# Periodic solutions of Hamiltonian systems coupling twist with an isochronous center

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**Abstract**

We extend the Poincaré–Birkhoff Theorem to a Hamiltonian system which couples two systems with fairly different behaviours; the first one involves a twist assumption, while the second one is generated from a nonresonant isochronous center. By a suitable change of variables we modify the second system into a perturbation of a nonresonant linear one, and then prove that there exist multiple periodic solutions.

## 1 Introduction and statement of the main result

The celebrated Poincaré–Birkhoff Theorem [25] has been recently extended in the framework of Hamiltonian systems to any even dimension [20], with possible coupling with some nonresonant asymptotically linear system [11]. It is the aim of this paper to show that, in this coupling, linearity can be replaced by positive homogeneity, still preserving the already established multiplicity results.

We consider a Hamiltonian system in $\mathbb{R}^{2N}$ of the type

$$\begin{cases} \dot{q} = \nabla_p \mathcal{H}(t,q,p) + \nabla_p P(t,q,p,w)\,, \\ \dot{p} = -\nabla_q \mathcal{H}(t,q,p) - \nabla_q P(t,q,p,w)\,, \\ J\dot{w} = \nabla \mathscr{H}(w) + \nabla_w P(t,q,p,w)\,, \end{cases} \tag{1.1}$$

where we assume that all the involved functions are continuous, and $T$-periodic in their first variable $t$. Here $N = M + L$, and we write $w = (u,v)$, with

$$q = (q_1,\dots,q_M) \in \mathbb{R}^M, \quad p = (p_1,\dots,p_M) \in \mathbb{R}^M,$$
$$u = (u_1,\dots,u_L) \in \mathbb{R}^L, \quad v = (v_1,\dots,v_L) \in \mathbb{R}^L\,.$$

We have denoted by $J = \begin{pmatrix} 0 & -I_L \\ I_L & 0 \end{pmatrix}$ the standard symplectic $2L \times 2L$ matrix. Moreover, we assume that $\mathscr{H} : \mathbb{R}^{2L} \to \mathbb{R}$ is of the type

$$\mathscr{H}(u,v) = \sum_{j=1}^{L} \mathscr{H}_j(u_j,v_j)\,,$$

for some functions $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$.

In order to introduce a *twist condition*, we first consider the case when $\mathcal{D}$ is a rectangle in $\mathbb{R}^M$, i.e.

$$\mathcal{D} = [a_1, b_1] \times \cdots \times [a_M, b_M]\,,$$

and we denote by $\mathring{\mathcal{D}}$ its interior. Here are our hypotheses.

*A*1. For every $i \in \{1, \ldots, M\}$, there is a $\mathscr{T}_i > 0$ for which the function $\mathcal{H}(t, q, p)$ is $\mathscr{T}_i$-periodic in the variable $q_i$.

*A*2. There exists an $M$-tuple $\sigma = (\sigma_1, \ldots, \sigma_M) \in \{-1, 1\}^M$ such that for every $C^1$-function $\mathcal{W} : [0, T] \to \mathbb{R}^{2L}$, all the solutions $(q, p)$ of system

$$\begin{cases} \dot q = \nabla_p \mathcal{H}(t, q, p) + \nabla_p P(t, q, p, \mathcal{W}(t)) \\ \dot p = -\nabla_q \mathcal{H}(t, q, p) - \nabla_q P(t, q, p, \mathcal{W}(t))\,, \end{cases} \tag{1.2}$$

starting with $p(0) \in \mathcal{D}$, are defined on $[0, T]$ and, for every $i \in \{1, \ldots, M\}$, we have

$$\begin{cases} p_i(0) = a_i \quad \Rightarrow \quad \sigma_i(q_i(T) - q_i(0)) < 0\,, \\ p_i(0) = b_i \quad \Rightarrow \quad \sigma_i(q_i(T) - q_i(0)) > 0\,. \end{cases}$$

