1/p}\sin_p\Big(\nu^{-1/p}\Big(t - \frac{\pi_p}{\mu^{1/p}}\Big)\Big) & \text{if } t \in \big[\pi_p\mu^{-1/p}, 2\pi/N\big]\,, \end{cases}$$

extended by periodicity to the whole real line. Concerning the behaviour of $\sin_p(t)$, we refer the reader to [13]. So, all the solutions of system $J\dot{z} = \nabla H(z)$ are periodic with the same minimal period $\mathcal{T} = 2\pi/N$ (see [13, 20]).

We assume the existence of the finite limits

$$h(\pm\infty) = \lim_{u\to\pm\infty} h(u)\,.$$

The following corollary generalizes the result in [9]; it is a consequence of Theorem 1.1.

Corollary 5.1. *In the above setting, assume moreover that $e(t)$ is 2π-periodic and*

$$p\sin\left(\frac{\pi}{p}\right)\big[h(+\infty) - h(-\infty)\big] > \int_0^{2\pi} e(t)\dot{\psi}_1(t+\theta)\,dt\,, \tag{5.3}$$

for every $\theta \in [0, 2\pi/N]$. Then (5.1) has a 2π-periodic solution.

Proof. We need to prove that $(A2)$ and $(A3)$ hold so to be able to apply Theorem 1.1. From the boundedness of G, it is easy to see that assumption $(A2)$ holds. Let us set $\psi = (\psi_1, \psi_2)$ with $\psi_2 = |\dot{\psi}_1|^{p-2}\dot{\psi}_1$ and take $d = (p-q)/(p+q) = -1 + \frac{2}{q} > -1$.

We see that

$$\gamma^{\frac{p+q}{2}d}\big\langle G\big(\tau, \gamma^q\psi_1(\alpha), \gamma^p\psi_2(\alpha)\big),\, \big(\gamma^{\frac{q-p}{2}}\dot{\psi}_1(\alpha), \gamma^{\frac{p-q}{2}}\dot{\psi}_2(\alpha)\big)\big\rangle \\ = \big[h\big(\gamma^q\dot{\psi}_1(\alpha)\big) - e(\tau)\big]\dot{\psi}_1(\alpha)\,. \quad (5.4)$$

This quantity is bounded, hence (1.5) holds. Moreover, recalling (5.2),

$$\int_{\{\dot{\psi}_1>0\}} \dot{\psi}_1(t)\,dt = N\int_{-\frac{1}{2}\pi_p\nu^{-1/p}}^{\frac{1}{2}\pi_p\mu^{-1/p}} \dot{\psi}_1(t)\,dt \\ = N\big(p-1\big)^{\frac{1}{p}}\big(\mu^{-1/p} + \nu^{-1/p}\big) = p\sin\left(\frac{\pi}{p}\right)\,,$$

and similarly

$$\int_{\{\dot{\psi}_1<0\}} \dot{\psi}_1(t)\,dt = N\int_{\frac{1}{2}\pi_p\mu^{-1/p}}^{\mathcal{T}-\frac{1}{2}\pi_p\nu^{-1/p}} \dot{\psi}_1(t)\,dt = -p\sin\left(\frac{\pi}{p}\right)\,.$$

From the above computations we can deduce that (1.6) is equivalent to (5.3). Hence $(A3)$ holds and the proof is completed. □

Concerning an application of Theorem 1.2, we propose the asymmetric scalar equation (5.1), where $h : \mathbb{R} \to \mathbb{R}$ and $e : \mathbb{R} \to \mathbb{R}$ are continuous and uniformly bounded. The positive constants μ, ν satisfy (5.2). Proceeding as in the first example, we have the following result.

Corollary 5.2. *In the above setting, assume moreover that $e(t)$ is bounded and*

$$h(+\infty) - h(-\infty) > \sup_{t\in\mathbb{R}} e(t) - \inf_{t\in\mathbb{R}} e(t)\,. \quad (5.5)$$

