Boundary value problems associated with Hamiltonian systems coupled with positively-(p, q)-homogeneous systems

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Abstract

We study the multiplicity of solutions for a two-point boundary value problem of Neumann type associated with a Hamiltonian system which couples a system with periodic Hamiltonian in the space variable with a second one with positively-(p, q)-homogeneous Hamiltonian. The periodic problem is also treated.

1 Introduction and statement of the main result

In the recent paper [7], a multiplicity result for a Neumann-type boundary value problem associated with a Hamiltonian system has been proved. It is the aim of this paper to extend this result to coupled systems, the first of which is of the type considered in [7], while the second one involves a positively-(p, q)-homogeneous and positive Hamiltonian function.

Denoting by J the standard symplectic matrix, our Hamiltonian system

$$J\dot{z} = \nabla_z H(t, z)\,,$$

when writing $z = ((x, y), (u, v)) \in \mathbb{R}^{2M} \times \mathbb{R}^{2L}$, is driven by a Hamiltonian function of the type

$$H(t, z) = \mathcal{H}(t, x, y) + \mathscr{H}(u, v) + P(t, x, y, u, v)\,.$$

To be more precise, we are dealing with the Hamiltonian system

$$\begin{cases} \dot{x} = \nabla_y \mathcal{H}(t, x, y) + \nabla_y P(t, x, y, u, v)\,, \\ \dot{y} = -\nabla_x \mathcal{H}(t, x, y) - \nabla_x P(t, x, y, u, v)\,, \\ \dot{u} = \nabla_v \mathscr{H}(u, v) + \nabla_v P(t, x, y, u, v)\,, \\ \dot{v} = -\nabla_u \mathscr{H}(u, v) - \nabla_u P(t, x, y, u, v)\,, \end{cases} \tag{1.1}$$

with Neumann-type boundary conditions

$$\begin{cases} y(a) = 0 = y(b)\,, \\ v(a) = 0 = v(b)\,. \end{cases} \tag{1.2}$$

We write

$$x = (x_1, \dots, x_M) \in \mathbb{R}^M, \quad y = (y_1, \dots, y_M) \in \mathbb{R}^M,$$
$$u = (u_1, \dots, u_L) \in \mathbb{R}^L, \quad v = (v_1, \dots, v_L) \in \mathbb{R}^L\,.$$

The functions $\mathcal{H} : [a,b] \times \mathbb{R}^{2M} \to \mathbb{R}$, $\mathscr{H} : \mathbb{R}^{2L} \to \mathbb{R}$ and $P : [a,b] \times \mathbb{R}^{2M+2L} \to \mathbb{R}$ are continuous, and continuously differentiable with respect to (x,y), (u,v) and (x,y,u,v), respectively.

Here are our hypotheses.

*A*1. For every $i = 1, \dots, M$ there exists $\kappa_i > 0$ such that the functions $\mathcal{H}(t,x,y)$ and $P(t,x,y,u,v)$ are κ_i-periodic in the variable x_i.

The periodicity assumption *A*1 naturally leads us to consider the torus

$$\mathbb{T}^M = (\mathbb{R}/\kappa_1\mathbb{Z}) \times \cdots \times (\mathbb{R}/\kappa_M\mathbb{Z})\,.$$

Indeed, in view of this assumption, the x component of the solutions could sometimes be interpreted as belonging to $\mathbb{T}^M$.

*A*2. The function $P(t,x,y,u,v)$ has a bounded gradient with respect to (x,y,u,v).

Assumption *A*2 guarantees that the coupling term $P(t,x,y,u,v)$ can be seen as some kind of not so large perturbation term.

*A*3. All the solutions of system (1.1) satisfying $y(a) = v(a) = 0$ are defined on $[a,b]$.

In view of the results in [5, 7], assumption *A*3 is surely satisfied if there exists a constant K_1 such that

$$|\nabla_x \mathcal{H}(t,x,y)| \le K_1(1 + |y|), \qquad \text{for every } (t,x,y) \in [a,b] \times \mathbb{T}^M \times \mathbb{R}^M\,.$$

*A*4. The function $\mathscr{H} : \mathbb{R}^{2L} \to \mathbb{R}$ is of the type

$$\mathscr{H}(u,v) = \sum_{j=1}^{L} \mathscr{H}_j(u_j, v_j)\,,$$

for some functions $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$ which are positively-(p_j, q_j)-homogeneous and positive, meaning that for some $p_j > 1$ and $q_j > 1$ with $\big(1/p_j\big)+\big(1/q_j\big) = 1$ we have

$$\mathscr{H}_j(\gamma^{q_j} r, \gamma^{p_j} s) = \gamma^{p_j+q_j} \mathscr{H}_j(r,s) > 0\,, \text{ for every } (r,s) \in \mathbb{R}^2 \setminus \{0\} \text{ and } \gamma > 0\,.$$

In this setting, the origin $(0,0)$ is an isochronous center for the planar autonomous system

$$\dot{u} = \nabla_v \mathscr{H}_j(u,v)\,, \qquad \dot{v} = -\nabla_u \mathscr{H}_j(u,v)\,. \tag{1.3}$$

For every $j \in \{1,\dots,L\}$, besides the origin all solutions of system (1.3) are periodic and have the same minimal period, which will be denoted by τ_j. Moreover, if $u_0 < 0$, for all solutions ζ of (1.3) starting with $\zeta(0) = (u_0, 0)$, there is a first time $\tau_{j+} > 0$ for which $v(\tau_{j+}) = 0$, while $v(t) > 0$ for all $t \in]0, \tau_{j+}[$, and this time τ_{j+} is independent of $u_0 < 0$. Similarly, if $u_0 > 0$, there is a first time $\tau_{j-} > 0$ for which $v(\tau_{j-}) = 0$, while $v(t) < 0$ for all $t \in]0, \tau_{j-}[$, and this time τ_{j-} is independent of $u_0 > 0$. Clearly enough, $\tau_j = \tau_{j+} + \tau_{j-}$.

Here is our main result.

Theorem 1.1. *Assume that A1 – A4 hold true. Let $\tau_{j+} = \tau_{j-}$ and*

$$\frac{b-a}{\tau_{j+}} \notin \mathbb{N}\,, \qquad \textit{for every } j \in \{1,\dots,L\}\,.$$

Then there are at least $M+1$ geometrically distinct solutions of the boundary value problem (1.1)-(1.2).

Notice that, when a solution has been found, infinitely many others appear by just adding an integer multiple of κ_i to the x_i-th component. We say that two solutions are *geometrically distinct* if they cannot be obtained from each other in this way.

Let us remark here that a sufficient condition for having satisfied the assumption $\tau_{j+} = \tau_{j-}$ is that the function $\mathscr{H}_j$ is even in v. This is a frequent case in the applications, where, e.g., $\mathscr{H}_j$ is quadratic in v.

Theorem 1.1 generalizes the result in [7], where the case $P \equiv 0$ was treated, dealing only with the system in (x,y). In order to prove it, we first consider the case when, writing $w = (u,v)$, the second Hamiltonian functions is of the type $\mathscr{H}(w) = \frac{1}{2}\langle \mathbb{A}w, w\rangle$, where $\mathbb{A}$ is a particular diagonal matrix. Then, by a symplectic change of variables, we are able to transform the positively-(p,q)-homogeneous Hamiltonian in the quadratic one.

We also study the periodic problem for such kind of Hamiltonian systems, and obtain a similar multiplicity result when a suitable *twist condition* is assumed. This part of the paper is related to the Poincaré–Birkhoff Theorem [15], and we exploit some results obtained in [4], where any symmetric matrix $\mathbb{A}$ can be considered, provided that a nonresonance condition is also assumed. We thus generalize to this setting some results obtained in [3, 8, 9].

At the end of the paper we will analyze the possibility of dealing with any symmetric matrix $\mathbb{A}$, provided that a nonresonance condition is assumed, also for the Neumann-type problem. However, we succeed doing this only in the case $L = 1$, while the case $L \geq 2$ remains an open problem.

Let us describe more in detail how the paper is organized.

In Section 2 we study the Neumann-type boundary value problem in the particular case when

$$\mathscr{H}(u, v) = \frac{1}{2} \sum_{j=1}^{L} \lambda_j (u_j^2 + v_j^2) \,,$$

for some positive constants $\lambda_1, \dots, \lambda_L$. The proof is variational, and it is modeled on the method developed in [7]. However, some delicate estimates are needed in order to prove the invertibility of the involved selfadjoint operator.

In Section 3 we provide the proof of Theorem 1.1. The idea is to construct a symplectic change of variables, so to reduce the problem to the one already treated in Section 2.

In Section 4 we study the periodic problem. Here we need to introduce a *twist condition*, which recalls the classical assumption in the Poincaré–Birkhoff Theorem. We obtain a similar multiplicity result as in Theorem 1.1 by applying a corollary of the main result in [4].

Some possible applications are given in Section 5. For example, we propose a system of the type

$$\begin{cases} \dot{x} = f(y) + E(t) \,, \quad \dot{y} = -A \sin x - \partial_x P(t, x, u) \,, \\ \dot{u} = |v|^{q-2} v \,, \quad \dot{v} = -\mu (u^+)^{p-1} + \nu (u^-)^{p-1} + \partial_u P(t, x, u) \,, \end{cases}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. The first two equations can be seen as a generalization of the pendulum equation (obtained when

$f(y) = y$), while the last two equations correspond to the scalar equation

$$\frac{d}{dt}\big(|\dot{u}|^{p-2}\dot{u}\big) + \mu(u^+)^{p-1} - \nu(u^-)^{p-1} = \partial_u P(t,x,u)\,.$$

Notice that the particular case $p = 2$ leads to a classical asymmetric oscillator. Both Neumann-type and periodic problems are analyzed.

Finally, in Section 6 we end with some further remarks and proposing an open problem.

In all the rest of the paper we will denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the Euclidean scalar product and norm on $\mathbb{R}^k$, for any $k \in \mathbb{N}$.