*A*3. For every $j \in \{1, \ldots, L\}$, the Hamiltonian function $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$ is positively homogeneous of degree 2 and positive, i.e.,

$$\mathscr{H}_j(\lambda\zeta) = \lambda^2 \mathscr{H}_j(\zeta) > 0\,, \quad \text{for every } \zeta \in \mathbb{R}^2 \setminus \{0\} \text{ and } \lambda > 0\,.$$

In this setting, the origin $(0, 0)$ is an isochronous center for the planar autonomous system

$$J\dot\zeta = \nabla \mathscr{H}_j(\zeta)\,. \tag{1.3}$$

For every $j \in \{1, \ldots, L\}$, all solutions of system (1.3) besides the origin are periodic and have the same minimal period, which will be denoted by $\tau_j$.

*A*4. The function $P(t, q, p, w)$ is $\mathscr{T}_i$-periodic in $q_i$, for every $i \in \{1, \ldots, M\}$, and has a bounded gradient with respect to $(q, p, w)$.

Here is our first result.

**Theorem 1.1.** *Assume that A*1 *– A*4 *hold true and*

$$\frac{T}{\tau_j} \notin \mathbb{N}\,, \qquad \textit{for every } j \in \{1, \ldots, L\}\,. \tag{1.4}$$

*Then there are at least $M + 1$ geometrically distinct $T$-periodic solutions of system* (1.1)*, with $p(0) \in \mathring{\mathcal{D}}$.*

Clearly enough, when a $T$-periodic solution $(q, p, w)$ of system (1.1) has been found, we may obtain infinitely many others by just adding an integer multiple of $\mathscr{T}_i$ to the $q_i$-th component. We say that two solutions are *geometrically distinct* if they cannot be obtained from each other in this way.

The above theorem extends the Poincaré–Birkhoff Theorem as stated in [20]. Its proof will be given is Section 2. By a suitable change of variables, similarly as in [1, 6, 15], we first modify the second system into a perturbation of a nonresonant linear one, and then use the results in [11] to prove the multiplicity of periodic solutions. Some variants of Theorem 1.1 and examples of applications will be provided in Section 3.

Different extensions of the results in [11, 20] have been proposed, coupling the twist assumption with the existence of well-ordered lower/upper solutions [10, 19], still obtaining multiplicity of periodic solutions. Our result represents a step forward in this field of investigation.

# 2 Proof of Theorem 1.1

In order to simplify the exposition, we first provide the proof for the case $M = L = 1$ (Sections 2.1 and 2.2), and then explain how to adapt it to the general case (Section 2.3). More precisely, in Section 2.1 we deal with the autonomous system (1.3) and show how to construct an appropriate symplectic change of variables to linearize it. Most of this part is taken from [1], but we prefer to report the arguments, for the reader's convenience. In Section 2.2 we use this change of variables to modify the original system so to be able to apply a result from [11]. We thus first consider the four-dimensional system

$$\begin{cases} \dot{q} = \partial_p \mathcal{H}(t,q,p) + \partial_p P(t,q,p,w)\,, \\ \dot{p} = -\partial_q \mathcal{H}(t,q,p) - \partial_q P(t,q,p,w)\,, \\ J\dot{w} = \nabla \mathscr{H}(w) + \nabla_w P(t,q,p,w)\,, \end{cases} \tag{2.1}$$

where all functions involved are continuous and $T$-periodic in $t$. Clearly enough, being $L = 1$, we have that $\mathscr{H} = \mathscr{H}_1$. Moreover, for simplicity, the period $\tau_1$ will simply be denoted by $\tau$.