Then (5.1) *has a bounded solution.*

Proof. Without loss of generality we can assume $\sup_{t\in\mathbb{R}} e(t) = -\inf_{t\in\mathbb{R}} e(t)$, simply replacing h by $h - \frac{1}{2}(\sup_{t\in\mathbb{R}} e(t) + \inf_{t\in\mathbb{R}} e(t))$. Following the lines of the previous proof, it remains to verify the validity of (1.7) in order to successfully apply Theorem 1.2. Once $\theta \in [0, 2\pi/N]$ is fixed, by (5.4) we have

$$\begin{aligned}
&\int_0^{2\pi/N} \liminf_{\substack{\lambda\to+\infty\\ s\to\theta}} \inf_{\tau\in\mathbb{R}} \Big[h(\gamma^q\dot\psi_1(t+s))\dot\psi_1(t+s) - e(\tau)\dot\psi_1(t+s)\Big]\,dt\\
&\quad= \int_{\{\dot\psi_1(\cdot+\theta)>0\}} \liminf_{\substack{\lambda\to+\infty\\ s\to\theta}} \Big[\big[h(\gamma^q\dot\psi_1(t+s))\dot\psi_1(t+s) - \|e\|_\infty\dot\psi_1(t+s)\big]\Big]dt\\
&\qquad+ \int_{\{\dot\psi_1(\cdot+\theta)<0\}} \liminf_{\substack{\lambda\to+\infty\\ s\to\theta}} \Big[\big[h(\gamma^q\dot\psi_1(t+s))\dot\psi_1(t+s) + \|e\|_\infty\dot\psi_1(t+s)\big]\Big]dt\\
&\quad= \int_{\{\dot\psi_1(\cdot+\theta)>0\}} \Big[\big[h(+\infty)\dot\psi_1(t+\theta) - \|e\|_\infty\dot\psi_1(t+\theta)\big]\Big]dt\\
&\qquad+ \int_{\{\dot\psi_1(\cdot+\theta)<0\}} \Big[\big[h(-\infty)\dot\psi_1(t+\theta) + \|e\|_\infty\dot\psi_1(t+\theta)\big]\Big]dt\\
&\quad= \frac{1}{N}\,p\sin\left(\frac{\pi}{p}\right)[h(+\infty) - h(-\infty) - 2\|e\|_\infty]\,.
\end{aligned}$$

So, we deduce (1.7) from (5.5). □

6 Final remarks

We conclude the paper with some remarks, and suggesting some open problems.

1. Concerning the statement of Theorem 1.1, in assumption $(A3)$ we can replace (1.5) and (1.6) respectively by

$$\gamma^{\frac{p+q}{2}d}\big\langle G(\tau,\gamma^q\psi_1(\alpha),\gamma^p\psi_2(\alpha)),\,(\gamma^{\frac{q-p}{2}}\dot\psi_1(\alpha),\gamma^{\frac{p-q}{2}}\dot\psi_2(\alpha))\big\rangle \le C\,, \tag{6.1}$$

and

$$\int_0^{2\pi} \limsup_{\substack{\gamma\to+\infty\\ s\to\theta}} \gamma^{\frac{p+q}{2}d}\big\langle G(t,\gamma^q\psi_1(t+s),\gamma^p\psi_2(t+s)),\\ (\gamma^{\frac{q-p}{2}}\dot\psi_1(t+s),\gamma^{\frac{p-q}{2}}\dot\psi_2(t+s))\big\rangle dt < 0\,.$$

2. Correspondingly, concerning Theorem 1.2, in assumption $(A4)$ we can replace (1.5) and (1.7) respectively by (6.1) and

$$\int_0^{\mathcal{T}} \limsup_{\substack{\gamma\to+\infty\\ s\to\theta}} \sup_{\tau\in\mathbb{R}} \gamma^{\frac{p+q}{2}d}\big\langle G(\tau,\gamma^q\psi_1(t+s),\gamma^p\psi_2(t+s))\,,\\ (\gamma^{\frac{q-p}{2}}\dot\psi_1(t+s),\gamma^{\frac{p-q}{2}}\dot\psi_2(t+s))\big\rangle dt < 0\,.$$

3. In [9], the existence of almost periodic solutions was also considered. It would be interesting to prove an analogue result in the setting of this paper.

4. It would be desirable to extend the results by Yang [20] to our setting, by introducing some kind of generalized Landesman–Lazer conditions.

5. Theorems 1.1 and 1.2 could possibly be adapted to systems in higher dimensions, coupling planar systems ruled by some positively-(p, q)-homogeneous and positive Hamiltonians.

6. The possible interaction of Frederickson–Lazer-type conditions with a Landesman–Lazer-type condition remains a field for further investigation.

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