2 Coupling with a linear system

In this section we consider a Hamiltonian system of the type

$$\begin{cases} \dot{x} = \nabla_y \mathcal{H}(t,x,y) + \nabla_y P(t,x,y,w)\,, \\ \dot{y} = -\nabla_x \mathcal{H}(t,x,y) - \nabla_x P(t,x,y,w)\,, \\ J\dot{w} = \mathbb{A}w + \nabla_w P(t,x,y,w)\,. \end{cases} \tag{2.1}$$

Here, the functions $\mathcal{H} : [a,b] \times \mathbb{R}^{2M} \to \mathbb{R}$ and $P : [a,b] \times \mathbb{R}^{2M+2L} \to \mathbb{R}$ are continuous, and continuously differentiable with respect to (x,y) and (x,y,w), respectively. We denote by J the standard symplectic matrix, i.e.,

$$J = \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right),$$

where I is the $L \times L$ identity matrix. (In the following, the same letter J will also be used to denote analogous symplectic matrices in any dimensions.) The $2L \times 2L$ matrix $\mathbb{A}$ is of the type

$$\mathbb{A} = \left(\begin{array}{c|c} \mathbb{B}_L & 0 \\ \hline 0 & \mathbb{B}_L \end{array}\right), \tag{2.2}$$

where

$$\mathbb{B}_L = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_L \end{bmatrix},$$

for some positive real numbers $\lambda_1, \dots, \lambda_L$. Writing

$$x = (x_1, \dots, x_M) \in \mathbb{R}^M, \quad y = (y_1, \dots, y_M) \in \mathbb{R}^M,$$

and $w = (u, v) \in \mathbb{R}^{2L}$, with

$$u = (u_1, \dots, u_L) \in \mathbb{R}^L, \quad v = (v_1, \dots, v_L) \in \mathbb{R}^L,$$

we consider the Neumann-type boundary conditions

$$\begin{cases} y(a) = 0 = y(b)\,, \\ v(a) = 0 = v(b)\,. \end{cases} \tag{2.3}$$

Here is the main result of this section.

Theorem 2.1. *Assume that A1 – A3 hold true, and*

$$\frac{b-a}{\pi}\,\lambda_j \notin \mathbb{N}\,, \qquad \text{for every } j \in \{1, \dots, L\}\,.$$

Then, the boundary value problem (2.1)-(2.3) *has at least* $M+1$ *geometrically distinct solutions.*

Proof. Without loss of generality, we may assume that $[a, b] = [0, \pi]$. By $A3$ and a standard compactness argument, there exists a constant $K_2 > 0$ such that, for any solution (x, y, w) of (2.1) satisfying $y(0) = v(0) = 0$, one has that

$$|y(t)| \le K_2\,, \qquad \text{for every } t \in [0, \pi]\,.$$

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that

$$\sigma(s) = \begin{cases} 1\,, & \text{if } |s| \le K_2\,, \\ 0\,, & \text{if } |s| \ge K_2 + 1\,, \end{cases}$$

and set

$$\widehat{\mathcal{H}}(t, x, y) = \sigma(|y|)\mathcal{H}(t, x, y)\,, \tag{2.4}$$

and consider the modified system

$$\begin{cases} \dot{x} = \nabla_y \widehat{\mathcal{H}}(t, x, y) + \nabla_y P(t, x, y, w)\,, \\ \dot{y} = -\nabla_x \widehat{\mathcal{H}}(t, x, y) - \nabla_x P(t, x, y, w)\,, \\ J\dot{w} = \mathbb{A}w + \nabla_w P(t, x, y, w)\,. \end{cases} \tag{2.5}$$

The new Hamiltonian function is thus

$$\widetilde{H}(t, x, y, w) = \widehat{\mathcal{H}}(t, x, y) + \tfrac{1}{2}\langle \mathbb{A}w, w \rangle + P(t, x, y, w)\,. \tag{2.6}$$

We will prove that the boundary value problem (2.5)-(2.3) has at least $M+1$ geometrically distinct solutions. By the above argument, these solutions will satisfy (2), hence they will be the solutions of (2.1)-(2.3) we are looking for.

The proof is variational, and it is based on a theorem by Szulkin recalled below. We will now introduce the function spaces and the needed functionals.

2.1 The function spaces

For any $\alpha \in]0,1[$, we define X_α as the set of those real valued functions $\tilde{x} \in L^2(0,\pi)$ such that

$$\tilde{x}(t) \sim \sum_{m=1}^{\infty} \tilde{x}_m \cos(mt)\,,$$

where $(\tilde{x}_m)_{m\geq 1}$ is a sequence in $\mathbb{R}$ satisfying

$$\sum_{m=1}^{\infty} m^{2\alpha} \tilde{x}_m^2 < \infty\,.$$

The space X_α is endowed with the inner product and the norm

$$\langle \tilde{x},\, \tilde{\phi} \rangle_{X_\alpha} = \sum_{m=1}^{\infty} m^{2\alpha} \tilde{x}_m \tilde{\phi}_m\,, \qquad ||\tilde{x}||_{X_\alpha} = \sqrt{\sum_{m=1}^{\infty} m^{2\alpha} \tilde{x}_m^2}\,.$$

For any $\beta \in]0,1[$, we define Y_β as the set of those real valued functions $y \in L^2(0,\pi)$ such that

$$y(t) \sim \sum_{m=1}^{\infty} y_m \sin(mt)\,,$$

where $(y_m)_{m\geq 1}$ is a sequence in $\mathbb{R}$ satisfying

$$\sum_{m=1}^{\infty} m^{2\beta} y_m^2 < \infty\,.$$

The space Y_β is endowed with the inner product and the norm

$$\langle y,\, \rho \rangle_{Y_\beta} = \sum_{m=1}^{\infty} m^{2\beta} y_m \rho_m\,, \qquad ||y||_{Y_\beta} = \sqrt{\sum_{m=1}^{\infty} m^{2\beta} y_m^2}\,.$$

From now on, we will consider functions x, y, u, v which can be written as

$$x(t) = \bar{x} + \tilde{x}(t)\,, \quad \bar{x} = \frac{1}{\pi}\int_0^\pi x(t)\,dt\,,$$

$$u(t) = \bar{u} + \tilde{u}(t)\,, \quad \bar{u} = \frac{1}{\pi}\int_0^\pi u(t)\,dt\,,$$

where $\tilde{x}$ and y belong to the spaces X_α^M and Y_β^M respectively, while functions $\tilde{u}$ and v belong to the spaces X_α^L and Y_β^L respectively.

Choose two positive numbers α, β such that

$$\alpha < \tfrac{1}{2} < \beta \quad \text{and} \quad \alpha + \beta = 1\,.$$

Consider the space $E = X_\alpha^M \times Y_\beta^M \times (\mathbb{R}^L \times X_\alpha^L) \times Y_\beta^L$, and the torus $\mathbb{T}^M = (\mathbb{R}/\kappa_1\mathbb{Z}) \times \cdots \times (\mathbb{R}/\kappa_M\mathbb{Z})$. The space E is endowed with the scalar product

$$\begin{aligned}\langle(\tilde{x}\,, y\,, \bar{u}\,, \tilde{u}\,, v),\, (\widetilde{X}\,, Y\,, \bar{u}\,, \widetilde{U}\,, V)\rangle_E =& \langle\tilde{x},\, \widetilde{X}\rangle_{X_\alpha^M} + \langle y,\, Y\rangle_{Y_\beta^M} + \\ &+ \langle\bar{u}\,, \bar{u}\rangle + \langle\tilde{u},\, \widetilde{U}\rangle_{X_\alpha^L} + \langle v,\, V\rangle_{Y_\beta^L}\,,\end{aligned}$$

and the corresponding norm

$$||(\tilde{x}\,, y\,, \bar{u}\,, \tilde{u}\,, v)||_E = \sqrt{||\tilde{x}||^2_{X_\alpha^M} + ||y||^2_{Y_\beta^M} + |\bar{u}|^2 + ||\tilde{u}||^2_{X_\alpha^L} + ||v||^2_{Y_\beta^L}}\,.$$

Since X_α, Y_β and $\mathbb{R}$ are separable Hilbert spaces [7, Proposition 2.3 and 2.6], the same is true for E.

By *A*1, the Hamiltonian function $\widetilde{H}$ in (2.6) is κ_i-periodic in x_i for $i = 1, \ldots, M$, hence writing $x(t) = \bar{x} + \tilde{x}(t)$, with

$$\bar{x} = \frac{1}{\pi}\int_0^\pi x(t)\,dt\,,$$

we can assume that $\bar{x} \in \mathbb{T}^M$ and look for solutions $(z, \bar{x}) \in E \times \mathbb{T}^M$, where

$$z = (\tilde{x}, y, \bar{u}, \tilde{u}, v)\,.$$

These solutions will be found as critical points of a suitable functional, by applying the following theorem of Szulkin [18] (see also [10, 13]).

Theorem 2.2 ([18])**.** *If $\varphi : E \times \mathbb{T}^M \to \mathbb{R}$ is a continuously differentiable functional of the type*

$$\varphi(z, \bar{x}) = \frac{1}{2}\langle\mathscr{L}z, z\rangle_E + \psi(z, \bar{x})\,,$$

where $\mathscr{L} : E \to E$ is a bounded selfadjoint invertible operator and $d\psi(E \times \mathbb{T}^M)$ is relatively compact, then φ has at least $M + 1$ critical points.

2.2 The functional and the bilinear form

We define a functional $\psi : E \times \mathbb{T}^M \to \mathbb{R}$ as

$$\begin{aligned}\psi(z, \bar{x}) &= \psi\big((\tilde{x}\,, y\,, \bar{u}\,, \tilde{u}, v)\,, \bar{x}\big) \\ &= \int_0^\pi \widetilde{H}\big(t\,, \bar{x} + \tilde{x}(t)\,, y(t)\,, \bar{u} + \tilde{u}(t)\,, v(t)\big)\,dt\,.\end{aligned}$$

In the following, we will treat $\mathbb{T}^M$ as being lifted to $\mathbb{R}^M$, so $E\times\mathbb{T}^M$ will often be identified with $E\times\mathbb{R}^M$. It has been shown in [7, Proposition 2.10] and [6, Proposition 19, Proposition 22] that ψ is continuously differentiable, and the gradient function $\nabla\psi$ has a relatively compact image. In what follows we introduce the operator $\mathscr{L}$.

We first consider the space

$$D = [\widetilde{C}^1([0,\pi])]^M \times [C_0^1([0,\pi])]^M \times F_L\,,$$

where

$$F_L = \big(\mathbb{R}^L \times [\widetilde{C}^1([0,\pi])]^L\big) \times [C_0^1([0,\pi])]^L\,,$$

and define a symmetric bilinear form $\mathcal{B} : D\times D \to \mathbb{R}$ as follows. For every $z = (\tilde{x}, y, \bar{u}, \tilde{u}, v)$ and $\mathcal{Z} = (\widetilde{X}, Y, \bar{u}, \widetilde{U}, V)$ in D,

$$\mathcal{B}(z,\mathcal{Z}) = \int_0^\pi \Big[\langle y', \widetilde{X}\rangle - \langle \tilde{x}', Y\rangle - \langle J\dot{w}, W\rangle + \langle \mathbb{A}w, W\rangle\Big]\,dt\,,$$

where $w = (\bar{u}+\tilde{u}, v), W = (\bar{u}+\widetilde{U}, V)$ are in F_L.