## 2.1 The autonomous system

By $A3$, we have that $\mathscr{H}(0) = 0$, and the Euler Identity holds true, i.e.,

$$\langle \nabla \mathscr{H}(w), w \rangle = 2\mathscr{H}(w)\,, \quad \text{for every } w \in \mathbb{R}^2. \tag{2.2}$$

Choose the positive constant

$$\gamma = \tfrac{1}{2} \min\{\mathscr{H}(w) : |w| = 1\}\,, \tag{2.3}$$

and let $\eta : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function such that $\eta'(s) \le 0$ for all $s \in \mathbb{R}$ and

$$\eta(s) = \begin{cases} 1\,, & \text{if } \ s \le 1\,, \\ 0\,, & \text{if } \ s \ge 2\,. \end{cases}$$

Set

$$\widehat{\mathscr{H}}(w) = \eta(|w|)\gamma|w|^2 + (1 - \eta(|w|))\mathscr{H}(w)\,, \tag{2.4}$$

and consider the new system

$$J\dot{w} = \nabla\widehat{\mathscr{H}}(w)\,. \tag{2.5}$$

Notice that $\widehat{\mathscr{H}}(w) = \mathscr{H}(w)$ when $|w|$ is large enough, and $\widehat{\mathscr{H}}(0) = 0$. We first show that the origin is still a global center for system (2.5). For any $w \neq 0$, we have

$$\nabla\widehat{\mathscr{H}}(w) = \Big(\gamma\eta'(|w|)|w| + 2\gamma\eta(|w|) - \frac{\eta'(|w|)}{|w|}\mathscr{H}(w)\Big)w + (1 - \eta(|w|))\nabla\mathscr{H}(w)\,.$$

Then, using (2.2) and (2.3), if $w \neq 0$ we have

$$\begin{aligned} \langle\nabla\widehat{\mathscr{H}}(w), w\rangle \,=\, & \eta'(|w|)|w|^3\bigg(\gamma - \mathscr{H}\Big(\frac{w}{|w|}\Big)\bigg) \\ & +2|w|^2\bigg(\eta(|w|)\,\gamma + (1 - \eta(|w|))\mathscr{H}\Big(\frac{w}{|w|}\Big)\bigg) > 0\,. \end{aligned}$$

This shows that the origin is a global center for system (2.5) (cf. [1, Lemma 2.1]). For any $w_0 \in \mathbb{R}^2 \setminus \{0\}$, we denote by $\hat{\tau}(w_0)$ the minimal period of the solution of (2.5) passing through $w_0$. (Notice that this solution is unique, even if we are not assuming $\nabla\mathscr{H}$ to be locally Lipschitz continuous, cf. [26].) The function $\hat{\tau} : \mathbb{R}^2\setminus\{0\} \to [0, +\infty[$ thus defined is continuously differentiable (see [1, Appendix]), and $\hat{\tau}(w) = \tau$ when $|w|$ is large enough.

Define

$$\delta^\star = [0, +\infty[\times\{0\}\,,$$

and a function $\xi : ]0, +\infty[\to]0, +\infty[$ as follows: for every $c > 0$, the level line $\{w \in \mathbb{R}^2 : \widehat{\mathscr{H}}(w) = c\}$ intersects $\delta^\star$ at the point $(\xi(c), 0)$. By the above arguments, such a point is unique.

Now define $H : \mathbb{R}^2 \to \mathbb{R}$ as

$$H(w) = \frac{1}{\tau}\int_0^{\widehat{\mathscr{H}}(w)} \hat{\tau}(\xi(c), 0)\, dc\,. \tag{2.6}$$

This function is continuously differentiable, and

$$\nabla H(w) = \frac{\hat{\tau}(w)}{\tau}\nabla\widehat{\mathscr{H}}(w)\,.$$

Hence, the origin is an *isochronous* center for the system

$$J\dot{w} = \nabla H(w)\,, \tag{2.7}$$

since all solutions except the equilibrium 0 have minimal period $\tau$. Moreover, $H(w) = \mathscr{H}(w)$ when $|w|$ is large enough, and

$$H(w) = \frac{\pi}{\tau}|w|^2 , \qquad \text{when } |w| \le 1 .$$

Now, for every $w_0 \in \mathbb{R}^2 \setminus \{0\}$, let $\zeta(t\,; w_0)$ be the solution of system (2.7) satisfying $\zeta(0\,; w_0) = w_0$, and define $\theta(w_0) \in [0, 2\pi[$ as the minimum time for which

$$\zeta\Big( -\frac{\tau}{2\pi}\theta(w_0)\,; w_0\Big) \in \delta^\star .$$

As shown in [1], the restricted function $\theta : \mathbb{R}^2 \setminus \delta^\star \to ]0, 2\pi[$ is continuously differentiable, and its gradient $\nabla\theta$ can be continuously extended to $\mathbb{R}^2 \setminus \{0\}$. We will still denote this extension by $\nabla\theta : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$.

