An extension of the Poincaré–Birkhoff Theorem coupling twist with lower and upper solutions

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Abstract

In 1983, Conley and Zehnder [11] proved a remarkable theorem on the periodic problem associated with a general Hamiltonian system, giving a partial answer to a conjecture by V.I. Arnold. Their pioneering paper has been extended in different directions by several authors. In 2017, Fonda and Ureña [30] established a deeper relation between the results in [11] and the Poincaré–Birkhoff Theorem. The main theorem in [30] was then extended in 2020 by Fonda and Gidoni to systems whose Hamiltonian function includes a nonresonant quadratic term. It is the aim of this paper to further extend this fertile theory to Hamiltonian systems which, besides the periodicity-twist conditions always required in the Poincaré–Birkhoff Theorem, also include a term involving a pair of well-ordered lower and upper solutions.

1 Introduction

In 1983, C.C. Conley and E.J. Zehnder [11] proved a remarkable result on the periodic problem associated with a general Hamiltonian system

$$J\dot{z} = \nabla H(t,z)\,, \tag{1}$$

giving a partial answer to a conjecture by V.I. Arnold [1, 2]. Here, $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ is the standard symplectic matrix, $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is T-periodic in t, and $\nabla H(t,z)$ denotes its gradient with respect to z. Let us write $z = (q,p)$, with $q = (q_1, \dots, q_N)$ and $p = (p_1, \dots, p_N)$, so that the system reads as

$$\dot{q} = \partial_p H(t,q,p)\,, \quad \dot{p} = -\partial_q H(t,q,p)\,.$$

Their paper contains two main results.

In a first theorem, assuming the Hamiltonian function $H(t,z)$ to be periodic in all variables $q_1, \dots, q_N$ and $p_1, \dots, p_N$, they prove that system (1) has at least $2N+1$ geometrically distinct T-periodic solutions.

In a second theorem, they assume H to be periodic in $q_1, \dots, q_N$ and to have a quadratic behaviour in p; namely, that there exist a constant $R > 0$ and a symmetric regular matrix $\mathbb{A}$ such that $H(t,q,p) = \frac{1}{2}\langle \mathbb{A}p, p\rangle$ + "lower order terms", when $|p| \geq R$. In this setting, they prove that system (1) has at least $N+1$ geometrically distinct T-periodic solutions.

They also mention a possible relation of this second result with the Poincaré–Birkhoff Theorem (the title of the paper is however a bit misleading). This theorem was first conjectured by Poincaré [46] in 1912, shortly before his death, and then proved by Birkhoff in [4, 5]. Both of them were interested in proving it because of its consequences in the existence of periodic solutions for some Hamiltonian systems originating from Celestial Mechanics. As mentioned by Zehnder in [50], Arnold called this theorem "the seed of symplectic topology" (cf. also [43]). See [24] for a brief historical account till the centennial anniversary of the theorem.

The pioneering results in [11] have been extended in different directions by several authors (see, e.g., [9, 15, 23, 33, 37, 38, 40, 41, 47, 49]).

More recently, a deeper relation between these results and the Poincaré–Birkhoff Theorem has been established by the first author jointly with A.J. Ureña [30]. Taking, e.g., $\mathcal{D} = \overline{B}(0,R)$, the closed ball with radius R, and assuming H to be periodic in $q_1, \dots, q_N$, the existence of $N+1$ geometrically distinct T-periodic solutions was established under the following hypothesis: there exist a constant $R > 0$ and a symmetric regular matrix $\mathbb{A}$ such that the solutions $z(t) = (q(t), p(t))$ of (1) starting with $p(0) \in \mathcal{D}$ are defined on $[0,T]$ and satisfy a "twist condition" like

$$\big\langle q(T) - q(0)\,, \mathbb{A}p(0) \big\rangle > 0\,, \quad \text{when } p(0) \in \partial\mathcal{D}\,.$$

It is easily checked that this result generalizes the second Conley–Zehnder Theorem described above. Variants of the twist condition were also proposed in [30, 31].

The results in [30] have been extended in [17] by the first author and P. Gidoni in order to include both the above quoted Conley–Zehnder theorems, assuming H to be periodic in $q_1, \dots, q_N$ and possibly also in $p_1, \dots, p_L$, for some $L \in \{1, \dots, N\}$, together with a very general twist condition, thus finding $N + L + 1$ periodic solutions. The same authors further extended the theory, in a second paper [18], to the case when the Hamiltonian function includes a nonresonant quadratic term. Possible resonance has also been investigated in [10], assuming some Ahmad–Lazer–Paul conditions.