Proposition 2.3. *The set D is a dense in E, and the bilinear form $\mathcal{B}$: $D\times D\to\mathbb{R}$ is continuous with respect to the topology of $E\times E$.*

Proof. We know by [7, Proposition 2.5 and 2.8] that D is a dense subspace of E. In order to prove the second part of the statement, let us write

$$\mathcal{B}(z,\mathcal{Z}) = \mathcal{B}_1\big((\tilde{x},y),(\widetilde{X},Y)\big) + \mathcal{B}_2(w,W)\,,$$

where

$$\mathcal{B}_1\big((\tilde{x},y),(\widetilde{X},Y)\big) = \int_0^\pi \Big(\langle y', \widetilde{X}\rangle - \langle \tilde{x}', Y\rangle\Big)dt\,, \tag{2.7}$$

and

$$\mathcal{B}_2(w,W) = \int_0^\pi \Big(-\langle J\dot{w}, W\rangle + \langle \mathbb{A}w, W\rangle\Big)dt\,. \tag{2.8}$$

It has been proved in [6, Section 3.4] that $\mathcal{B}_1$ is continuous with respect to the topology of $X^L_\alpha\times Y^L_\beta$. We need to prove that $\mathcal{B}_2$ is continuous with respect to the topology of $\mathbb{R}^L\times X^L_\alpha\times Y^L_\beta$. For $w = (w_1,\ldots,w_L)$ and $W = (W_1,\ldots,W_L)$ in F_L we have

$$\int_0^\pi \langle J\dot{w}, W\rangle dt = \sum_{j=1}^L \int_0^\pi \langle J\dot{w}_j, W_j\rangle dt\,, \tag{2.9}$$

and, writing $w_j = (\bar{u}_j + \tilde{u}_j, v_j), W_j = (\overline{U}_j + \widetilde{U}_j, V_j)$,

$$\int_0^\pi \langle J\dot{w}_j, W_j\rangle dt = \int_0^\pi \dot{u}_j V_j dt - \int_0^\pi \dot{v}_j \overline{U}_j dt - \int_0^\pi \dot{v}_j \widetilde{U}_j dt\,. \qquad (2.10)$$

We decompose the involved functions as

$$v_j = \sum_{m=1}^\infty v_m^j \sin(mt)\,, \quad V_j = \sum_{m=1}^\infty V_m^j \sin(mt)\,,$$

$$\tilde{u}_j = \sum_{m=1}^\infty \tilde{u}_m^j \cos(mt)\,, \quad \widetilde{U}_j = \sum_{m=1}^\infty \widetilde{U}_m^j \cos(mt)\,.$$

By the boundary condition $v(0) = 0 = v(\pi)$, we see that

$$\int_0^\pi \dot{v}_j \overline{U}_j dt = 0\,.$$

Recalling that $\alpha + \beta = 1$, we have

$$\begin{aligned}\left|\int_0^\pi \dot{u}_j V_j dt\right| &= \frac{\pi}{2}\left|\sum_{m=1}^\infty -m\,\tilde{u}_m^j V_m^j\right| \\ &\le \frac{\pi}{2}\sum_{m=1}^\infty \left|m^\alpha \tilde{u}_m^j m^\beta V_m^j\right| \\ &\le \frac{\pi}{2}||\tilde{u}_j||_{X_\alpha}\,||V_j||_{Y_\beta}\,,\end{aligned}$$

and

$$\begin{aligned}\left|\int_0^\pi \dot{v}_j \widetilde{U}_j dt\right| &= \frac{\pi}{2}\left|\sum_{m=1}^\infty m\,v_m^j \widetilde{U}_m^j\right| \\ &\le \frac{\pi}{2}\sum_{m=1}^\infty \left|m^\alpha \widetilde{U}_m^j m^\beta v_m^j\right| \\ &\le \frac{\pi}{2}||\widetilde{U}_j||_{X_\alpha}\,||v_j||_{Y_\beta}\,.\end{aligned}$$

Going back to (2.10), for each $j = 1, \ldots, L$, we thus have

$$\left|\int_0^\pi \langle J\dot{w}_j, W_j\rangle dt\right| \le \frac{\pi}{2}||w_j||_{\mathbb{R}\times X_\alpha\times Y_\beta}\,||W_j||_{\mathbb{R}\times X_\alpha\times Y_\beta}\,.$$

Hence, by (2.9),

$$\left|\int_0^\pi \langle J\dot{w}, W\rangle dt\right| \le \frac{\pi}{2}||w||_{\mathbb{R}^L\times X_\alpha^L\times Y_\beta^L}\,||W||_{\mathbb{R}^L\times X_\alpha^L\times Y_\beta^L}\,.$$

We have thus proved the continuity of the first part of the bilinear form defined in (2.8).

For the second part, we can write

$$\int_0^\pi \langle \mathbb{A}w, W\rangle dt = \sum_{j=1}^L \lambda_j \int_0^\pi \langle w_j, W_j\rangle \, dt \,, \tag{2.11}$$

where

$$\begin{aligned}\int_0^\pi \langle w_j, W_j\rangle \, dt &= \int_0^\pi \big\langle (\bar{u}_j + \tilde{u}_j, v_j), (\overline{U}_j + \widetilde{U}_j, V_j)\big\rangle \, dt \\ &= \int_0^\pi (\bar{u}_j + \tilde{u}_j)(\overline{U}_j + \widetilde{U}_j) dt + \int_0^\pi v_j V_j \, dt \,.\end{aligned}$$

Now for every $j = 1, \dots, L$, we have

$$\begin{aligned}&\left|\int_0^\pi (\bar{u}_j + \tilde{u}_j)(\overline{U}_j + \widetilde{U}_j) dt\right| \le \left|\int_0^\pi \bar{u}_j \overline{U}_j\right| + \left|\int_0^\pi \tilde{u}_j \widetilde{U}_j\right| \\ &\le \pi|\bar{u}_j||\overline{U}_j| + \left|\frac{\pi}{2}\sum_{m=1}^\infty \tilde{u}_m^j \widetilde{U}_m^j\right| \\ &\le \pi|\bar{u}_j||\overline{U}_j| + \frac{\pi}{2}\sum_{m=1}^\infty \left|m^\alpha \tilde{u}_m^j m^\alpha \widetilde{U}_m^j\right| \\ &\le \pi|\bar{u}_j||\overline{U}_j| + \frac{\pi}{2}||\tilde{u}_j||_{X_\alpha}||\widetilde{U}_j||_{X_\alpha} \,,\end{aligned}$$

while

$$\begin{aligned}\left|\int_0^\pi v_j V_j dt\right| &= \left|\frac{\pi}{2}\sum_{m=1}^\infty v_m^j V_{m_j}\right| \\ &\le \frac{\pi}{2}\sum_{m=1}^\infty \left|m^\beta v_m^j m^\beta V_m^j\right| \\ &\le \frac{\pi}{2}||v_j||_{Y_\beta}||V_j||_{Y_\beta} \,.\end{aligned}$$

Thus we have

$$\left|\int_0^\pi \langle w_j, W_j\rangle dt\right| \le \pi||w_j||_{\mathbb{R}\times X_\alpha\times Y_\beta}||W_j||_{\mathbb{R}\times X_\alpha\times Y_\beta} \,,$$

and, going back to (2.11),

$$\begin{aligned}\left|\int_0^\pi \langle \mathbb{A}w, W\rangle dt\right| &= \left|\sum_{j=1}^L \lambda_j \int_0^\pi \langle w_j, W_j\rangle dt\right| \\ &\le \sum_{j=1}^L \lambda_j \left|\int_0^\pi \langle w_j, W_j\rangle dt\right|\end{aligned}$$

$$\leq \sum_{j=1}^{L} \pi\lambda_j ||w_j||_{\mathbb{R}\times X_\alpha \times Y_\beta} ||W_j||_{\mathbb{R}\times X_\alpha\times Y_\beta}$$
$$\leq \pi\lambda ||w||_{\mathbb{R}^L\times X_\alpha^L\times Y_\beta^L} ||W||_{\mathbb{R}^L\times X_\alpha^L\times Y_\beta^L}\,,$$

where $\lambda = \max\{\lambda_1, \dots, \lambda_L\}$. This shows that also the second part of the bilinear form $\mathcal{B}_2 : D \times D \to \mathbb{R}$ in (2.8) is continuous, and the proof is complete. □

The bilinear form $\mathcal{B} : D \times D \to \mathbb{R}$ can thus be extended in a unique way to a continuous symmetric bilinear form $\mathcal{B} : E \times E \to \mathbb{R}$, for which we maintain the same notation. A bounded selfadjoint operator $\mathscr{L} : E \to E$ can thus be defined by

$$\langle \mathscr{L} z, \mathcal{Z} \rangle_E = \mathcal{B}(z, \mathcal{Z})\,,$$

for z and $\mathcal{Z}$ in E. Referring to (2.7) and (2.8), we can write

$$\mathscr{L}(\tilde{x}, y, \bar{u}, \tilde{u}, v) = (\mathscr{L}_1(\tilde{x}, y), \mathscr{L}_2(w))\,,$$

where

$$\langle \mathscr{L}_1(\tilde{x}, y), (\widetilde{X}, Y)\rangle_{X_\alpha^M\times Y_\beta^M} = \mathcal{B}_1\big((\tilde{x}, y), (\widetilde{X}, Y)\big)\,,$$

and

$$\langle \mathscr{L}_2(w), W\rangle_{\mathbb{R}^L\times X_\alpha^L\times Y_\beta^L} = \mathcal{B}_2(w, W)\,,$$

for every $z = (\tilde{x}, y, \bar{u}, \tilde{u}, v)$ and $\mathcal{Z} = (\widetilde{X}, Y, \overline{U}, \widetilde{U}, V)$ in E with $w = (\bar{u}, \tilde{u}, v)$, and $W = (\overline{U}, \widetilde{U}, V)$. It has been proved in [7, Proposition 2.14] that

$$||\mathscr{L}_1(\tilde{x}, y)||_{X_\alpha^M\times Y_\beta^M} = \frac{\pi}{2}||(\tilde{x}, y)||_{X_\alpha^M\times Y_\beta^M}\,. \tag{2.12}$$

We now need the following.