Notice that, if $w(t) \neq 0$ is a solution of system (2.7), then $w(t_0) \in \delta^\star$ for some $t_0 \in ]t - \tau, t\,]$. Thus $\theta(w(t)) = \frac{2\pi}{\tau}(t - t_0)$ for all $t \in ]t_0, t_0 + \tau[$, hence

$$\frac{d}{dt}\theta(w(t)) = \frac{2\pi}{\tau} , \qquad \text{for every } t \in ]t_0, t_0 + \tau[ .$$

Now define $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Lambda(w) = \begin{cases} \sqrt{\frac{\tau}{\pi}H(w)}\big(\cos\theta(w), \, -\sin\theta(w)\big), & \text{if } w \neq 0 , \\ 0 , & \text{if } w = 0 . \end{cases}$$

As a consequence of [1, Proposition 2.2], we have that the map $\Lambda$ is a symplectic diffeomorphism and satisfies

$$\det \Lambda'(w) = 1 , \qquad \text{for every } \; w \in \mathbb{R}^2 .$$

By the change of variables $z = \Lambda(w)$, system (2.7) becomes

$$J\dot{z} = \frac{2\pi}{\tau} z .$$

Hence, this change of variables considerably simplifies the problem, leading to a linear system. The sequel of the proof of Theorem 1.1 will be based on the idea of applying this change of variables to the general system (2.1), so to obtain the coupling of a twisting system with a linear one.

## 2.2 Back to the original system

By the global existence assumption in $A2$, there exists a constant $C > 0$ such that, for any solution $(q, p)$ of (1.2) starting with $p(0) \in [a, b]$, one has that

$$|p(t)| \le C , \qquad \text{for every } t \in [0, T] .$$

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-function such that

$$\sigma(s) = \begin{cases} 1 , & \text{if } \; s \le C , \\ 0 , & \text{if } \; s > C + 1 , \end{cases}$$

and set

$$\widehat{\mathcal{H}}(t,q,p) = \sigma(|p|)\mathcal{H}(t,q,p)\,. \tag{2.8}$$

Then $\widehat{\mathcal{H}}$ has a bounded gradient with respect to $(q,p)$.

In addition, we define

$$\widetilde{P}(t,q,p,z) = P(t,q,p,\Lambda^{-1}(z))\,.$$

**Lemma 2.1.** *The function $\widetilde{P}$ has a bounded gradient with respect to $(q,p,z)$.*

*Proof.* By $A4$, both

$$\partial_q \widetilde{P}(t,q,p,z) = \partial_q P(t,q,p,\Lambda^{-1}(z))\,, \quad \partial_p \widetilde{P}(t,q,p,z) = \partial_p P(t,q,p,\Lambda^{-1}(z))$$

are bounded, and

$$\begin{aligned} \nabla_z \widetilde{P}(t,q,p,z) &= \left[(\Lambda^{-1}(z))'\right]^{tr} \nabla_w P(t,q,p,\Lambda^{-1}(z)) \\ &= \left[(\Lambda'(\Lambda^{-1}(z)))^{tr}\right]^{-1} \nabla_w P(t,q,p,\Lambda^{-1}(z))\,. \end{aligned}$$

