Periodic solutions of Hamiltonian systems with symmetries

Alessandro Fonda

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Abstract. After a brief historical account, starting with the celebrated Poincaré–Birkhoff Theorem, we provide a multiplicity result for periodic solutions of some Hamiltonian systems whose Hamiltonian function $H(t, x, y)$ is periodic in the space variables x, and even in the variables (t, y). Our result is based on a recent theorem by R. Ortega and the author, and it does not require any twist condition on the solutions of the system.

1 A historical introduction

In this section we will provide a brief historical account on the development of the theory of periodic solutions of Hamiltonian systems which originated from the celebrated Poincaré–Birkhoff Theorem. We will start from the original conjecture by Poincaré, concerning fixed points of a planar area-preserving homeomorphism, and follow the efforts done to prove it and generalize it over more than one hundred years so to obtain a powerful tool to be used in the applications. We will try to explain the relation with the Lusternik–Schnirelmann theory and its generalizations, which have been used to extend the theorem to higher dimensions, so to provide multiple periodic solutions for Hamiltonian systems. Finally, we will emphasize a recent result of the author with R. Ortega on a two-point boundary value problem, which however is strongly related with the Poincaré–Birkhoff Theorem, and which we will then use to prove the multiplicity of periodic solutions to some Hamiltonian systems with symmetries.

1.1 The Poincaré–Birkhoff Theorem

Among the most influential mathematicians in history we acknowledge Jules Henri Poincaré, who passed away 111 years ago, on July 17th, 1912. Just three months before his death, in the May 1912 issue of the *Rendiconti del Circolo Matematico di Palermo* he published his paper "Sur un théorème de géométrie", which asserts the existence of at least two fixed points for an area-preserving homeomorphism of a planar circular annulus onto itself, such that the points of the inner circle Γ_1 are moved along Γ_1 in the clockwise sense and the points of the outer circle Γ_2 are moved along Γ_2 in the counter-clockwise sense [101].

However, Poincaré was not able to prove this result, and he tried to justify himself with these words (in our translation):

> *I have never presented such an incomplete work to the public; therefore, I think it necessary to briefly explain the reasons which convinced me to publish it, and, above all, those which drove me to start it. I have already proved in the past the existence of periodic solutions for the three body problem; however, the result was still unsatisfactory [...] While thinking at this problem, I convinced myself that the answer should depend on the truth or falseness of a geometric theorem [...] So, I was led to research the veracity of this theorem, but I met some unexpected difficulties [...] It seems that, in such a situation, I should refrain from any publication until I have solved the problem; but, after all the pointless efforts made over many months, I thought that the wiser choice was to leave the problem to mature, while resting for some years; this would have been fine if I had been sure to be able to take it up again one day; but at my age I cannot be so sure. On the other hand, the importance of the subject is too great and the quantity of results so far obtained too considerable, to resign myself to let them definitively unfruitful [...] I think that these considerations are sufficient to justify me.*

The existence of one fixed point was proved by George David Birkhoff [10] the year later, in 1913, while the proof of the existence of a second fixed point was provided by Birkhoff himself [12] only in 1925 (see also [14, 26] for a modern exposition). Since then, the "théorème de géométrie" is known as the Poincaré–Birkhoff Theorem.

Applications of the Poincaré–Birkhoff Theorem to dynamical systems coming from nonlinear mechanics and geometry were already suggested by Poincaré in [101] and studied by Birkhoff in [11, 13]. As mentioned by Zehnder in [125], Arnold considered this theorem as "the seed of symplectic topology".

1.2 First extensions in the planar case

In the case of planar Hamiltonian systems one often looks for the existence of periodic solutions as fixed points of the Poincaré map. However, a major difficulty in the application of the Poincaré–Birkhoff Theorem is the construction of invariant annular regions. Hence, a modification of the theorem not assuming the invariance conditions for the annulus and its inner and outer boundaries became necessary for the applications.

In 1976, Jacobowitz [80, 81], following a suggestion of Moser [97], proposed a modified version of the Poincaré–Birkhoff Theorem for a topological pointed disc, showing how to apply it to the search of periodic solutions to some superlinear second order differential equations. Applications in this direction were also given by Hartman [77] and Butler [27]. Based on Jacobowitz's result, W.-Y. Ding [41, 42] obtained a new version of the Poincaré–Birkhoff Theorem where the boundary invariance assumption was removed. Moreover, in [42] the circular annulus was replaced by a topological annulus whose inner and outer boundaries Γ_1 and Γ_2 are Jordan curves, assuming only Γ_1 to be star-shaped. It has been shown by Martins and Ureña in [92] that this star-shapedness assumption is not eliminable. Later on, Le Calvez and Wang in [86] provided an example showing that a star-shapedness assumption on the outer boundary is also needed, thus proving that Ding's theorem, hence most probably also Jacobowitz's, were not correct.