Lemma 2.4. *There exist positive constants $\alpha, \beta, \widetilde{\delta}$ with $\alpha < \frac{1}{2} < \beta$, and $\alpha + \beta = 1$ such that*

$$||\mathscr{L}_2(w)||_{\mathbb{R}^L\times X_\alpha^L\times Y_\beta^L} \geq \widetilde{\delta}\,||w||_{\mathbb{R}^L\times X_\alpha^L\times Y_\beta^L}\,, \tag{2.13}$$

for every $w \in \mathbb{R}^L \times X_\alpha^L \times Y_\beta^L$.

Proof. We first assume $L = 1$. Let $(\overline{\zeta}, \widetilde{\zeta}, \xi) \in \mathbb{R} \times X_\alpha \times Y_\beta$ be such that $\mathscr{L}_2(w) = (\overline{\zeta}, \widetilde{\zeta}, \xi)$, so that

$$\mathcal{B}_2(w, W) = \langle (\overline{\zeta}, \widetilde{\zeta}, \xi), W\rangle_{\mathbb{R}\times X_\alpha\times Y_\beta}\,, \tag{2.14}$$

for every $W = (U,V) \in \mathbb{R} \times X_\alpha \times Y_\beta$. Recalling that $w = (\bar{u}, \tilde{u}, v)$, we decompose

$$\tilde{u} = \sum_{m=1}^{\infty} u_m \cos(mt)\,, \quad v = \sum_{m=1}^{\infty} v_m \sin(mt)\,,$$

$$\widetilde{\zeta} = \sum_{m=1}^{\infty} \zeta_m \cos(mt)\,, \quad \xi = \sum_{m=1}^{\infty} \xi_m \sin(mt)\,.$$

By taking first $V = 0$ and then $U = 0$ in (2.14), and using (2.8), we obtain the following identities

$$\begin{cases} \overline{\zeta} = \lambda_1 \pi \bar{u}\,, \\ \zeta_m m^{2\alpha} = \frac{\pi}{2}[\lambda_1 u_m + m v_m]\,, \\ \xi_m m^{2\beta} = \frac{\pi}{2}[m u_m + \lambda_1 v_m]\,. \end{cases} \tag{2.15}$$

Thus we have

$$\zeta_m m^{\alpha} = \frac{\pi}{2}[\lambda_1 m^{-\alpha} u_m + m^{\beta} v_m]\,, \quad \xi_m m^{\beta} = \frac{\pi}{2}[m^{\alpha} u_m + \lambda_1 m^{-\beta} v_m]\,,$$

and, by using the Young inequality,

$$\begin{aligned} \zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} =& \frac{\pi^2}{4}\Big[\lambda_1^2 m^{-2\alpha} u_m^2 + m^{2\beta} v_m^2 + m^{2\alpha} u_m^2 \\ &+ \lambda_1^2 m^{-2\beta} v_m^2 + 2\lambda_1 [m^{\alpha-\beta} + m^{\beta-\alpha}] u_m v_m\Big] \\ \geq& \frac{\pi^2}{4}\Big[\lambda_1^2 m^{-2\alpha} u_m^2 + m^{2\beta} v_m^2 + m^{2\alpha} u_m^2 \\ &+ \lambda_1^2 m^{-2\beta} v_m^2 - \lambda_1 [m^{\alpha-\beta} + m^{\beta-\alpha}]\big(u_m^2 + v_m^2\big)\Big] \\ =& \frac{\pi^2}{4} m^{-4\alpha}\Big[(\lambda_1 - m)\big(\lambda_1 - m^{4\alpha-1}\big)\Big] m^{2\alpha} u_m^2 \\ &+ \frac{\pi^2}{4} m^{-4\beta}\Big[(\lambda_1 - m)\big(\lambda_1 - m^{4\beta-1}\big)\Big] m^{2\beta} v_m^2\,. \end{aligned} \tag{2.16}$$

By hypothesis, we know that there exists a positive integer n_1 such that

$$n_1 < \lambda_1 < n_1 + 1\,.$$

We now discuss separately the cases for $n_1 = 0$ and $n_1 \geq 1$.

Case 1. If $n_1 = 0$, then $0 < \lambda_1 < 1$, and so $\lambda_1 < m$ for all $m \geq 1$. Now for $m = 1$, (2.16) implies that

$$\zeta_1^2 + \xi_1^2 \geq \frac{\pi^2}{4}(\lambda_1 - 1)^2 (u_1^2 + v_1^2)\,. \tag{2.17}$$

For $m \geq 2$, we have

$$(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1}) > (1-m)(1-m^{4\alpha-1}) = (m-1)(m^{4\alpha-1}-1)\,.$$

By writing $m^{-4\alpha} = m^{-1}m^{-4\alpha+1}$, and choosing α such that

$$\frac{1}{4}\left(\frac{\log(4/3)}{\log 2}+1\right) < \alpha < \frac{1}{2}\,,$$

we have

$$\begin{aligned} m^{-4\alpha}(\lambda_1 - m)\big(\lambda_1 - m^{4\alpha-1}\big) &> \left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^{4\alpha-1}}\right) \\ &\geq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{2^{4\alpha-1}}\right) \geq \frac{1}{8} \geq \frac{\lambda_1^2}{8}\,, \end{aligned}$$

since $\lambda_1 < 1$. Similarly, since $\beta > \frac{1}{2} > \alpha$, we get

$$m^{-4\beta}(\lambda_1 - m)\big(\lambda_1 - m^{4\beta-1}\big) \geq \frac{\lambda_1^2}{8},$$

and thus (2.16) implies that

$$\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} \geq \frac{\pi^2}{4}\frac{\lambda_1^2}{8}\big[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\big]\,. \tag{2.18}$$

Combining (2.17), (2.18), and the first identity in (2.15), we have

$$\begin{aligned} &||\mathscr{L}_2(\bar{u},\tilde{u},v)||^2_{\mathbb{R}\times X_\alpha\times Y_\beta} = |\bar{\zeta}|^2 + ||\widetilde{\zeta}||^2_{X_\alpha} + ||\widetilde{\zeta}||^2_{Y_\beta} \\ &= \pi^2\lambda_1^2|\bar{u}|^2 + (\zeta_1^2+\xi_1^2) + \sum_{m=2}^{\infty}\big(\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta}\big) \\ &\geq \frac{\pi^2}{4}\frac{\lambda_1^2}{8}\left[|\bar{u}|^2 + \left(1-\frac{1}{\lambda_1}\right)^2[u_1^2+v_1^2] + \sum_{m=2}^{\infty}\big(u_m^2 m^{2\alpha} + v_m^2 m^{2\beta}\big)\right] \\ &\geq \widetilde{\delta}^2\left[|\bar{u}|^2 + \sum_{m=1}^{\infty}\big(u_m^2 m^{2\alpha} + v_m^2 m^{2\beta}\big)\right] = \widetilde{\delta}^2||(\bar{u},\tilde{u},v)||^2_{\mathbb{R}\times X_\alpha\times Y_\beta}\,, \end{aligned}$$

where

$$\widetilde{\delta} = \frac{\pi}{8}\lambda_1 \min\left\{1, \left|1-\frac{1}{\lambda_1}\right|\right\}.$$

This implies that (2.13) holds in this case, for $L = 1$.

Case 2. If $n_1 \geq 1$, then for $m \in \{1,\ldots,n_1\}$ we have $\lambda_1 - m \geq \lambda_1 - n_1 > 0$, and so

$$\lambda_1 - m^{4\alpha-1} \geq \lambda_1 - m \geq \lambda_1 - n_1 > 0\,.$$

This implies that

$$m^{-4\alpha}(\lambda_1 - m)\big(\lambda_1 - m^{4\alpha-1}\big) \geq n_1^{-4\alpha}(\lambda_1 - n_1)^2 \geq n_1^{-4\beta}(\lambda_1 - n_1)^2 \,.$$

By choosing β such that

$$\frac{1}{2} < \beta < \frac{1}{4}\left(\frac{\log\big(\frac{1}{2}(\lambda_1 + n_1)\big)}{\log n_1} + 1\right), \tag{2.19}$$

we obtain that $\lambda_1 - m^{4\beta-1} \geq \lambda_1 - n_1^{4\beta-1} > \frac{1}{2}(\lambda_1 - n_1) > 0$, and so

$$m^{-4\beta}(\lambda_1 - m)\big(\lambda_1 - m^{4\beta-1}\big) \geq n_1^{-4\beta}\frac{1}{2}(\lambda_1 - n_1)^2 \,.$$

Thus, for $m \in \{1, \dots, n_1\}$, (2.16) and (2.19) imply that

$$\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} \geq \frac{\pi^2}{8} n_1^{-4\beta}(\lambda_1 - n_1)^2 \big[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\big] \,. \tag{2.20}$$

For $m = n_1 + 1$, we have $\lambda_1 - m = \lambda_1 - (n_1 + 1) < 0$, and so

$$\lambda_1 - m^{4\beta-1} = \lambda_1 - (n_1 + 1)^{4\beta-1} < \lambda_1 - (n_1 + 1) < 0 \,.$$

This implies that

$$m^{-4\beta}(\lambda_1 - m)\big(\lambda_1 - m^{4\beta-1}\big) \geq (n_1 + 1)^{-4\beta}(\lambda_1 - (n_1 + 1))^2 \,.$$

By choosing α such that

$$\frac{1}{4}\left(\frac{\log\big(\frac{1}{2}(\lambda_1 + n_1 + 1)\big)}{\log(n_1 + 1)} + 1\right) \leq \alpha < \frac{1}{2} \,, \tag{2.21}$$

we obtain

$$\lambda_1 - m^{4\alpha-1} = \lambda_1 - (n_1 + 1)^{4\alpha-1} \leq \frac{1}{2}\big(\lambda_1 - (n_1 + 1)\big) < 0 \,,$$

and so

$$\begin{aligned} m^{-4\alpha}(\lambda_1 - m)\big(\lambda_1 - m^{4\alpha-1}\big) &\geq (n_1 + 1)^{-4\alpha}\frac{1}{2}\big(\lambda_1 - (n_1 + 1)\big)^2 \\ &\geq (n_1 + 1)^{-4\beta}\frac{1}{2}\big(\lambda_1 - (n_1 + 1)\big)^2 \,. \end{aligned}$$

Thus, for $m = n_1 + 1$, (2.16) and (2.21) imply that

$$\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} \geq \frac{\pi^2}{8}(n_1 + 1)^{-4\beta}(\lambda_1 - (n_1 + 1))^2 \big[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\big] \,. \tag{2.22}$$