By $A4$ again, $\nabla_w P(t,q,p,w)$ is bounded, so it is sufficient to show that $(\Lambda'(w))^{-1}$ is bounded. For $|w|$ large enough, we have that $H(w) = \mathscr{H}(w)$. By denoting $c(w) = \cos\theta(w)$ and $s(w) = \sin\theta(w)$, with $w = (u,v) \in \mathbb{R}^2$, we have

$$\Lambda'(w) = \begin{bmatrix} a_{11}(w) & a_{12}(w) \\ a_{21}(w) & a_{22}(w) \end{bmatrix}\,,$$

where

$$\begin{aligned} a_{11}(w) &= \sqrt{\frac{\tau}{\pi}}\left(\frac{\partial_u \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,c(w) - \sqrt{\mathscr{H}(w)}\,\partial_u\theta(w)\,s(w)\right), \\ a_{12}(w) &= \sqrt{\frac{\tau}{\pi}}\left(\frac{\partial_v \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,c(w) - \sqrt{\mathscr{H}(w)}\,\partial_v\theta(w)\,s(w)\right), \\ a_{21}(w) &= \sqrt{\frac{\tau}{\pi}}\left(-\frac{\partial_u \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,s(w) - \sqrt{\mathscr{H}(w)}\,\partial_u\theta(w)\,c(w)\right), \\ a_{22}(w) &= \sqrt{\frac{\tau}{\pi}}\left(-\frac{\partial_v \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,s(w) - \sqrt{\mathscr{H}(w)}\,\partial_v\theta(w)\,c(w)\right). \end{aligned}$$

Recalling that $\det \Lambda'(w) = 1$, the inverse matrix is

$$(\Lambda'(w))^{-1} = \begin{bmatrix} a_{22}(w) & -a_{12}(w) \\ -a_{21}(w) & a_{11}(w) \end{bmatrix}\,.$$

From the definition of $\theta$, for $w \neq 0$ and $\lambda > 0$ we see that $\theta(\lambda w) = \theta(w)$, hence $\nabla\theta(\lambda w) = \lambda^{-1}\,\nabla\theta(w)$. Therefore, recalling that $\mathscr{H}$ is positively homogeneous of degree 2, we easily conclude that all the elements of the matrix $(\Lambda'(w))^{-1}$ are bounded, uniformly in $w \in \mathbb{R}^2$, cf. [1, Eq. (3.3)]. This proves that the map $\widetilde{P}$ has a bounded gradient with respect to $z$. □

Now we consider the modified system

$$\begin{cases} \dot{q} = \partial_p \widehat{\mathcal{H}}(t,q,p) + \partial_p \widetilde{P}(t,q,p,z)\,, \\ \dot{p} = -\partial_q \widehat{\mathcal{H}}(t,q,p) - \partial_q \widetilde{P}(t,q,p,z)\,, \\ J\dot{z} = \dfrac{2\pi}{\tau} z + \nabla_z \widetilde{P}(t,q,p,z)\,, \end{cases} \tag{2.9}$$

where we have applied the change of variables $z = \Lambda(w)$. The new Hamiltonian function is defined as

$$\widetilde{H}(t,q,p,z) = \widehat{\mathcal{H}}(t,q,p) + \frac{\pi}{\tau}|z|^2 + \widetilde{P}(t,q,p,z)\,.$$

Using $A2$ and (1.4), we conclude by [11, Corollary 2.4] that the modified system (2.9) has at least two geometrically distinct $T$-periodic solutions such that $p(0) \in \,]a,b[\,$.

Recalling that $\Lambda$ is a diffeomorphism, we can apply the inverse change of variables $w = \Lambda^{-1}(z)$ and obtain the two solutions of system (2.1) we were looking for. This concludes the proof in the case $M = L = 1$.

## 2.3 Conclusion of the proof

We now briefly explain how the above proof for the case $M = L = 1$ can be adapted to the general case. For every $j \in \{1, \dots, L\}$, there exists a symplectic diffeomorphism $\Lambda_j : \mathbb{R}^2 \to \mathbb{R}^2$ such that the change of variables $\rho = \Lambda_j(\zeta)$ transforms system (1.3) into

$$J\dot{\rho} = \frac{2\pi}{\tau_j}\rho\,.$$

Define $\Lambda : \mathbb{R}^{2L} \to \mathbb{R}^{2L}$ by

$$\Lambda(u,v) = \left(\Lambda_1(u_1,v_1), \dots, \Lambda_L(u_L,v_L)\right)\,.$$

Then $\Lambda$ is a symplectic diffeomorphism. Let us define

$$\widetilde{P}(t,q,p,z) = P(t,q,p,\Lambda^{-1}(z))\,.$$

As in Lemma 2.1, we can show that the function $\widetilde{P}$ has a bounded gradient with respect to $(q,p,z)$.