These general existence results have found so far several applications (see [7, 16, 18, 25, 26, 28, 30]), thus generalizing some previously established results for second order equations (cf. [8, 13, 14, 22, 27, 35, 36]). They have even been extended to the study of infinite-dimensional systems [6, 19].

It is the aim of this paper to further extend this fertile theory to systems which, besides the periodicity-twist conditions illustrated above, also present a pair of well-ordered lower and upper solutions. In order to better explain this situation, let the considered Hamiltonian system be of the type

$$\begin{cases} \dot q = \partial_p H(t,q,p,u,v)\,, & \dot p = -\partial_q H(t,q,p,u,v)\,, \\ \dot u = \partial_v H(t,q,p,u,v)\,, & \dot v = -\partial_u H(t,q,p,u,v)\,. \end{cases} \tag{2}$$

Here $H : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ is T-periodic in t, periodic in q, and has a twist involving the (q,p) variables; at the same time, we also assume that there exist some constant lower/upper solutions $\alpha \le \beta$ involving the (u,v) variables. In this situation we are able to prove the existence of two geometrically distinct T-periodic solutions. The result will then be extended to higher dimensional systems.

The reader is surely familiar with the method of lower/upper solutions in the case of scalar equations like, e.g.,

$$\ddot{u} = f(t, u) .$$

This method has a long history, dating back to the pioneering papers [44, 45, 48] (see the book [12] for a detailed exposition). We recall that the T-periodic functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are said to be a lower solution and an upper solution, respectively, if

$$\ddot{\alpha}(t) \geq f(t, \alpha(t)) , \qquad \ddot{\beta}(t) \leq f(t, \beta(t)) ,$$

for every $t \in [0, T]$. Recently the theory has been extended to periodic planar systems in [20, 29]; this will be the approach adopted here, even if we will only be able to deal with the case of constant lower and upper solutions.

Let us briefly explain why we need to reduce to constant lower/upper solutions. The standard proof procedure in lower/upper solutions theorems is as follows: a) modify the problem below $\alpha(t)$ and above $\beta(t)$, in the u component, by the use of a truncating function; b) show that the modified problem has a solution; c) prove that the u component of this solution stays between $\alpha(t)$ and $\beta(t)$. The technical difficulty encountered in the present paper is that we need to maintain the Hamiltonian structure, hence the modification of the problem has to be made in the Hamiltonian function itself, being careful to preserve the differentiability of the new function. Once the modification has been made, we are allowed to apply the results in [18]. We believe that this technical difficulty could be overcome, but for now the case of nonconstant lower/upper solutions remains an open problem.

Another open problem arises in the case of non-well-ordered lower and upper solutions. Assuming some nonresonance conditions with respect to the higher part of the spectrum, this case is usually treated by topological degree methods. We do not know how to adapt this type of technique to our situation.

In order to maintain a friendly exposition we preferred writing this paper following an increasing order of complexity, first presenting the main ideas in the simplest situation, then extending them to more general systems. The paper is thus organized as follows.

In Section 2 we state our result in the simple case of system (2). The proof is provided in Section 3. Then, in Section 4, we provide some variants of the first theorem. In particular, we state a version of the theorem involving a topological annulus, in the spirit of Poincaré's original statement. In Section 5 we illustrate several examples of applications.

In Section 6 we extend the result to higher dimensions, thus generalizing both the Conley–Zehnder theorems presented above. The proof is provided in Section 7. In Section 8 we extend the higher dimensional result to systems whose Hamiltonian function further involves a quadratic term and some examples of possible applications are given in Section 9. In Section 10 we provide a further application to the study of periodic solutions to perturbations of completely integrable systems.

Finally, in Section 11 we establish the most general result of this paper, where the twist condition is stated as an "avoiding cones condition".

2 Statement of the first result

In this section we consider the Hamiltonian system (2), where the Hamiltonian function $H : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ is assumed to be continuous, T-periodic in t, and continuously differentiable in the q, p, u, v variables.

Let us state our assumptions.