Lastly, for $m \geq n_1 + 2$, by choosing α such that

$$\frac{1}{4}\bigg(\frac{\log\big(\frac{2(n_1+1)(n_1+2)}{2n_1+3}\big)}{\log(n_1+2)} + 1\bigg) \leq \alpha < \frac{1}{2}\,, \tag{2.23}$$

we have

$$\begin{aligned}(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1}) &> (n_1 + 1 - m)(n_1 + 1 - m^{4\alpha-1})\\ &= (m - (n_1+1))(m^{4\alpha-1} - (n_1+1))\,,\end{aligned}$$

and, writing $m^{-4\alpha} = m^{-1}\, m^{-4\alpha+1}$,

$$\begin{aligned}m^{-4\alpha}(\lambda_1 - m)\big(\lambda_1 - m^{4\alpha-1}\big) &> \bigg(1 - \frac{n_1+1}{m}\bigg)\bigg(1 - \frac{n_1+1}{m^{4\alpha-1}}\bigg)\\ &\geq \bigg(1 - \frac{n_1+1}{n_1+2}\bigg)\bigg(1 - \frac{n_1+1}{(n_1+2)^{4\alpha-1}}\bigg)\\ &\geq \frac{1}{2}\bigg(1 - \frac{n_1+1}{n_1+2}\bigg)^2 = \frac{1}{2}\frac{1}{(n_1+2)^2}\,.\end{aligned}$$

Similarly, since $\beta > \frac{1}{2} > \alpha$, we obtain

$$m^{-4\beta}(\lambda_1 - m)\big(\lambda_1 - m^{4\beta-1}\big) \geq \frac{1}{2}\frac{1}{(n_1+2)^2}\,.$$

Hence for $m \geq n_1 + 2$, (2.16) and (2.23) imply that

$$\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} \geq \frac{\pi^2}{8}\frac{1}{(n_1+2)^2}\big[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\big]\,. \tag{2.24}$$

Combining (2.20), (2.22), (2.24), and the first identity in (2.15) we have

$$\begin{aligned}&||\mathscr{L}_2(\bar{u},\tilde{u},v)||^2_{\mathbb{R}\times X_\alpha\times Y_\beta} = |\bar{\zeta}|^2 + ||\widetilde{\zeta}||^2_{X_\alpha} + ||\widetilde{\zeta}||^2_{Y_\beta}\\ &= \pi^2\lambda_1^2|\bar{u}|^2 + \sum_{m=1}^{n_1}\big[\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta}\big] +\\ &\quad + \big[\zeta_{n_1+1}^2(n_1+1)^{2\alpha} + \xi_{n_1+1}^2(n_1+1)^{2\beta}\big] + \sum_{m=n_1+2}^{\infty}\big[\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta}\big]\\ &\geq \frac{\pi^2}{8}|\bar{u}|^2 + \frac{\pi^2}{8}(n_1+1)^{-4\beta}\bigg[\bigg(1 - \frac{n_1}{\lambda_1}\bigg)^2\sum_{m=1}^{n_1}\big[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\big]\\ &\quad + \bigg(1 - \frac{n_1+1}{\lambda_1}\bigg)^2\Big[(n_1+1)^{2\alpha}u_{n_1+1}^2 + (n_1+1)^{2\beta}v_{n_1+1}^2\Big]\bigg] +\\ &\quad + \frac{\pi^2}{8}\frac{1}{(n_1+2)^2}\sum_{m=n_1+2}^{\infty}\big[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\big]\\ &\geq \widetilde{\delta}^2\bigg[|\bar{u}|^2 + \sum_{m=1}^{\infty}\big(u_m^2 m^{2\alpha} + v_m^2 m^{2\beta}\big)\bigg] = \widetilde{\delta}^2||(\bar{u},\tilde{u},v)||^2_{\mathbb{R}\times X_\alpha\times Y_\beta}\,,\end{aligned}$$

where

$$\widetilde{\delta} = \frac{\pi}{2\sqrt{2}} \min\left\{\frac{1}{n_1+2}, (n_1+1)^{-2\beta}\Big|1-\frac{n_1+1}{\lambda_1}\Big|, (n_1+1)^{-2\beta}\Big|1-\frac{n_1}{\lambda_1}\Big|\right\}.$$

This implies that (2.13) holds also in this case, for $L = 1$.

Finally, by using (2.9) and (2.11), we can easily see that (2.13) holds for any $L \geq 1$. □

By combining (2.12) and (2.13) in Lemma 2.4, we can say that the selfadjoint operator $\mathscr{L} : E \to E$ is invertible, and the inverse operator $\mathscr{L}^{-1} : E \to E$ is continuous.

By Theorem 2.2, we conclude that the functional φ has at least $M+1$ critical points. Arguing as in [6, Proposition 24], it can be seen that these critical points correspond to the solutions of the boundary value problem (2.5)-(2.3) that we are looking for. The proof of Theorem 2.1 is thus completed. □

3 Proof of Theorem 1.1

Without loss of generality, we may assume that $[a, b] = [0, \pi]$. We start assuming $L = 1$, and we first work on the planar system (1.3) so to transform it, by a symplectic change of variables, into a linear one. We will follow the approach developed in [1, 8, 11].

3.1 A symplectic change of variables

By using $A4$, we have that $\mathscr{H}(0,0) = 0$ and the generalized Euler Identity holds true, i.e.,

$$\left\langle \nabla\mathscr{H}(u,v), \left(\frac{u}{p}, \frac{v}{q}\right)\right\rangle = \mathscr{H}(u,v)\,. \tag{3.1}$$

Choose the positive constant

$$\Upsilon = \min\left\{\frac{1}{|w|^2}\mathscr{H}(w) : 1 \leq |w| \leq 2\right\}\,, \tag{3.2}$$

and let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $\eta'(s) \leq 0$ for all $s \in \mathbb{R}$ and

$$\eta(s) = \begin{cases} 1\,, & \text{if } s \leq 1\,, \\ 0\,, & \text{if } s \geq 2\,. \end{cases}$$

For $w=(u,v)$, set

$$\widehat{\mathscr{H}}(w)=\eta(|w|)\Upsilon|w|^2+(1-\eta(|w|))\mathscr{H}(w)\,, \tag{3.3}$$

and consider the new system

$$J\dot{w}=\nabla\widehat{\mathscr{H}}(w)\,. \tag{3.4}$$

Notice that $\widehat{\mathscr{H}}(0)=0$. For every $w\neq 0$, we have

$$\nabla\widehat{\mathscr{H}}(w)=\Big(\Upsilon\eta'(|w|)|w|+2\Upsilon\eta(|w|)-\frac{\eta'(|w|)}{|w|}\mathscr{H}(w)\Big)w+(1-\eta(|w|))\nabla\mathscr{H}(w)\,.$$

Then, using (3.1) and (3.2), if $w=(u,v)$ is such that $1\leq|w|\leq 2$, we have

$$\begin{aligned}\left\langle\nabla\widehat{\mathscr{H}}(w),\left(\frac{u}{p},\frac{v}{q}\right)\right\rangle &= \eta'(|w|)|w|\left(\frac{u^2}{p}+\frac{v^2}{q}\right)\big(\Upsilon-\frac{1}{|w|^2}\mathscr{H}(w)\big)\\ &\quad+2\eta(|w|)\,\Upsilon\left(\frac{u^2}{p}+\frac{v^2}{q}\right)+(1-\eta(|w|))\mathscr{H}(w)>0\,.\end{aligned}$$

This implies that $\nabla\widehat{\mathscr{H}}(w)\neq 0$, for $1\leq|w|\leq 2$. For $0<|w|\leq 1$, the Hamiltonian function $\widehat{\mathscr{H}}$ is quadratic, so that $\nabla\widehat{\mathscr{H}}(w)\neq 0$. Lastly, for $|w|\geq 2$, we have $\nabla\widehat{\mathscr{H}}(w)=\nabla\mathscr{H}(w)$, and it is clear from (3.1) that $\nabla\mathscr{H}(w)\neq 0$. Hence $\nabla\widehat{\mathscr{H}}(w)\neq 0$ for every $w\neq 0$, and this shows that every non-zero solution of system (3.4) does not pass through the origin, and by Poincaré–Bendixson theory, all the solutions of system (3.4) are periodic. Thus the origin is still a global center for the system (3.4).

Now for any $w_0\in\mathbb{R}^2\setminus\{0\}$, we denote by $\widehat{T}(w_0)$ the minimal period of the solution of (3.4) passing through w_0. We notice here that this solution is unique, even if we are not assuming $\nabla\mathscr{H}$ to be locally Lipschitz continuous, cf. [16]. The function $\widehat{T}:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ thus defined is continuously differentiable (see [1]).

Define

$$\delta^\star=[0,+\infty[\times\{0\}\,,$$

and a function $\xi:]0,+\infty[\to]0,+\infty[$ as follows: for every $r>0$, the level line $\{w\in\mathbb{R}^2:\widehat{\mathscr{H}}(w)=r\}$ intersects $\delta^\star$ at the point $(\xi(r),0)$. Such a point is unique, because for every $(\xi,0)\in\delta^\star$ with $\xi\neq 0$ we have

$$\left\langle\nabla\widehat{\mathscr{H}}(\xi,0),\left(\frac{\xi}{p},0\right)\right\rangle>0\,,$$

which implies that

$$\left\langle \nabla\widehat{\mathscr{H}}(\xi,0),(\xi,0)\right\rangle > 0\,.$$

Thus, if $w(t_0) = (u(t_0), v(t_0)) = (u(t_0), 0)$ is such that $u(t_0) > 0$, then $v'(t_0) < 0$, and so it is impossible for the level line $\{w \in \mathbb{R}^2 : \widehat{\mathscr{H}}(w) = r\}$ to intersect $\delta^\star$ at two different points.