Meanwhile, Ding's theorem had been used by many authors to prove existence and multiplicity of periodic solutions of some non-autonomous Hamiltonian systems. See, e.g., [15, 16, 20, 22, 27, 28, 36, 37, 38, 39, 40, 46, 48, 51, 52, 56, 62, 72, 78, 80, 91, 102, 103, 107, 108, 109, 110, 122, 123, 124]. Fortunately, all these papers deal with annuli for which both boundaries are star-shaped, and Rebelo [107] was able to save the situation proving a safer version of the Poincaré–Birkhoff Theorem where such an assumption was made. We refer to [61, 85] for further references and a complete historical account until the year 2012, the centennial of Poincaré's original paper.

Let us now state a modern version of the Poincaré–Birkhoff Theorem providing the existence and multiplicity of periodic solutions to a planar Hamiltonian system

$$x' = \partial_y H(t,x,y)\,, \qquad y' = -\partial_x H(t,x,y)\,, \tag{1}$$

where $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, with continuous partial derivatives $\partial_x H(t,x,y)$ and $\partial_y H(t,x,y)$. The following theorem is contained in [70].

Theorem 1. *Assume $H(t,x,y)$ to be T-periodic in t for some $T > 0$, and τ-periodic with respect to x, for some $\tau > 0$. Let $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}$ be two continuous and T-periodic functions, with $\gamma_1(x) < \gamma_2(x)$ for every $x \in \mathbb{R}$, satisfying the following property: All solutions (x,y) of (1) starting with $y(0) \in [\gamma_1(x(0)), \gamma_2(x(0))]$ are defined on $[0,T]$ and are such that*

$$\begin{cases} y(0) = \gamma_1(x(0)) \quad \Rightarrow \quad x(T) - x(0) < 0\,, \\ y(0) = \gamma_2(x(0)) \quad \Rightarrow \quad x(T) - x(0) > 0\,. \end{cases} \tag{2}$$

Then, system (1) has at least two geometrically distinct T-periodic solutions (x,y), with $y(0) \in \,]\gamma_1(x(0)), \gamma_2(x(0))[\,$.

Let us specify what we mean by *geometrically distinct* solutions. By the periodicity assumption, once a solution $z(t) = (x(t), y(t))$ of (1) has been found, infinitely many others appear by just adding an integer multiple of 2π to $x(t)$. We will call *geometrically distinct* two solutions which cannot be obtained from each other in this way.

Assumption (2) is usually called *twist condition*, and the same conclusion also holds if the inequalities in (2) are reversed. The original Poincaré's setting on an annulus can be recovered by a change of variables, choosing some suitable polar coordinates. Notice that in Theorem 1 no uniqueness assumption is made for the solutions of initial value problems (hence the Poincaré map could be multivalued).

As an illustrative example, consider the planar system

$$x' = \psi(y + E(t))\,, \qquad y' = g(x)\,, \tag{3}$$

where all functions involved are continuous on the whole real line $\mathbb{R}$.

Corollary 2. *Let the following sign assumption hold.*

$$\exists d > 0 : \quad |\sigma| \geq d \quad \Rightarrow \quad \sigma\psi(\sigma) > 0\,.$$

Moreover, let the function $E(t)$ be T-periodic, and the function $g(x)$ be 2π-periodic, with $\int_0^{2\pi} g(s)\,ds = 0$. Then, system (3) has at least two geometrically distinct T-periodic solutions.

Proof. Since $g(x)$ is bounded, Theorem 1 directly applies, taking as constant the functions $\gamma_1(x) = -R$ and $\gamma_2(x) = R$, with $R > 0$ large enough. $\square$

Corollary 3. *Let $\phi : I \to \mathbb{R}$ be an increasing homeomorphism, with I an open interval containing 0, and $\phi(0) = 0$. Moreover, let the function $g(x)$ be continuous and 2π-periodic, with $\int_0^{2\pi} g(s)\,ds = 0$, and the function $e(t)$ be T-periodic, with $\int_0^T e(t)\,dt = 0$. Then, the equation*

$$(\phi(x'))' = g(x) + e(t)\,. \tag{4}$$

has at least two geometrically distinct T-periodic solutions.