Assumption 1 (Periodicity)**.** *The function $H(t, q, p, u, v)$ is periodic in q.*

To fix the ideas, we will assume the period in q to be equal to 2π. Under this setting, T-periodic solutions of (2) appear in equivalence classes made of those solutions whose components $q(t)$ differ by an integer multiple of 2π. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

Assumption 2 (Lower and upper solutions)**.** *There exist some constants $\delta > 0$ and $\alpha \le \beta$ such that*

$$v\,\partial_v H(t, q, p, u, v) > 0\,, \quad \textit{when } u \in [\alpha - \delta, \alpha] \cup [\beta, \beta + \delta] \textit{ and } v \neq 0\,, \tag{3}$$

and

$$\begin{cases} \partial_u H(t, q, p, u, 0) \ge 0\,, & \textit{when } u \in [\alpha - \delta, \alpha]\,, \\ \partial_u H(t, q, p, u, 0) \le 0\,, & \textit{when } u \in [\beta, \beta + \delta]\,. \end{cases} \tag{4}$$

The above assumption comes from the definition of lower and upper solutions given in [20, 29]. We require here that these lower and upper solutions are constant. More precisely, all constants in $[\alpha - \delta, \alpha]$ are lower solutions, and all constants in $[\beta, \beta + \delta]$ are upper solutions. In the sequel, the constant $\delta > 0$ provided by Assumption 2 will be used without further mention.

Assumption 3 (Nagumo condition)**.** *There exist $d > 0$ and two continuous functions $f, \varphi : [d, +\infty[\, \to \,]0, +\infty[$, with*

$$\int_d^{+\infty} \frac{f(s)}{\varphi(s)}\, ds = +\infty\,,$$

satisfying the following property. If $u \in [\alpha - \delta, \beta + \delta]$, then

$$\begin{cases} \partial_v H(t, q, p, u, v) \ge f(v)\,, & \textit{when } v \ge d\,, \\ \partial_v H(t, q, p, u, v) \le -f(-v)\,, & \textit{when } v \le -d\,, \end{cases}$$

and

$$|\partial_u H(t, q, p, u, v)| \le \varphi(|v|)\,, \quad \textit{when } |v| \ge d\,.$$

Assumption 4 (Linear growth)**.** *For every $K > 0$ there is a constant $C_K > 0$ such that*

$$|\partial_q H(t, q, p, u, v)| \le C_K(|p| + 1)\,, \quad \textit{when } u \in [\alpha - \delta, \beta + \delta] \textit{ and } |v| \le K\,.$$

Remark 1. *Notice that, under the above assumption, for any two continuous functions* $U, V : [0, T] \to \mathbb{R}$*, with*

$$\alpha - \delta \leq U(t) \leq \beta + \delta\,, \quad \text{for every } t \in [0, T]\,, \tag{5}$$

the solutions of the system

$$\dot{q} = \partial_p H(t, q, p, U(t), V(t))\,, \quad \dot{p} = -\partial_q H(t, q, p, U(t), V(t)) \tag{6}$$

are defined on $[0, T]$*. Indeed, let* $(q(t), p(t))$ *be a solution of system* (6) *starting at time* $t = 0$ *from some* $(q(0), p(0)) = (q_0, p_0)$*. This solution is defined on a maximal interval of future existence* $[0, T] \cap [0, \omega[$*. Set* $K = \|V\|_\infty$*. By Assumption 4,*

$$|\dot{p}(t)| \leq C_K(|p(t)| + 1)\,, \quad \text{for every } t \in [0, T] \cap [0, \omega[\,,$$

which, combined with the Gronwall Lemma, yields

$$|p(t)| \leq (|p_0| + 1)\, e^{C_K T}\,, \quad \text{for every } t \in [0, T] \cap [0, \omega[\,.$$

Then, since H *is periodic in* q*, we have that there is a constant* $C > 0$ *depending only on* U*,* V *and* $|p_0|$*, such that*

$$|\dot{q}(t)| \leq C\,, \quad \text{for every } t \in [0, T] \cap [0, \omega[\,.$$

Hence,

$$|q(t)| \leq |q_0| + CT\,, \quad \text{for every } t \in [0, T] \cap [0, \omega[\,.$$

This implies that the solution $(q(t), p(t))$ *must be defined on* $[0, T]$*, i.e., that* $\omega > T$*.*

Here is our first result.