Now define $\widehat{K} : \mathbb{R}^2 \to \mathbb{R}$ as

$$\widehat{K}(w) = \frac{1}{\tau}\int_0^{\widehat{\mathscr{H}}(w)} \widehat{T}(\xi(r),0)\,dr\,.$$

This function is continuously differentiable, and

$$\nabla\widehat{K}(w) = \frac{\widehat{T}(w)}{\tau}\nabla\widehat{\mathscr{H}}(w)\,.$$

Hence, the origin is an *isochronous* center for the system

$$J\dot{w} = \nabla\widehat{K}(w)\,, \tag{3.5}$$

since all solutions except the equilibrium 0 are periodic with minimal period τ. Moreover,

$$\widehat{K}(w) = \frac{\pi}{\tau}|w|^2, \text{ if } |w| \le 1\,.$$

Now, for every $w_0 \in \mathbb{R}^2 \setminus \{0\}$, let $\zeta(t\,; w_0)$ be the solution of system (3.5) satisfying $\zeta(0\,; w_0) = w_0$, and define $\theta(w_0) \in [0, 2\pi[$ as the minimum time for which

$$\zeta\Big(-\frac{\tau}{2\pi}\theta(w_0)\,; w_0\Big) \in \delta^\star.$$

As shown in [1], the restricted function $\theta : \mathbb{R}^2 \setminus \delta^\star \to]0, 2\pi[$ is continuously differentiable, and its gradient $\nabla\theta$ can be continuously extended to $\mathbb{R}^2 \setminus \{0\}$. We will still denote this extension by $\nabla\theta : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$.

Hence, by [1, Proposition 2.2.], there exists a symplectic diffeomorphism $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Lambda(w) = \begin{cases} \sqrt{\dfrac{\tau}{\pi}\widehat{K}(w)}\big(\cos\theta(w),\, -\sin\theta(w)\big), & \text{if } w \neq 0\,,\\ 0\,, & \text{if } w = 0\,, \end{cases}$$

such that, by the change of variable $z = \Lambda(w)$, system (3.5) is changed to the linear one

$$J\dot{z} = \frac{2\pi}{\tau}z\,.$$

3.2 The proof in the case $L = 1$

By $A3$ and a standard compactness argument, we can modify the function $\mathcal{H}$ as in (2.4) so to obtain the modified system

$$\begin{cases} \dot{x} = \nabla_y \widehat{\mathcal{H}}(t,x,y) + \nabla_y P(t,x,y,u,v)\,, \\ \dot{y} = -\nabla_x \widehat{\mathcal{H}}(t,x,y) - \nabla_x P(t,x,y,u,v)\,, \\ \dot{u} = \nabla_v \mathscr{H}(u,v) + \nabla_v P(t,x,y,u,v)\,, \\ \dot{v} = -\nabla_u \mathscr{H}(u,v) - \nabla_u P(t,x,y,u,v)\,. \end{cases} \tag{3.6}$$

Using the argument in [5, Section 3], it can be seen that all the solutions of this system are globally defined. Moreover, those satisfying the boundary conditions

$$\begin{cases} y(0) = 0 = y(\pi)\,, \\ v(0) = 0 = v(\pi) \end{cases} \tag{3.7}$$

are solutions of the original system (1.1).

Recalling the change of variables $\Lambda(w) = z$ in Section 3.1, we define a map

$$\widetilde{P}(t,x,y,z) = P(t,x,y,\Lambda^{-1}(z))\,.$$

Lemma 3.1. *The function $\widetilde{P}$ has a bounded gradient with respect to (q,p,z).*

Proof. Clearly, by $A2$ both

$$\partial_x \widetilde{P}(t,x,y,z) = \partial_x P(t,x,y,\Lambda^{-1}(z))\,, \quad \partial_y \widetilde{P}(t,x,y,z) = \partial_y P(t,x,y,\Lambda^{-1}(z))$$

are bounded and denoting by $\mathbb{M}^*$, the transpose of a matrix $\mathbb{M}$,

$$\begin{aligned} \nabla_z \widetilde{P}(t,x,y,z) &= \left[(\Lambda^{-1}(z))'\right]^* \nabla_w P(t,x,y,\Lambda^{-1}(z)) \\ &= \left[(\Lambda'(\Lambda^{-1}(z)))^*\right]^{-1} \nabla_w P(t,x,y,\Lambda^{-1}(z)))\,. \end{aligned}$$

Again by $A2$, $\nabla_w P(t,x,y,w)$ is bounded, so it is sufficient to show that $(\Lambda'(w))^{-1}$ is bounded. For $|w|$ large enough, we have that $\widehat{K}(w) = \mathscr{H}(w)$. By denoting $c(w) = \cos\theta(w)$ and $s(w) = \sin\theta(w)$, we have

$$\Lambda'(w) = \begin{bmatrix} a_{11}(w) & a_{12}(w) \\ a_{21}(w) & a_{22}(w) \end{bmatrix}\,,$$

where

$$a_{11}(w) = \sqrt{\frac{\tau}{\pi}}\left(\frac{\partial_u \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,c(w) - \sqrt{\mathscr{H}(w)}\partial_u\theta(w)s(w)\right),$$

$$a_{12}(w) = \sqrt{\frac{\tau}{\pi}}\left(\frac{\partial_v \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,c(w) - \sqrt{\mathscr{H}(w)}\partial_v\theta(w)s(w)\right),$$

$$a_{21}(w) = \sqrt{\frac{\tau}{\pi}}\left(-\frac{\partial_u \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,s(w) - \sqrt{\mathscr{H}(w)}\partial_u\theta(w)c(w)\right),$$

$$a_{22}(w) = \sqrt{\frac{\tau}{\pi}}\left(-\frac{\partial_v \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}\,s(w) - \sqrt{\mathscr{H}(w)}\partial_v\theta(w)c(w)\right).$$

Recalling that Λ is symplectic, so $\det \Lambda'(w) = 1$, the inverse matrix is

$$(\Lambda'(w))^{-1} = \begin{bmatrix} a_{22}(w) & -a_{12}(w) \\ -a_{21}(w) & a_{11}(w) \end{bmatrix}.$$

From the definition of θ, for $w \neq 0$ and $\gamma > 0$ we see that $\theta(\gamma^q u, \gamma^p v) = \theta(u, v)$. Indeed, if $w(t) = (u(t), v(t))$ is a solution of system (3.5), then $w_\gamma = (\gamma^q u, \gamma^p v)$ is also a solution of system (3.5) with the vertical speed of $\gamma^p \dot{v}(t)$. Hence, if $w(t)$ needs a time $\frac{\tau}{2\pi}\theta(u_0, v_0)$ to go from $\delta^\star$ to (u_0, v_0) (it has a vertical speed $\dot{v}(t)$), then the time for $w_\gamma(t)$ to go from $\delta^\star$ to $(\gamma^q u_0, \gamma^p v_0)$ must be the same, since its vertical speed is just γ^p times the vertical speed of $w(t)$. Thus we have

$$\partial_u\theta(\gamma^q u, \gamma^p v)\gamma^q = \partial_u\theta(u, v), \qquad \partial_v\theta(\gamma^q u, \gamma^p v)\gamma^p = \partial_v\theta(u, v),$$

for every $\gamma > 0$. For $w = (u, v)$ with $|w| \geq 2$, since $\mathscr{H}$ is positively-(p, q)-homogeneous, the following identities have been proved in [5]:

$$\frac{\partial \mathscr{H}}{\partial u}(\gamma^q u, \gamma^p v) = \gamma^{q(p-1)}\frac{\partial \mathscr{H}}{\partial u}(u, v) = \gamma^p\frac{\partial \mathscr{H}}{\partial u}(u, v),$$

$$\frac{\partial \mathscr{H}}{\partial v}(\gamma^q u, \gamma^p v) = \gamma^{p(q-1)}\frac{\partial \mathscr{H}}{\partial v}(u, v) = \gamma^q\frac{\partial \mathscr{H}}{\partial v}(u, v).$$

Thus we have

$$|a_{22}(w)| \leq \sqrt{\frac{\tau}{\pi}}\left(\frac{|\partial_v \mathscr{H}(w)|}{2\sqrt{\mathscr{H}(w)}} + \sqrt{\mathscr{H}(w)}\,|\partial_v\theta(w)|\right)$$

$$= \sqrt{\frac{\tau}{\pi}}\frac{|w|^q\left|\partial_v\mathscr{H}\left(\frac{u}{|w|^q}, \frac{v}{|w|^p}\right)\right|}{2|w|^{p+q}\sqrt{\mathscr{H}\left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}}\right)}} +$$

$$+ \sqrt{\frac{\tau}{\pi}}\frac{|w|^{p+q}}{|w|^{p(p+q)}}\sqrt{\mathscr{H}\left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}}\right)}\partial_v\theta\left(\frac{u}{|w|^{q(p+q)}}, \frac{v}{|w|^{p(p+q)}}\right)$$

$$
\leq \sqrt{\frac{\tau}{\pi}} \frac{\left|\partial_v \mathscr{H}\left(\frac{u}{|w|^q}, \frac{v}{|w|^p}\right)\right|}{2\sqrt{\mathscr{H}\left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}}\right)}} +
$$

$$
+ \sqrt{\frac{\tau}{\pi}} \sqrt{\mathscr{H}\left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}}\right)} \partial_v \theta\left(\frac{u}{|w|^{q(p+q)}}, \frac{v}{|w|^{p(p+q)}}\right).
$$

Define three types of sets as follow:

$$
S = \left\{ \left(\frac{u}{|w|^q}, \frac{v}{|w|^p}\right) : w = (u, v), |w| \geq 1 \right\},
$$

$$
S' = \left\{ \left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}}\right) : w = (u, v), |w| \geq 1 \right\},
$$

and

$$
S'' = \left\{ \left(\frac{u}{|w|^{q(p+q)}}, \frac{v}{|w|^{p(p+q)}}\right) : w = (u, v), |w| \geq 1 \right\}.
$$

It is easy to see that the sets S, S', and S'' are subsets of the closed unit ball $\overline{B}(0,1)$ of $\mathbb{R}^2$. This implies that $|a_{22}(w)|$ is bounded, since the functions $\mathscr{H}$ and θ are C^1. Similarly we can show that all the other elements of the matrix $(\Lambda'(w))^{-1}$ are bounded, which thus proves that the map $\widetilde{P}$ has a bounded gradient with respect to z. □

Now we consider the modified system

$$
\begin{cases}
\dot{x} = \nabla_y \widehat{\mathcal{H}}(t, x, y) + \nabla_y \widetilde{P}(t, x, y, \xi, \zeta), \\
\dot{y} = -\nabla_x \widehat{\mathcal{H}}(t, x, y) - \nabla_x \widetilde{P}(t, x, y, \xi, \zeta), \\
\dot{\xi} = \frac{2\pi}{\tau}\zeta + \partial_\zeta \widetilde{P}(t, x, y, \xi, \zeta), \\
\dot{\zeta} = -\frac{2\pi}{\tau}\xi - \partial_{\xi_j} \widetilde{P}(t, x, y, \xi, \zeta),
\end{cases} \tag{3.8}
$$

where $z = (\xi, \zeta)$. By the assumption $\tau_+ = \tau_-$, the boundary conditions become

$$
\begin{cases}
y(0) = 0 = y(\pi), \\
\zeta(0) = 0 = \zeta(\pi).
\end{cases} \tag{3.9}
$$

Thus, by taking $\lambda_1 = \frac{2\pi}{\tau}$, all the assumptions of Theorem 2.1 are satisfied, so that the boundary value problem (3.8)-(3.9) has at least $M+1$ geometrically distinct solutions.