Proof. Setting $\psi = \phi^{-1}$ and $E(t) = \int_0^t e(\tau)\,d\tau$, equation (4) can be written in the form of system (3), and Corollary 2 directly applies. $\square$

The case $\phi(y) = y$ was first proved in [95] by a variational method, taking as a model the forced pendulum equation. For the relativistic pendulum, when $\phi(y) = y/\sqrt{1 - y^2}$, the result has been proved in [7, 25]. See also [65].

1.3 The relation with Lusternik–Schnirelmann theory

The theory of Lusternik and Schnirelmann was first published in Russian in 1930. The French translation [90] appeared in the "Exposés sur l'analyse mathématique et ses applications", published under the direction of J. Hadamard, who introduces it with the following words (our translation):

> [...] *we will admire the novelty and breadth of views, the power and fecundity of ideas expressed. We considered it appropriate not to allow the reader ignore a work of this value.*

As the authors say in their introduction, they were motivated by some problems raised by Poincaré in a field connecting Analysis and Topology, a domain where the most important advances at that time had been achieved by Birkhoff.

In the following years, the ideas of Lusternik and Schnirelmann were extended and generalized in several directions. In 1964, Schwartz [112] provided a first infinite-dimensional version of the theory, thus starting the development of variational methods, in view of the applications to different boundary value problems. Indeed, both ODE's and PDE's could be handled using those methods, providing several multiplicity results. In 1978, Rabinowitz [105] showed us how the periodic problem for a Hamiltonian system could be treated using a variational method. In this case, a major difficulty lies in the fact that the associated functional is strongly indefinite. On the other hand, there have been many attempts to generalize the Poincaré–Birkhoff Theorem to higher dimensions starting with Birkhoff himself, who considered this an *outstanding question* [12, 13]. In the sixties Arnold proposed some famous conjectures, some of which are still open problems. Since then, various higher-dimensional versions of the Poincaré–Birkhoff Theorem were claimed, for maps which are close to the identity and also for monotone twist maps [2, 3, 24, 98, 99, 100, 121].

Using a different approach, Conley and Zehnder [34, Theorem 3] proved another possible version of the Poincaré–Birkhoff Theorem in higher dimensions. They obtained the multiplicity of periodic solutions for a Hamiltonian system assuming that the C^2-smooth Hamiltonian function $H = H(t, x, y)$ is periodic in t and in the space variables x_k, and quadratic in y on a neighborhood of infinity. Precisely, they assumed

$$|y| \geq R \quad \Rightarrow \quad H(t, x, y) = \tfrac{1}{2}\langle \mathbb{B}y, y\rangle + \langle a, y\rangle \,, \tag{5}$$

for some $R > 0$, some vector $a \in \mathbb{R}^N$ and some regular symmetric matrix $\mathbb{B}$. Remarkably, their result does not need the Poincaré time-map to be close to the identity, nor to have a monotone twist.

The development of infinite-dimensional Lusternik–Schnirelmann methods would allow Szulkin [113, Theorem 4.2] to generalize the Conley and Zehnder theorem by replacing the term $\langle a, y\rangle$ by nonlinearities $G(t, x, y)$ with bounded first-order derivatives. Further results along these lines can be found in [8, 9, 31, 47, 57, 58, 75, 79, 82, 83, 87, 93, 94, 96, 106, 114].

1.4 A higher dimensional Poincaré–Birkhoff Theorem for Hamiltonian systems

Despite all this ample literature it seems that, for the time being, there is still no genuine generalization of the Poincaré–Birkhoff theorem to higher dimensions. However, in 2017, the author together with A.J. Ureña proved a higher dimensional version of the Poincaré–Birkhoff Theorem while considering the Hamiltonian system

$$x' = \nabla_y H(t,x,y)\,, \qquad y' = -\nabla_x H(t,x,y)\,, \tag{6}$$

where $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is a continuous function, with continuous partial gradients $\nabla_x H(t,x,y)$ and $\nabla_y H(t,x,y)$. Here,

$$x = (x_1, \dots, x_N)\,, \qquad y = (y_1, \dots, y_N)\,.$$

Let us see how Theorem 1 can be generalized in the simpler case when $\gamma_1(x) = a$ and $\gamma_2(x) = b$ are constant functions.