Theorem 2. *Let Assumptions 1, 2, 3 and 4 hold. Assume that there exist* $a < b$ *and* $\rho > 0$ *with the following property: For any two continuous functions* $U, V : [0, T] \to \mathbb{R}$ *satisfying* (5)*, the solutions of system* (6) *are such that*

$$\begin{cases} q(T) - q(0) < 0\,, & \text{when } p(0) \in [a - \rho, a]\,, \\ q(T) - q(0) > 0\,, & \text{when } p(0) \in [b, b + \rho]\,. \end{cases} \tag{7}$$

Then, there exist at least two geometrically distinct T*-periodic solutions of system* (2)*, such that* $p(0) \in \,]a, b[$ *and*

$$\alpha \leq u(t) \leq \beta\,, \quad \text{for every } t \in \mathbb{R}\,. \tag{8}$$

The same conclusion holds if (7) *is replaced by*

$$\begin{cases} q(T) - q(0) > 0\,, & \text{when } p(0) \in [a - \rho, a]\,, \\ q(T) - q(0) < 0\,, & \text{when } p(0) \in [b, b + \rho]\,. \end{cases}$$

3 The proof of Theorem 2

The proof is based on [18, Corollary 2.4], and will be divided in two steps. In the first step we analyze the dynamics focusing on the (u, v) variables. In the second one, we draw our attention on the (q, p) variables.

3.1 Working with the (u, v) coordinates

Let us focus our attention on the couple of variables (u, v). At first, we are going to modify the original problem (2) outside some suitably chosen set $\mathcal{V} \subseteq \mathbb{R}^4$. We will then prove that all the T-periodic solutions of the modified system must be such that $z(t) = (q(t), p(t), u(t), v(t)) \in \mathcal{V}$ for every $t \in [0, T]$; hence, such solutions will solve the original problem (2), too.

Assumption 3 permits us to apply the reasoning in [29, Theorem 3.1] (see also [20, Lemma 15]) in order to find two continuously differentiable functions $\gamma_\pm : \mathbb{R} \to \mathbb{R}$ satisfying

$$\gamma_-(s) < -d < d < \gamma_+(s)\,, \tag{9}$$

for every $s \in [\alpha - \delta, \beta + \delta]$, and

$$-\partial_u H\big(t, q, p, u, \gamma_+(u)\big) > \partial_v H\big(t, q, p, u, \gamma_+(u)\big)\ \gamma_+'(u)\,, \tag{10}$$

$$-\partial_u H\big(t, q, p, u, \gamma_-(u)\big) < \partial_v H\big(t, q, p, u, \gamma_-(v)\big)\ \gamma_-'(u)\,, \tag{11}$$

for every $(t, q, p, u) \in [0, T] \times [0, 2\pi] \times \mathbb{R} \times [\alpha - \delta, \beta + \delta]$. Correspondingly, we introduce the set

$$\mathcal{V} = \{z = (q, p, u, v) \mid \alpha \le u \le \beta\,,\, \gamma_-(u) < v < \gamma_+(u)\}\,.$$

Now we can choose a constant $\widehat{d} > d$ satisfying

$$-\widehat{d} < \gamma_-(s) < \gamma_+(s) < \widehat{d}\,,$$

for every $s \in [\alpha - \delta, \beta + \delta]$.

We consider a continuously differentiable function $\zeta : \mathbb{R} \to \mathbb{R}$ such that

$$\zeta(u) = \begin{cases} \alpha - \delta\,, & \text{if } u \le \alpha - 2\delta\,, \\ u\,, & \text{if } \alpha \le u \le \beta\,, \\ \beta + \delta\,, & \text{if } u \ge \beta + 2\delta\,, \end{cases} \tag{12}$$

and

$$\zeta'(u) > 0\,,\ \text{when } u \in]\alpha - 2\delta, \beta + 2\delta[\,. \tag{13}$$

Notice that

$$\alpha - \delta \le \zeta(u) \le \beta + \delta\,, \quad \text{for every } u \in \mathbb{R}\,.$$

Then, we introduce a continuously differentiable function $\chi : [\alpha-\delta, \beta+\delta] \times \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

$$\chi(u, v) = \begin{cases} -\widehat{d}\,, & \text{if } v < -\widehat{d} - 1\,, \\ v\,, & \text{if } \gamma_-(u) \le v \le \gamma_+(u)\,, \\ \widehat{d}\,, & \text{if } v > \widehat{d} + 1\,. \end{cases} \tag{14}$$

Moreover, we assume that

$$\partial_v \chi(u, v) > 0\,,\ \text{ when } |v| < \widehat{d} + 1\,. \tag{15}$$