Recalling that Λ is a diffeomorphism, we can apply the inverse change of variables $w = \Lambda^{-1}(z)$, and obtain the $M+1$ geometrically distinct solutions of system (3.6) satisfying the boundary conditions (3.7) we were looking for. This completes the proof of Theorem 1.1 in the case $L = 1$. $\square$

3.3 The proof in the higher dimensional case

We now consider the case $L \geq 2$, for which we will follow briefly the lines of the proof in the previous section. We can define $\widehat{\mathscr{H}_j}$ as in (3.3) and consider the new system

$$J\dot{\zeta} = \nabla \widehat{\mathscr{H}_j}(\zeta)\,.$$

We can define $\widehat{K}_j : \mathbb{R}^2 \to \mathbb{R}$ as

$$\widehat{K}_j(\zeta) = \frac{1}{\tau_j} \int_0^{\widehat{\mathscr{H}_j}(\zeta)} \widehat{T}_j(\xi_j(r), 0)\, dc\,,$$

so that the origin is an isochronous center for the system

$$J\dot{\zeta} = \nabla \widehat{K}_j(\zeta)\,, \tag{3.10}$$

i.e., for every $j \in \{1, \dots, L\}$, all solutions of system (3.10) except the origin are periodic and have the same minimal period τ_j. Now, for every $j \in \{1, \dots, L\}$, there exists a symplectic diffeomorphism $\Lambda_j : \mathbb{R}^2 \to \mathbb{R}^2$ such that by the change of variables $\rho = \Lambda_j(\zeta)$, system (3.10) becomes

$$J\dot{\rho} = \frac{2\pi}{\tau_j}\rho\,.$$

By the use of a cut-off function, we modify the Hamiltonian $\mathcal{H}$ like in (2.4), so that the new Hamiltonian $\widehat{\mathcal{H}}$ has a bounded gradient with respect to (x, y).

Defining $\Lambda : \mathbb{R}^{2L} \to \mathbb{R}^{2L}$ by

$$\Lambda(u, v) = \left(\Lambda_1(u_1, v_1), \dots, \Lambda_L(u_L, v_L)\right)\,,$$

we see that Λ is a symplectic diffeomorphism. By writing

$$\widetilde{P}(t, x, y, z) = P(t, x, y, \Lambda^{-1}(z))\,,$$

as in Lemma 3.1, we can show that the function $\widetilde{P}$ has a bounded gradient with respect to (x, y, z).

We apply the change of variables $z = \Lambda(w)$ and write $z = (\xi, \zeta)$ with

$$\xi = (\xi_1, \dots, \xi_L), \quad \zeta = (\zeta_1, \dots, \zeta_L),$$

so to obtain the modified system

$$\begin{cases} \dot{x} = \nabla_y \widehat{\mathcal{H}}(t,x,y) + \nabla_y \widetilde{P}(t,x,y,z), \\ \dot{y} = -\nabla_x \widehat{\mathcal{H}}(t,x,y) - \nabla_x \widetilde{P}(t,x,y,z), \\ \dot{\xi}_j = \dfrac{2\pi}{\tau_j}\zeta_j + \partial_{\zeta_j}\widetilde{P}(t,x,y,z), \quad j = 1, \dots, L, \\ \dot{\zeta}_j = -\dfrac{2\pi}{\tau_j}\xi_j - \partial_{\xi_j}\widetilde{P}(t,x,y,z), \quad j = 1, \dots, L. \end{cases} \tag{3.11}$$

Moreover, since $\tau_{j_+} = \tau_{j_-}$, the boundary conditions become the same as those in (3.9). Hence, by taking $\lambda_j = \frac{2\pi}{\tau_j}$, Theorem 2.1 implies that the modified system (3.11) has at least $M+1$ geometrically distinct solutions satisfying the boundary conditions (3.9).

Recalling that Λ is a diffeomorphism, we can apply the inverse change of variables $w = \Lambda^{-1}(z)$ and obtain the solutions of problem (1.1)-(1.2) we are looking for. □

4 The periodic problem

In this section, we consider the Hamiltonian system (1.1), where besides the regularity assumptions already made on the functions involved, we assume that all these functions are T-periodic in t. While maintaining assumptions $A1$, $A2$ and $A4$ we will reinforce assumption $A3$ by a *twist condition*, and for this we first recall some definitions.

By a convex body of $\mathbb{R}^M$, we mean a closed convex bounded subset $\mathcal{D}$ of $\mathbb{R}^M$ having nonempty interior. If in addition, $\mathcal{D}$ has a smooth boundary, then we denote the unit outward normal at $\zeta \in \partial\mathcal{D}$ by $\nu_{\mathcal{D}}(\zeta)$. Moreover, we say that $\mathcal{D}$ is strongly convex if for any $p \in \partial\mathcal{D}$, the map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ defined by $\mathcal{F}(\xi) = \langle \xi - p, \nu_{\mathcal{D}}(p)\rangle$ has a unique maximum point at $\xi = p$. Below is our *twist condition*.

B3′. There are a strongly convex body $\mathcal{D}$ of $\mathbb{R}^M$ having a smooth boundary and a symmetric regular $M \times M$ matrix $\mathbb{B}$ such that for every C^1-function $\mathcal{W} : [0,T] \to \mathbb{R}^{2L}$, all the solutions (x,y) of system

$$\begin{cases} \dot{x} = \nabla_y \mathcal{H}(t,x,y) + \nabla_y P(t,x,y,\mathcal{W}(t)), \\ \dot{y} = -\nabla_x \mathcal{H}(t,x,y) - \nabla_x P(t,x,y,\mathcal{W}(t)), \end{cases} \tag{4.1}$$

starting with $y(0) \in \mathcal{D}$ are defined on $[0, T]$, and

$$y(0) \in \partial\mathcal{D} \quad \Rightarrow \quad \langle x(T) - x(0) \, , \, \mathbb{B}\nu_{\mathcal{D}}(y(0)) \rangle > 0 \, .$$

Here is our first result for the periodic problem.

Theorem 4.1. *Assume that A1, A2, B3′ and A4 hold true, and let*

$$\frac{T}{\tau_j} \notin \mathbb{N} \, , \qquad \textit{for every } j \in \{1, \dots, L\} \, .$$

Then there are at least $M + 1$ *geometrically distinct* T*-periodic solutions of system* (1.1)*, with* $y(0) \in \mathring{\mathcal{D}}$.

Proof. Following the lines of the proof of Theorem 1.1, we modify the problem so to have a coupling with a perturbed linear system. Then, [4, Corollary 2.4] applies (instead of Theorem 2.1), and the proof is readily completed. □

We can state some variants of Theorem 4.1 replacing the twist assumption $B3'$ by $B3''$ or by $B3'''$ given below.

$B3''$. There exists a convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, such that for $\sigma \in \{-1, 1\}$ and for every C^1-function $\mathcal{W} : [0, T] \to \mathbb{R}^{2L}$, all the solutions (x, y) of system (4.1) starting with $y(0) \in \mathcal{D}$ are defined on $[0, T]$, and

$$y(0) \in \partial\mathcal{D} \quad \Rightarrow \quad x(T) - x(0) \notin \{\sigma\lambda\nu_{\mathcal{D}}(y(0)) : \lambda \geq 0\} \, .$$

$B3'''$. Let $\mathcal{D}$ be a rectangle in $\mathbb{R}^M$, i.e.

$$\mathcal{D} = [c_1, d_1] \times \cdots \times [c_M, d_M] \, .$$

There exists an M-tuple $\sigma = (\sigma_1, \dots, \sigma_M) \in \{-1, 1\}^M$ such that for every C^1-function $\mathcal{W} : [0, T] \to \mathbb{R}^{2L}$, all the solutions (x, y) of system (4.1) starting with $y(0) \in \mathcal{D}$ are defined on $[0, T]$, and, for every $i = 1, \dots, M$, we have

$$\begin{cases} y_i(0) = c_i \quad \Rightarrow \quad \sigma_i(x_i(T) - x_i(0)) < 0 \, , \\ y_i(0) = d_i \quad \Rightarrow \quad \sigma_i(x_i(T) - x_i(0)) > 0 \, . \end{cases}$$

The proofs of such results are similar to those of [8, Theorem 4.2, Theorem 4.3], so we avoid them for briefness.

5 Some possible applications

As an example of application of Theorem 1.1, we consider the following system for $L = M = 1$:

$$\begin{cases} \dot{x} = f(y) + E(t)\,, \quad \dot{y} = -A\sin x - \partial_x P(t,x,u)\,, \\ \dot{u} = |v|^{q-2}v\,, \quad \dot{v} = -\mu(u^+)^{p-1} + \nu(u^-)^{p-1} + \partial_u P(t,x,u)\,, \end{cases} \tag{5.1}$$

with the Neumann-type boundary conditions

$$\begin{cases} y(a) = 0 = y(b)\,, \\ v(a) = 0 = v(b)\,. \end{cases} \tag{5.2}$$

Here we use the notation $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. We assume that the constants A, μ, ν are positive, and the functions $f : \mathbb{R} \to \mathbb{R}$, $E : [a,b] \to \mathbb{R}$ and $P : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. Assume further that $P(t,x,u)$ is 2π-periodic in x, continuously differentiable in (x,u), and that it has a bounded gradient with respect to (x,u). Since $\sin x$ and $\partial_x P(t,x,u)$ are bounded, assumption $A3$ clearly holds.

On the other hand, notice that the last two equations in system (5.1) correspond to the scalar equation

$$\frac{d}{dt}\big(|\dot{u}|^{p-2}\dot{u}\big) + \mu(u^+)^{p-1} - \nu(u^-)^{p-1} = \partial_u P(t,x,u)\,.$$

If we define $\mathscr{H}$ by

$$\mathscr{H}(u,v) = \frac{|v|^q}{q} + \frac{1}{p}\big(\mu(u^+)^p + \nu(u^-)^p\big)\,,$$

then $\mathscr{H}$ is positively-(p,q)-homogeneous and positive, and all the solutions of system $J\dot{w} = \nabla\mathscr{H}(w)$ with $w = (u,v)$ are periodic with the same minimal period

$$\tau = \pi_p(\mu^{-1/p} + \nu^{-1/p})\,, \tag{5.3}$$

(see [12, 17]), where

$$\pi_p = \frac{2(p-1)^{1/p}}{p\sin(\pi/p)}\pi.$$

We thus get the following immediate consequence of Theorem 1.1.