Theorem 4. *Let $H(t,x,y)$ be T-periodic in t for some $T > 0$, and that, for every $k = 1, \dots, N$, it is τ_k-periodic with respect to the variable x_k, for some $\tau_k > 0$. Let $\mathscr{D}$ be a convex body in $\mathbb{R}^N$ with a smooth boundary, and assume that all the solutions (x,y) of (6) starting with $y(0) \in \mathscr{D}$ are defined on $[0,T]$ and are such that*

$$y(0) \in \partial\mathscr{D} \quad \Rightarrow \quad \langle x(T) - x(0)\,, \nu_{\mathscr{D}}(y(0))\rangle > 0\,. \tag{7}$$

Then, system (6) has at least $N+1$ geometrically distinct T-periodic solutions (x,y), with $y(0) \in \operatorname{int}(\mathscr{D})$.

In the above statement, $\nu_{\mathscr{D}}(y)$ denotes the outward unit normal vector to $\mathscr{D}$, for any $y \in \partial\mathscr{D}$. The twist condition (7) can be generalized in several different directions. See [33, 53, 54, 70, 71] for further details.

This result has already found several applications (see [17, 19, 23, 29, 32, 35, 44, 45, 49, 50, 63, 64, 66, 67, 68, 69, 76, 88, 89, 104, 116, 117, 118, 119, 120]). It was applied to vortex dynamics [5] and to Keplerian dynamics [18]. Finally, it has also been extended to infinite-dimensional Hamiltonian systems [21, 55].

1.5 A new functional setting

A crucial step in the application of variational methods is the choice of the space where to define the functional to be studied. For the T-periodic problem associated with the Hamiltonian system (6), the natural space seems to be the one made of those functions $z = (x,y)$ with both x and y belonging to $H_T^{1/2}$.

Recently the author jointly with R. Ortega [60] has obtained the following multiplicity result for a two-point boundary value problem associated with system (6) with two-point boundary conditions

$$y(a) = 0 = y(b)\,. \tag{8}$$

Theorem 5. *Let $H : [a,b] \times \mathbb{R}^{2N} \to \mathbb{R}$ be a continuous function, with continuous partial gradients $\nabla_x H(t,x,y)$ and $\nabla_y H(t,x,y)$, and assume that, for every $k = 1, \dots, N$, it is τ_k-periodic with respect to the variable x_k, for some $\tau_k > 0$. Assume moreover that all solutions of* (6) *starting with $y(a) = 0$ are defined on $[a,b]$. Then, problem* (6)-(8) *has at least $N+1$ geometrically distinct solutions.*

Surprisingly enough, *no twist condition is assumed* in the above statement. The main novelty in the proof of Theorem 5 lies in the fact that, while for the periodic problem x and y are usually both taken in the same space $H_T^{1/2}$, in [60] we have assumed x and y to belong to some complementary spaces, which are closely related to fractional Sobolev spaces.

When the Hamiltonian function has the special form $H(t,x,y) = \frac{1}{2}|y|^2 + G(t,x)$, problem (6)-(8) becomes a Neumann boundary value problem for a second order differential equation. We can find a multiplicity result for the Neumann problem by Castro [30] in 1980 and a similar one by Rabinowitz [106] in 1988. Both papers use variational methods.

The aim of this paper is to provide the existence and multiplicity of T-periodic solutions for a Hamiltonian system of the type (6) whose Hamiltonian function is T-periodic in t and presents some symmetries. More precisely, we will assume $H(t,x,y)$ to be even, both in t and in y. In this setting, no twist condition will be needed.

We will then extend our result to infinite dimensional systems, passing through a finite dimensional approximation. However, the passage to the limit in the dimension will eventually only guarantee the existence of one periodic solution. We refer to [4, 21, 43, 55, 59, 84, 111] for related results in infinite dimension.

2 Finite-dimensional systems

2.1 The main result

We consider the Hamiltonian system (6), where $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is a continuous function, with continuous partial gradients $\nabla_x H(t,x,y)$ and $\nabla_y H(t,x,y)$. As above, we use the notation $z = (x,y)$, with

$$x = (x_1, \dots, x_N)\,, \qquad y = (y_1, \dots, y_N)\,.$$

Here are our assumptions.

(A_1) *The function $H(t,x,y)$ is T-periodic in the variable t, for some $T > 0$.*

(A_2) *For every $k \in \{1, \dots, N\}$ there is a $\tau_k > 0$ such that the function $H(t,x,y)$ is τ_k-periodic in the variable x_k.*

Let us now introduce the notation

$$\mathbb{T}^N = \prod_{k=1}^{N} [0, \tau_k] \, .$$

In view of (A_2), the following assumptions will be made only for those x belonging to $\mathbb{T}^N$.