Let $\widehat{H} : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ be defined as

$$\widehat{H}(t, q, p, u, v) = H\big(t, q, p, \zeta(u), \chi(\zeta(u), v)\big) + \mathscr{H}(u, v)\,,$$

with

$$\mathscr{H}(u,v) = \frac{1}{2}\left[\left[(v-\widehat{d}\,)^+\right]^2 + \left[(v+\widehat{d}\,)^-\right]^2 - \left[(u-\beta)^+\right]^2 - \left[(u-\alpha)^-\right]^2\right],$$

where, as usual, $\xi^+ = \max\{\xi, 0\}$ and $\xi^- = \max\{-\xi, 0\}$. We consider the modified Hamiltonian system

$$\begin{cases} \dot q = \partial_p \widehat{H}(t,q,p,u,v)\,, & \dot p = -\partial_q \widehat{H}(t,q,p,u,v)\,, \\ \dot u = \partial_v \widehat{H}(t,q,p,u,v)\,, & \dot v = -\partial_u \widehat{H}(t,q,p,u,v)\,. \end{cases} \tag{16}$$

Notice that $\widehat{H} = H$ in the closure of the set $\mathcal{V}$. Our aim is to prove the following a priori bound.

Lemma 3. *If $z = (q,p,u,v)$ is a T-periodic solution of* (16)*, then $z(t) \in \mathcal{V}$ for every $t \in [0,T]$, hence it solves* (2)*.*

The proof of this lemma needs some preparation, hence it will be provided at the end of the section.

In what follows, for every $z = (q,p,u,v)$ we introduce the notation

$$\tilde z = (q,p,\zeta(u),\chi(\zeta(u),v))\,.$$

We can explicitly compute

$$\partial_q \widehat{H}(t,z) = \partial_q H\big(t,\tilde z\big)\,, \qquad \partial_p \widehat{H}(t,z) = \partial_p H\big(t,\tilde z\big)\,, \tag{17}$$

$$\partial_u \widehat{H}(t,z) = [\partial_u H\big(t,\tilde z\big) + \partial_v H\big(t,\tilde z\big)\,\partial_u \chi(\zeta(u),v)]\,\zeta'(u) + \partial_u \mathscr{H}(u,v)\,, \tag{18}$$

$$\partial_v \widehat{H}(t,z) = \partial_v H\big(t,\tilde z\big)\,\partial_v \chi(\zeta(u),v) + \partial_v \mathscr{H}(u,v)\,. \tag{19}$$

Proposition 4. *Any solution of* (16) *satisfies, for every $t_0 \in \mathbb{R}$,*

$$\begin{aligned} [u(t_0) < \alpha \ \ \text{and}\ \ v(t_0) = 0] \quad &\Rightarrow \quad \dot v(t_0) < 0\,, \\ [u(t_0) > \beta \ \ \text{and}\ \ v(t_0) = 0] \quad &\Rightarrow \quad \dot v(t_0) > 0\,. \end{aligned}$$

Proof. We first note that, as an immediate consequence of (3), we have

$$\partial_v H(t,q,p,u,0) = 0\,, \quad \text{when } u \in [\alpha-\delta,\alpha] \cup [\beta,\beta+\delta]\,. \tag{20}$$

We prove the first implication, the second one being similar. Let $t_0 \in \mathbb{R}$ be such that $u(t_0) < \alpha$ and $v(t_0) = 0$. Let $z(t)$ be a solution of (16), with $z(t_0) = (q(t_0), p(t_0), u(t_0), 0)$. Then, $\tilde z(t_0) = (q(t_0), p(t_0), \zeta(u(t_0)), 0)$. By (18) and (20) we have that

$$\begin{aligned} \dot v(t_0) = -\partial_u \widehat{H}(t_0, z(t_0)) &= -\partial_u H(t_0, \tilde z(t_0))\zeta'(u(t_0)) - \partial_u \mathscr{H}(u(t_0), 0) \\ &= -\partial_u H(t_0, \tilde z(t_0))\zeta'(u(t_0)) + u(t_0) - \alpha\,, \end{aligned} \tag{21}$$

so that

$$\dot v(t_0) < -\partial_u H(t_0, \tilde z(t_0))\zeta'(u(t_0))\,.$$

Then, (4) and (13) give the negative sign when $u(t_0) \in \,]\alpha - 2\delta, \alpha[$. On the other hand, $\zeta'(u)$ vanishes when $u \le \alpha - 2\delta$ and the conclusion easily follows from (21). ☐