By a cut-off function $\sigma$, we modify the Hamiltonian $\mathcal{H}$ like in (2.8), setting

$$\widehat{\mathcal{H}}(t,q,p) = \sigma(|p|)\mathcal{H}(t,q,p)\,,$$

so that the new Hamiltonian $\widehat{\mathcal{H}}$ has a bounded gradient with respect to $(q,p)$.

We now consider the modified system

$$\begin{cases} \dot{q} = \nabla_p \widehat{\mathcal{H}}(t,q,p) + \nabla_p \widetilde{P}(t,q,p,z)\,, \\ \dot{p} = -\nabla_q \widehat{\mathcal{H}}(t,q,p) - \nabla_q \widetilde{P}(t,q,p,z)\,, \\ J\dot{z}_j = \dfrac{2\pi}{\tau_j} z_j + \nabla_{z_j} \widetilde{P}(t,q,p,z)\,, \quad j = 1, \dots, L\,, \end{cases} \tag{2.10}$$

where we have applied the change of variables $z = \Lambda(w)$. Using $A2$ and (1.4), we conclude by [11, Corollary 2.4] that the modified system (2.10) has at least $M+1$ geometrically distinct $T$-periodic solutions, such that $p(0) \in \mathring{\mathcal{D}}$.

Recalling that $\Lambda$ is a diffeomorphism, we can apply the inverse change of variables $w = \Lambda^{-1}(z)$ and obtain the solutions of system (1.1) we are looking for. □

# 3 Some variants and applications

We now consider some variants of Theorem 1.1. To this aim, let us first recall some definitions. A closed convex bounded subset $\mathcal{D}$ of $\mathbb{R}^M$ having nonempty interior $\mathring{\mathcal{D}}$ is said to be a convex body of $\mathbb{R}^M$. If we assume that $\mathcal{D}$ has a smooth boundary, then we denote the unit outward normal at $\xi \in \partial\mathcal{D}$ by $\nu_{\mathcal{D}}(\xi)$. Moreover, we say that $\mathcal{D}$ is strongly convex if for any $p \in \partial\mathcal{D}$, the map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ defined by $\mathcal{F}(\xi) = \langle \xi - p, \nu_{\mathcal{D}}(p)\rangle$ has a unique maximum point at $\xi = p$.

Let us first state the following "avoiding rays" assumption.

*A2′*. There exists a convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, such that for $\sigma \in \{-1, 1\}$ and for every $C^1$-function $\mathcal{W} : [0,T] \to \mathbb{R}^{2L}$, all the solutions $(q, p)$ of system (1.2) starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$, and

$$p(0) \in \partial\mathcal{D} \quad \Rightarrow \quad q(T) - q(0) \notin \{\sigma\lambda\, \nu_{\mathcal{D}}(p(0)) : \lambda \geq 0\}\,.$$

**Theorem 3.1.** *If in the statement of Theorem 1.1 we replace assumption $A2$ by $A2'$, the same conclusion holds.*

*Proof.* The argument is the same as the one in the proof of Theorem 1.1, with the only difference that instead of applying [11, Corollary 2.4], we apply [11, Corollary 2.1]. □

Let us now state the following "indefinite twist" assumption.

*A2″*. There are a strongly convex body $\mathcal{D}$ of $\mathbb{R}^M$ having a smooth boundary and a symmetric regular $M \times M$ matrix $\mathbb{A}$ such that for every $C^1$-function $\mathcal{W} : [0,T] \to \mathbb{R}^{2L}$, all the solutions $(q,p)$ of system (1.2) starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$, and

$$p(0) \in \partial\mathcal{D} \quad \Rightarrow \quad \langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}(p(0))\rangle > 0\,.$$

**Theorem 3.2.** *If in the statement of Theorem 1.1 we replace assumption $A2$ by $A2''$, the same conclusion holds.*

*Proof.* Apply [11, Corollary 2.3] instead of [11, Corollary 2.4]. □

There are many possible applications of our result. The twist condition is encountered in the framework of scalar second order differential equations, with many different possible behaviours of the retraction forces (see, e.g., [8] for an updated list of references). Concerning the positively homogeneous case, various particular cases have been treated, starting with the pioneering papers by Fucik [21] and Dancer [3] (see also, e.g., [7] and the references therein).