(A_3) *One has*

$$\nabla_y H\big(\tfrac{T}{2}, x, 0\big) = 0 \, , \qquad \text{for every } x \in \mathbb{T}^N.$$

(A_4) *The function $H(t,x,y)$ is even in (t,y), i.e.,*

$$H(-t, x, -y) = H(t, x, y) \, , \qquad \text{for every } (t,x,y) \in \mathbb{R} \times \mathbb{T}^N \times \mathbb{R}^N.$$

(A_5) *The solutions $z = (x,y)$ of* (6) *such that $z(0) \in \mathbb{T}^N \times \{0\}$ are defined on the whole time interval $\big[0, \frac{T}{2}\big]$.*

Here is our main result.

Theorem 6. *Let the assumptions (A_1)-(A_5) hold true. Then, system* (6) *has at least $N+1$ geometrically distinct T-periodic solutions (x,y). These solutions satisfy*

$$(x(-t), y(-t)) = (x(t), -y(t)) \, , \qquad \textit{for every } t \in \mathbb{R} \, ,$$

and

$$y\big(n\tfrac{T}{2}\big) = 0 \, , \qquad \textit{for every } n \in \mathbb{Z} \, .$$

Proof. By Theorem 5, assumptions (A_2) and (A_5) guarantee the existence of $N+1$ geometrically distinct solutions (x,y) of (6) satisfying the two-point boundary condition

$$y(0) = 0 = y\big(\tfrac{T}{2}\big) \, . \tag{9}$$

Let $z = (x,y)$ be one of such solutions, defined on the interval $\big[0, \frac{T}{2}\big]$. We will extend it to the whole line $\mathbb{R}$ so to obtain the T-periodic solution we are looking for.

Before doing this, notice that

$$x'(0) = 0 = x'\big(\tfrac{T}{2}\big) \, .$$

Indeed, by (A_4),

$$\nabla_x H(-t, x, -y) = \nabla_x H(t, x, y) \, , \qquad \nabla_y H(-t, x, -y) = -\nabla_y H(t, x, y) \, , \tag{10}$$

hence

$$x'(0) = \nabla_y H(0, x(0), y(0)) = \nabla_y H(0, x(0), 0) = 0 \, ;$$

moreover, by (A_3),

$$x'\big(\tfrac{T}{2}\big) = \nabla_y H\big(\tfrac{T}{2}, x\big(\tfrac{T}{2}\big), y\big(\tfrac{T}{2}\big)\big) = \nabla_y H\big(\tfrac{T}{2}, x\big(\tfrac{T}{2}\big), 0\big) = 0 \, .$$

First of all, we extend $z(t) = (x(t), y(t))$ to the interval $\big[-\frac{T}{2}, \frac{T}{2}\big]$ by setting

$$(x(-t), y(-t)) = (x(t), -y(t)) \, .$$

It is easy to see that this function is continuously differentiable on the whole interval $\big[-\frac{T}{2}, \frac{T}{2}\big]$.

Now, since $x(t)$ is even on $\left[-\frac{T}{2}, \frac{T}{2}\right]$ and $y\left(-\frac{T}{2}\right) = y\left(\frac{T}{2}\right) = 0$, we have that

$$z\left(-\tfrac{T}{2}\right) = z\left(\tfrac{T}{2}\right).$$

We can then extend $z(t)$ to $\mathbb{R}$ by T-periodicity, thus obtaining a continuous function. Let us prove that it is differentiable. We just need to show that it is such at $t = \frac{T}{2}$. Indeed,

$$\begin{aligned}
\lim_{t\to\frac{T}{2}^+}(x'(t), y'(t)) &= \lim_{t\to-\frac{T}{2}^+}(x'(t+T), y'(t+T)) \\
&= \lim_{t\to-\frac{T}{2}^+}(x'(t), y'(t)) \\
&= \lim_{t\to-\frac{T}{2}^+}(-x'(-t), y'(-t)) \\
&= \lim_{t\to\frac{T}{2}^-}(-x'(t), y'(t)) = \left(0, y'\left(\tfrac{T}{2}\right)\right),
\end{aligned}$$

hence

$$\lim_{t\to\frac{T}{2}^+}(x'(t), y'(t)) = \lim_{t\to\frac{T}{2}^-}(x'(t), y'(t)) = \left(0, y'\left(\tfrac{T}{2}\right)\right).$$