Corollary 5.1. *In the above setting, assume moreover that*

$$\frac{(\mu\nu)^{1/p}}{\mu^{1/p} + \nu^{1/p}} \neq \frac{n\pi_p}{2\pi}\,, \qquad \text{for every } n \in \mathbb{N}\,.$$

Then problem (5.1)-(5.2) *has at least two geometrically distinct solutions.*

Remark 5.2. *Surprisingly enough, besides continuity, in the above corollary no further assumption is needed on the function f.*

Concerning the periodic problem, as a first example of application of Theorem 4.1 we consider the system

$$\begin{cases} \ddot{x} + A\sin x = e(t) + \partial_x P(t,x,u)\,, \\ \frac{d}{dt}\big(|\dot{u}|^{p-2}\dot{u}\big) + \mu(u^+)^{p-1} - \nu(u^-)^{p-1} = \partial_u P(t,x,u)\,, \end{cases} \tag{5.4}$$

where the constants A, μ, ν are positive. Assume that $P(t,x,u)$ is T-periodic in t and 2π-periodic in x, and that it has a bounded gradient with respect to (x,u). Setting $E(t) = \int_0^t e(s)\,ds$, system (5.4) is equivalent to

$$\begin{cases} \dot{x} = y + E(t)\,, \quad \dot{y} = -A\sin x + \partial_x P(t,x,u)\,, \\ \dot{u} = |v|^{q-2}v\,, \quad \dot{v} = -\mu(u^+)^{p-1} + \nu(u^-)^{p-1} + \partial_u P(t,x,u)\,. \end{cases} \tag{5.5}$$

Assuming $e(t)$ to be T-periodic with

$$\int_0^T e(t)\,dt = 0\,,$$

the function $E(t)$ is T-periodic, as well.

Let us verify that the first two equations in (5.5) satisfy the twist condition $B3'''$, with $M = 1$. Notice that there exists $K_3 > 0$ such that, for every C^1-function $\mathcal{U} : [0,T] \to \mathbb{R}$, all the solutions (x,y) of the system

$$\dot{x} = y + E(t)\,, \quad \dot{y} = -A\sin x + \partial_x P(t,x,\mathcal{U}(t))$$

are defined on $[0,T]$ and satisfy

$$|\dot{y}(t)| \le K_3\,, \qquad \text{for every } t \in [0,T]\,.$$

Define $d = K_3T + \|E\|_\infty + 1$ and $c = -(K_3T + \|E\|_\infty + 1)$. Then, if $y(0) = d$, we have

$$\dot{x}(t) = y(t) + E(t) = y(0) + \int_0^t \dot{y}(s)\,ds + E(t) \ge d - K_3T - \|E\|_\infty > 0\,,$$

for every $t \in [0,T]$, and so $x(T) - x(0) > 0$. Similarly, if $y(0) = c$, then $x(T) - x(0) < 0$, which shows that the twist condition is satisfied.

As a consequence of Theorem 4.1 we then immediately have the following.

Corollary 5.3. *In the above setting, assume moreover that*

$$\frac{(\mu\nu)^{1/p}}{\mu^{1/p}+\nu^{1/p}} \neq \frac{n\pi_p}{T}, \qquad \textit{for every } n \in \mathbb{N}.$$

Then system (5.4) *has at least two geometrically distinct T-periodic solutions.*

A variant of the previous example is provided by the system

$$\begin{cases} \ddot{x} + A\sin x = e(t) + \partial_x P(t,x,u), \\ \dot{u} = \nu(v^-)^{q-1} - \mu(v^+)^{q-1}, \\ \dot{v} = \mu(u^+)^{p-1} - \nu(u^-)^{p-1} - \partial_u P(t,x,u). \end{cases} \tag{5.6}$$

where, being $w=(u,v)$, one has $w^+=(u^+,v^+)$ and $w^-=(u^-,v^-)$. Assuming μ,ν to be positive, if we define $\mathscr{H}$ by

$$\mathscr{H}(u,v) = \frac{1}{q}\big(\mu(v^+)^q + \nu(v^-)^q\big) + \frac{1}{p}\big(\mu(u^+)^p + \nu(u^-)^p\big),$$

then $\mathscr{H}$ is positively-(p,q)-homogeneous and positive, and all the solutions of system $J\dot{w} = \nabla\mathscr{H}(w)$ with $w=(u,v)$ are periodic having the same minimal period τ, which can be compute as follows.

We first consider the dynamics in the first quadrant, i.e., when $u>0$ and $v>0$. In this case we can write $J\dot{w} = \nabla\mathscr{H}(w)$ as

$$\dot{u} = \mu v^{q-1}, \qquad \dot{v} = -\mu u^{p-1},$$

leading to the equation

$$\frac{d}{dt}\big(|\dot{u}|^{p-2}\dot{u}\big) + \mu^p u^{p-1} = 0.$$

Then, recalling (5.3), the time needed to pass from the positive v-axis to the positive u-axis is

$$\tau_1 = \frac{1}{4}\pi_p 2(\mu^p)^{-\frac{1}{p}} = \frac{\pi_p}{2\mu}.$$

Similarly, in the fourth quadrant, where $u>0$ and $v<0$, the system becomes

$$\dot{u} = -\nu|v|^{q-2}v, \qquad \dot{v} = \mu u^{p-1},$$

leading to the equation

$$\frac{d}{dt}\big(|\dot{u}|^{p-2}\dot{u}\big) + \mu\nu^{p-1}u^{p-1} = 0.$$

So, the time needed to pass from the positive u-axis to the negative v-axis is

$$\tau_2 = \frac{1}{4}\pi_p 2(\mu\nu^{p-1})^{-\frac{1}{p}} = \frac{\pi_p}{2\mu^{\frac{1}{p}}\nu^{\frac{1}{q}}}\,.$$

In a similar way, we obtain that the time needed to pass from the negative v-axis to the negative u-axis is

$$\tau_3 = \frac{\pi_p}{2\nu}$$

and the time needed to pass from the negative u-axis to the positive v-axis is

$$\tau_4 = \frac{\pi_p}{2\mu^{\frac{1}{q}}\nu^{\frac{1}{p}}}\,.$$

Hence,

$$\tau = \tau_1 + \tau_2 + \tau_3 + \tau_4 = \frac{\pi_p}{2}\left(\frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\mu^{\frac{1}{p}}\nu^{\frac{1}{q}}} + \frac{1}{\mu^{\frac{1}{q}}\nu^{\frac{1}{p}}}\right).$$

We thus get the following consequence of Theorem 4.1.

Corollary 5.4. *In the above setting, assume moreover that*

$$\frac{\pi_p}{2}\left(\frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\mu^{\frac{1}{p}}\nu^{\frac{1}{q}}} + \frac{1}{\mu^{\frac{1}{q}}\nu^{\frac{1}{p}}}\right) \neq \frac{T}{n}\,, \qquad \textit{for every } n \in \mathbb{N}\setminus\{0\}\,.$$

Then system (5.6) *has at least two geometrically distinct T-periodic solutions.*

Both Corollary 5.3 and Corollary 5.4 generalize a classical theorem of Mawhin and Willem [14] on the multiplicity of periodic solutions for the pendulum equation.

6 Final remarks

In Theorem 2.1, dealing with the Neumann problem, we have only considered a diagonal matrix $\mathbb{A}$ like in (2.2). However, for the T-periodic problem, the first author with Gidoni in [4] where able to deal with *any* symmetric matrix $\mathbb{A}$, provided that the nonresonance condition $\sigma(J\mathbb{A}) \cap \frac{2\pi}{T}i\mathbb{Z} = \emptyset$ is assumed. We are confident that a similar result should also holds for the Neumann problem, but we have been able to prove it only when $L = 1$ and the matrix has a positive determinant. Here is our result.

Theorem 6.1. *Assume $L = 1$ and that A1 – A3 hold true. Let $\mathbb{A}$ be a symmetric 2×2 matrix such that $\det \mathbb{A} > 0$. If the non-resonance condition $\sigma(J\mathbb{A}) \cap \frac{\pi}{b-a} i\mathbb{Z} = \emptyset$ holds, then there are at least $M+1$ geometrically distinct solution of the boundary value problem* (2.1)-(2.3) *.*

Proof. Consider the planar Hamiltonian system

$$J\dot{w} = \mathbb{A}w\,. \tag{6.1}$$

We can diagonalize $\mathbb{A}$ by a symplectic transformation. Indeed, there exist a matrix $\mathbb{U}$ with $\det \mathbb{U} = 1$ and a diagonal matrix $\mathbb{D}$ such that

$$\mathbb{A} = \mathbb{U}^{-1}\mathbb{D}\mathbb{U}\,.$$

Since $\det \mathbb{U} = 1$, and the dimension is 2, the change of variables $\varrho = \mathbb{U}w$ is symplectic. Hence, system (6.1) is transformed into the new Hamiltonian system

$$J\dot{\varrho} = \mathbb{D}\varrho\,, \tag{6.2}$$

with

$$\mathbb{D} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

for some α, β such that $\alpha\beta > 0$. Now, the symplectic change of variables $\varpi = \mathbb{M}\varrho$, with

$$\mathbb{M} = \begin{pmatrix} \sqrt[4]{\frac{\alpha}{\beta}} & 0 \\ 0 & \sqrt[4]{\frac{\beta}{\alpha}} \end{pmatrix}.$$

transforms system (6.2) into

$$J\dot{\varpi} = \lambda\varpi\,,$$

with $\lambda = \pm\sqrt{\alpha\beta}$, according to the signs of α and β. However, if $\lambda < 0$, a final change of variables $t \mapsto -t$ will lead to a positive λ. We can now use this procedure and apply Theorem 2.1 to conclude the proof. □

The case $L \geq 2$ remains an open problem.

As a final remark, we recall that, for the periodic problem, Chen and Qian in [2] proved a multiplicity result, coupling *resonant* linear components with twisting components by using Ahmad-Lazer-Paul type resonance condition. In our case, a similar result can be expected for Neumann problem without any twist assumption. The problem remains open for further investigation.

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