As a first example, in the case $M = L = 1$, consider the coupling of a pendulum-like equation with an asymmetric oscillator, i.e.,

$$\begin{cases} \ddot{q} + A \sin q = e(t) + \partial_q P(t, q, u)\,, \\ \ddot{u} + \mu u^+ - \nu u^- = \partial_u P(t, q, u)\,, \end{cases} \tag{3.1}$$

where the constants $A, \mu, \nu$ are positive. In the above, we have used the notation $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. Assume that $P(t, q, u)$ is $T$-periodic in $t$ and $2\pi$-periodic in $q$, and that it has a bounded gradient with respect to $(q, u)$. Setting $E(t) = \int_0^t e(s)\, ds$, system (3.1) is equivalent to

$$\begin{cases} \dot{q} = p + E(t)\,, \quad \dot{p} = -A \sin q + \partial_q P(t, q, u)\,, \\ \dot{u} = v\,, \quad \dot{v} = -\mu u^+ + \nu u^- + \partial_u P(t, q, u)\,. \end{cases}$$

Assuming $e(t)$ to be $T$-periodic with

$$\int_0^T e(t)\, dt = 0\,,$$

the function $E(t)$ is $T$-periodic, as well.

Let us verify the twist condition $A2$. Notice that there exists $K > 0$ such that, for every $C^1$-function $\mathcal{U} : [0, T] \to \mathbb{R}$, all the solutions $(q, p)$ of the system

$$\dot{q} = p + E(t)\,, \quad \dot{p} = -A \sin q + \partial_q P(t, q, \mathcal{U}(t))$$

are defined on $[0, T]$ and satisfy

$$|\dot{p}(t)| \le K\,, \qquad \text{for every } t \in [0, T]\,.$$

Define $b = KT + \|E\|_\infty + 1$ and $a = -(KT + \|E\|_\infty + 1)$. Then, if $p(0) = b$, we have

$$\dot{q}(t) = p(t) + E(t) = p(0) + \int_0^t \dot{p}(s)\, ds + E(t) \ge b - KT - \|E\|_\infty > 0\,,$$

for every $t \in [0, T]$, and so $q(T) - q(0) > 0$. Similarly, if $p(0) = a$, then $q(T) - q(0) < 0$. Assumption $A2$ is thus satisfied.

On the other hand, if we define $\mathscr{H}$ by

$$\mathscr{H}(u, v) = \frac{1}{2} v^2 + \frac{\mu}{2} (u^+)^2 + \frac{\nu}{2} (u^-)^2\,,$$

then $\mathscr{H}$ is positive, positively homogeneous of degree 2, and all the solutions of system $J\dot{w} = \nabla\mathscr{H}(w)$ with $w = (u, v)$ are periodic with a fixed period

$$\tau = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\,.$$

All the assumptions of Theorem 1.1 are satisfied, and we can thus state the following.

**Corollary 3.3.** *In the above setting, assume moreover that*

$$\frac{\sqrt{\mu\nu}}{\sqrt{\mu}+\sqrt{\nu}} \neq \frac{n\pi}{T}\,, \qquad \textit{for every } n \in \mathbb{N}\,.$$

*Then system* (3.1) *has at least two geometrically distinct $T$-periodic solutions.*