Finally, let us prove that $z(t)$ is a solution of (6) on the whole line $\mathbb{R}$. We know that it is a solution on $\left[0, \frac{T}{2}\right]$ and that it is differentiable on $\mathbb{R}$. Then, on $\left[-\frac{T}{2}, 0\right]$, by (10) we have

$$\begin{aligned}
x'(t) &= -x'(-t) = -\nabla_y H(-t, x(-t), y(-t)) \\
&= -\nabla_y H(-t, x(t), -y(t)) = \nabla_y H(t, x(t), y(t)),
\end{aligned}$$

and

$$\begin{aligned}
y'(t) &= y'(-t) = -\nabla_x H(-t, x(-t), y(-t)) \\
&= -\nabla_x H(-t, x(t), -y(t)) = -\nabla_x H(t, x(t), y(t)).
\end{aligned}$$

We have thus proved that $z(t)$ is a solution of (6) on $\left[-\frac{T}{2}, 0\right]$. Hence, it is a solution on $\left[-\frac{T}{2}, \frac{T}{2}\right]$. Now, by (A_1), the T-periodicity of $z(t)$ guarantees that it is a solution on the whole $\mathbb{R}$.

We have thus proved that any solution of (6) on $\left[0, \frac{T}{2}\right]$ satisfying (9) generates a T-periodic solution of (6). As already said above, we have $N+1$ of them, and they are geometrically distinct. The proof is thus completed. □

We now introduce a substitute of assumption (A_5).

(A_5') There exists a constant $\bar{c} > 0$ such that

$$|\nabla_x H(t, x, y)| \le \bar{c}(1 + |y|), \qquad \text{for every } (t, x, y) \in [0, T] \times \mathbb{T}^N \times \mathbb{R}.$$

We thus have the following.

Corollary 7. *Let the assumptions* (A_1)-(A_4) *and* (A_5') *hold true. Then, the same conclusion of Theorem 6 holds.*

Proof. By the periodicity assumption in the components of the x variable, the solutions (x, y) of (6) can be interpreted as if $x(t)$ belongs to $\mathbb{T}^N$, for every t. By (A_5') and the compactness of $\mathbb{T}^N$, one can easily deduce that (A_5) holds, hence Theorem 6 applies. $\square$

Remark 8. *Notice that no twist condition is needed in the above statements.*

Remark 9. *The results of this section should be compared with those in [1, 6, 115], involving Hamiltonian systems with symmetries. See also the references therein.*

2.2 Some corollaries

Let $N = 1$ and consider the planar system

$$x' = f(t, y)\,, \qquad y' = g(t, x)\,, \tag{11}$$

where the functions $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. The Hamiltonian function is

$$H(t, x, y) = \int_0^y f(t, \sigma)\, d\sigma + \int_0^x g(t, s)\, ds\,.$$

Corollary 10. *Let the following assumptions hold true.*

$(\mathcal{A}_1)$ *f and g are T-periodic in t.*

$(\mathcal{A}_2)$ *g is τ-periodic in x, with $\int_0^\tau g(t, s)\, ds = 0$, for every t.*

$(\mathcal{A}_3)$ $f\big(\frac{T}{2}, 0\big) = 0$.

$(\mathcal{A}_4)$ *$f(-t, -y) = -f(t, y)$ and $g(-t, x) = g(t, x)$, for every t, x, y.*

Then, system (11) *has at least two geometrically distinct T-periodic solutions (x, y). These solutions satisfy*

$$(x(-t), y(-t)) = (x(t), -y(t))\,, \qquad \text{for every } t \in \mathbb{R}\,,$$

and

$$y\big(n\tfrac{T}{2}\big) = 0\,, \qquad \text{for every } n \in \mathbb{Z}\,.$$

Proof. Assumptions (A_1)-(A_4) are direct consequences of $(\mathcal{A}_1)$-$(\mathcal{A}_4)$, respectively. Moreover, assumption (A_5') is surely satisfied, since g is bounded. Hence, Corollary 7 applies. $\square$

We immediately get the following result for system (3).

Corollary 11. *Assume that $E(t)$ is T-periodic and odd, with $E(\frac{T}{2}) = 0$. Let ψ be an odd function, and let $g(x)$ be 2π-periodic, with $\int_0^{2\pi} g(s)\, ds = 0$. Then, the same conclusion of Corollary 10 holds for system* (3).

Proof. Setting $f(t,y) = \psi(y + E(t))$, we have that

$$f(-t,-y) = \psi(-y + E(-t)) = \psi(-y - E(t)) = -\psi(y + E(t)) = -f(t,y)\,.$$

Then, Corollary 10 applies. □

Let us compare Corollary 11 with Corollary 2. We notice that in Corollary 11 no sign assumption is needed on ψ, which however is assumed to be an odd function. Concerning the function $E(t)$, in Corollary 11 it is assumed to be odd, with $E(\frac{T}{2}) = 0$, while in Corollary 2 one only needs it to be T-periodic.

3 Infinite-dimensional systems

In this section we deal with an infinite-dimensional Hamiltonian system on a separable real Hilbert space $\mathcal{H}$. Precisely, we consider the system (6), where $H : \mathbb{R} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is assumed to be continuous in (t, x, y) and continuously differentiable in $z = (x, y)$.

It will often be convenient to identify the space $\mathcal{H}$ with ℓ^2, the space of real sequences $\xi = (\xi_k)_{k\geq1}$ such that $\sum_{k=1}^{\infty} \xi_k^2 < \infty$, endowed with the usual scalar product

$$\langle \xi, \tilde{\xi} \rangle_{\mathcal{H}} = \sum_{k=1}^{\infty} \xi_k \tilde{\xi}_k\,,$$

and the associated norm $\|\xi\|_{\mathcal{H}} = \sqrt{\langle \xi, \xi \rangle_{\mathcal{H}}}$. We will also denote by $\|\cdot\|_{\mathcal{H}\times\mathcal{H}}$ the usual norm in the product space $\mathcal{H} \times \mathcal{H}$.

We can thus rewrite (6) as a system of infinitely many scalar ODE's, i.e.,

$$\begin{cases} x_k' = \dfrac{\partial H}{\partial y_k}(t, (x_1, x_2, \dots), (y_1, y_2, \dots))\,, \\ y_k' = -\dfrac{\partial H}{\partial x_k}(t, (x_1, x_2, \dots), (y_1, y_2, \dots))\,, \end{cases} \qquad k = 1, 2, \dots, \tag{12}$$

where $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ belong to ℓ^2.

Here are our assumptions.

$(\mathscr{A}_1)$ *The function $H(t, x, y)$ is T-periodic in the variable t, for some $T > 0$.*

$(\mathscr{A}_2)$ *There exists a sequence of positive real numbers $(\tau_k)_{k\geq1}$ in ℓ^2 such that, for every $k \geq 1$, the function $H(t, x, y)$ is τ_k-periodic in the variable x_k.*

It will be convenient to use the notation

$$\mathbb{T}^\infty = \prod_{k=1}^{\infty} [0, \tau_k]\,.$$

This is a compact set homeomorphic to the Hilbert cube. In view of $(\mathscr{A}_2)$, the following assumptions will be made only for those x belonging to $\mathbb{T}^\infty$.

($\mathscr{A}_3$) *One has*
$$\nabla_y H\big(\tfrac{T}{2}, x, 0\big) = 0\,, \qquad \text{for every } x \in \mathbb{T}^\infty.$$

($\mathscr{A}_4$) *The function $H(t,x,y)$ is even in (t,y), i.e.,*
$$H(-t,x,-y) = H(t,x,y)\,, \qquad \text{for every } (t,x,y) \in [0,T] \times \mathbb{T}^\infty \times \mathcal{H}\,.$$

($\mathscr{A}_5$) There exists a constant $\bar{c} > 0$ such that
$$|\nabla_x H(t,x,y)| \leq \bar{c}(1 + |y|)\,, \qquad \text{for every } (t,x,y) \in [0,T] \times \mathbb{T}^\infty \times \mathcal{H}\,.$$

($\mathscr{A}_6$) *There exists a constant $L > 0$ such that*
$$\|\nabla_z H(t,z_1) - \nabla_z H(t,z_2)\|_{\mathcal{H}\times\mathcal{H}} \leq L\,\|z_1 - z_2\|_{\mathcal{H}\times\mathcal{H}}\,, \\ \text{for every } t \in [0,T] \text{ and } z_1, z_2 \in \mathbb{T}^\infty \times \overline{B}(0, \bar{c}Te^{\bar{c}T})\,.$$

(We have denoted by $\overline{B}(0,R)$ the closed ball in $\mathcal{H}$ centered at 0 with radius R.)

Here is our result.