The above example can easily be extended to a system of the type

$$\begin{cases} \ddot{q}_i + A_i \sin q_i = e_i(t) + \partial_{q_i} P(t, q, u)\,, \quad i = 1, \dots, M\,, \\ \ddot{u}_j + \mu_j u_j^+ - \nu_j u_j^- = \partial_{u_j} P(t, q, u)\,, \quad j = 1, \dots, L\,, \end{cases}$$

where all constants $A_i, \mu_j, \nu_j$ are positive. Assume that $P(t, q, u)$ is $T$-periodic in $t$ and $2\pi$-periodic in each $q_i$ , it has a bounded gradient with respect to $(q, u)$, and that the functions $e_i(t)$ are $T$-periodic with

$$\int_0^T e_i(t)\,dt = 0\,, \qquad \text{for every } i = 1, \dots, M\,.$$

In this case, one gets the existence of $M + 1$ geometrically distinct $T$-periodic solutions.

A variant of system (3.1) in the above example is given by

$$\begin{cases} \ddot{q} + A \sin q = e(t) + \partial_q P(t, q, u)\,, \\ J\dot{w} + \mu w^+ - \nu w^- = \partial_u P(t, q, u)\,, \end{cases} \tag{3.2}$$

where, being $w = (u, v)$, one has $w^+ = (u^+, v^+)$ and $w^- = (u^-, v^-)$. Assuming $\mu\nu > 0$, as shown in [7], the minimal period of the isochronous system $J\dot{w} + \mu w^+ - \nu w^- = 0$ is

$$\tau = \frac{\pi}{2}\left(\frac{1}{\sqrt{|\mu|}} + \frac{1}{\sqrt{|\nu|}}\right)^2.$$

We thus get the following.

**Corollary 3.4.** *In the above setting, assume moreover that*

$$\frac{\mu\nu}{(\sqrt{|\mu|}+\sqrt{|\nu|})^2} \neq \frac{n\pi}{2T}\,, \qquad \textit{for every } n \in \mathbb{N}\,.$$

*Then system* (3.2) *has at least two geometrically distinct $T$-periodic solutions.*

One can similarly provide a generalization of Corollary 3.4 to higher dimensions as well. We avoid the details, for briefness. All the above results generalize a classical theorem in [24] by Mawhin and Willem on the multiplicity of periodic solutions for the pendulum equation.

Several other situations can be tackled using our results. Here we sketch some possible examples. Consider, e.g., a system of the type

$$\begin{cases} \ddot{q}_i + g_i(t, q_i) = \partial_{q_i} P(t, q, w)\,, \quad i = 1, \dots, M\,, \\ J\dot{w}_j = \nabla \mathscr{H}_j(w_j) + \nabla_{w_j} P(t, q, w)\,, \quad j = 1, \dots, L\,, \end{cases}$$

where the functions $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$ are as above, and the functions $g_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and $T$-periodic in their first variable.

When the retraction functions $g_i$ have a superlinear growth at infinity, one can follow the approach in [4, 16] to prove the existence of infinitely many $T$-periodic solutions. Indeed, the large amplitude solutions of the first system rotate around the origin faster and faster, with a time of rotation going to zero with the amplitude. Passing to suitably rotating polar coordinates, we recover the necessary twist condition (see [16] for the details).

A similar argument applies when the functions $g_i$ have different behaviours near the origin and near infinity, thus generating a twisting behaviour in the associated phase planes, like in [23], or for systems involving a parameter (see, e.g., [2] and the references therein), where the same situation is recovered after a change of coordinates. Perturbation results can also be obtained, as shown, e.g., in [9]. Let us finally mention the possibility of treating, with the same techniques, problems with one or more singularities (see, e.g., [13, 17] and the references therein). In all these mentioned situations, multiplicity of $T$-periodic solutions is obtained.

On the other hand, if the functions $g_i$ have a sublinear growth at infinity, we can use the approach developed in [5, 18] to get an infinite number of *subharmonic* solutions, i.e., periodic solutions having as minimal period an integer multiple of $T$. In this case, the large amplitude solutions of the first system rotate around the origin very slowly, with a time of rotation going to infinity.

Similar results for two-point boundary value problems have been recently obtained in [12, 14, 22] (see also [8]). The main novelty there is that no twist condition is needed. The extension of Theorem 1.1 to this setting will appear elsewhere.

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