Theorem 12. *Let the assumptions ($\mathscr{A}_1$)-($\mathscr{A}_6$) hold true. Then, system* (12) *has a T-periodic solution. This solution (x,y) satisfies*
$$(x(-t), y(-t)) = (x(t), -y(t))\,, \qquad \textit{for every } t \in \mathbb{R}\,,$$
and
$$y\big(n\tfrac{T}{2}\big) = 0\,, \qquad \textit{for every } n \in \mathbb{Z}\,.$$

Remark 13. *This result should be compared with [21, Theorem 2.1], where a twist condition involving some finite dimensional approximating systems was required. Here we do not need any twist condition. The same as in [21], the multiplicity of solutions remains an open problem.*

Proof. We proceed as in [21]. First of all we notice that any solution $z(t) = (x(t), y(t))$ of (12) starting with $z(0) \in \mathbb{T}_\infty \times \{0\}$ is defined on $[0,T]$ and satisfies
$$\|y(t)\| \leq \bar{c}Te^{\bar{c}T}, \qquad \text{for every } t \in [0,T]\,.$$
This follows from the compactness of $\mathbb{T}^\infty$, the linear growth assumption ($\mathscr{A}_5$) and Gronwall's Lemma. (Here, and in the sequel of the proof, we use a simplified notation for the norms.) Moreover, by ($\mathscr{A}_6$), such a solution is therein unique.

For every integer $N \geq 1$ we define the projection $P_N : \ell^2 \to \mathbb{R}^N$ as
$$P_N(\xi_1, \xi_2, \dots) = (\xi_1, \xi_2, \dots, \xi_N)\,,$$
and the immersion $I_N : \mathbb{R}^N \to \ell^2$ as
$$I_N(\eta_1, \eta_2, \dots, \eta_N) = (\eta_1, \eta_2, \dots, \eta_N, 0, 0, \dots)\,.$$

Let $H_N : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined as

$$H_N(t,u,v) = H(t, I_N u, I_N v)\,, \tag{13}$$

and consider the Hamiltonian system

$$u' = \nabla_v H_N(t,u,v)\,, \qquad v' = -\nabla_u H_N(t,u,v)\,, \tag{14}$$

where $u = (u_1,\dots,u_N)$ and $v = (v_1,\dots,v_N) \in \mathbb{R}^N$. Since

$$\begin{aligned} \nabla_u H_N(t,u,v) &= P_N \nabla_x H(t, I_N u, I_N v)\,, \\ \nabla_v H_N(t,u,v) &= P_N \nabla_y H(t, I_N u, I_N v)\,, \end{aligned}$$

it can be verified that, for N large enough, all the assumptions of Corollary 7 hold true for system (14). Hence, there exists a T-periodic solution $w^N(t) = (u^N(t), v^N(t))$ of (14) with $w^N(0) = w^N\left(\frac{T}{2}\right) \in \mathbb{T}^N \times \{0\}$, and

$$\|I_N v^N(t)\| \le \bar{c} T e^{\bar{c}T}, \qquad \text{for every } t \in [0,T]\,.$$

We now define the operators $\mathscr{I}_N : \mathbb{R}^N \times \mathbb{R}^N \to \ell^2 \times \ell^2$ as

$$\mathscr{I}_N(u,v) = (I_N u, I_N v) = ((u_1,\dots,u_N,0,\dots),(v_1,\dots,v_N,0,\dots))\,,$$

and set $z_0^N = \mathscr{I}_N w^N(0)$. We thus have a sequence $(z_0^N)_N$ in the compact set $\mathbb{T}^\infty \times \{0\}$, which therefore has a subsequence, still denoted by $(z_0^N)_N$, which converges to some $z_0 \in \mathbb{T}^\infty \times \{0\}$. In view of the arguments at the beginning of the proof, the solution $z(t)$ of (6) starting from $z(0) = z_0$ is uniquely defined on $[0,T]$. Hence, following the lines of the proof in [21, Theorem 2.1], it is possible to prove that

$$\lim_{N\to\infty} \mathscr{I}_N w^N(t) = z(t)\,, \quad \text{uniformly in } t \in [0,T]\,.$$

Being $w^N(0) = w^N(T)$, we have that $z(0) = z(T)$. Extending $z(t)$ be T-periodicity, by $(\mathscr{A}_1)$ we have thus found the periodic solution we were looking for. $\square$

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Author's address:

Alessandro Fonda
Dipartimento di Matematica e Geoscienze
Università di Trieste
P.le Europa 1
I-34127 Trieste
Italy
e-mail: a.fonda@units.it

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