Proposition 5. *Any solution of* (16) *satisfies, for every* $t_0 \in \mathbb{R}$,

$$\begin{aligned}
[u(t_0) \leq \alpha \ \ \textit{and} \ \ v(t_0) < 0] \quad &\Rightarrow \quad \dot{u}(t_0) < 0\,, \\
[u(t_0) \leq \alpha \ \ \textit{and} \ \ v(t_0) > 0] \quad &\Rightarrow \quad \dot{u}(t_0) > 0\,, \\
[u(t_0) \geq \beta \ \ \textit{and} \ \ v(t_0) < 0] \quad &\Rightarrow \quad \dot{u}(t_0) < 0\,, \\
[u(t_0) \geq \beta \ \ \textit{and} \ \ v(t_0) > 0] \quad &\Rightarrow \quad \dot{u}(t_0) > 0\,.
\end{aligned}$$

Proof. Let us prove the first assertion, the proof of the others being similar. Let $t_0 \in \mathbb{R}$ be such that $u(t_0) \leq \alpha$ and $v(t_0) < 0$. Since $\zeta(u) \in [\alpha - \delta, \alpha]$ when $u \leq \alpha$ and $\chi(u, v) < 0$ when $v < 0$, recalling (3), (15), and (19), if $-\widehat{d} - 1 < v(t_0) < 0$ we get

$$\begin{aligned}
\dot{u}(t_0) &= \partial_v H(t_0, \tilde{z}(t_0)) \partial_v \chi(\zeta(u(t_0)), v(t_0)) - (v(t_0) + \widehat{d}\,)^- \\
&\leq \partial_v H(t_0, \tilde{z}(t_0)) \partial_v \chi(\zeta(u(t_0)), v(t_0)) < 0\,.
\end{aligned}$$

On the other hand, if $v(t_0) \leq -\widehat{d} - 1$, then $\partial_v \chi(\zeta(u(t_0)), v(t_0)) = 0$, so that $\dot{u}(t_0) = -(v(t_0) + \widehat{d}\,)^- < 0$. □

We define the open sets

$$\begin{aligned}
A_{NW} &= \{z \in \mathbb{R}^4 \mid u < \alpha\,,\ v > 0\}\,, \qquad A_{NE} = \{z \in \mathbb{R}^4 \mid u > \beta\,,\ v > 0\}\,, \\
A_{SW} &= \{z \in \mathbb{R}^4 \mid u < \alpha\,,\ v < 0\}\,, \qquad A_{SE} = \{z \in \mathbb{R}^4 \mid u > \beta\,,\ v < 0\}\,.
\end{aligned}$$

As a consequence of the previous propositions, the following can be easily proved.

Proposition 6. *For every solution* z *of* (16) *the following assertions hold:*

if there is $t_0 \in \mathbb{R}$ *such that* $z(t_0) \in A_{NW}$ *then* $z(t) \in A_{NW}$ *for every* $t < t_0$,
if there is $t_0 \in \mathbb{R}$ *such that* $z(t_0) \in A_{NE}$ *then* $z(t) \in A_{NE}$ *for every* $t > t_0$,
if there is $t_0 \in \mathbb{R}$ *such that* $z(t_0) \in A_{SW}$ *then* $z(t) \in A_{SW}$ *for every* $t > t_0$,
if there is $t_0 \in \mathbb{R}$ *such that* $z(t_0) \in A_{SE}$ *then* $z(t) \in A_{SE}$ *for every* $t < t_0$.

Hence, A_{NE} and A_{SW} are positively invariant sets, while A_{SE} and A_{NW} are negatively invariant.

Proposition 7. *Any solution of* (16) *satisfies, for every* $t_0 \in \mathbb{R}$,

$$\begin{aligned}
[\alpha \leq u(t_0) \leq \beta \ \ \textit{and} \ \ v(t_0) > d] \quad &\Rightarrow \quad \dot{u}(t_0) > 0\,, \\
[\alpha \leq u(t_0) \leq \beta \ \ \textit{and} \ \ v(t_0) < -d] \quad &\Rightarrow \quad \dot{u}(t_0) < 0\,.
\end{aligned}$$

Proof. Let us prove the first implication. Since $u(t_0) \in [\alpha, \beta]$ and $v(t_0) > d$, we have that $\zeta(u(t_0)) = u(t_0)$ and

$$\dot{u}(t_0) = \partial_v H(t_0, \tilde{z}(t_0)) \partial_v \chi(u(t_0), v(t_0)) + (v(t_0) - \widehat{d}\,)^+\,.$$

If $v(t_0) \geq \widehat{d} + 1$, then $\partial_v \chi(u(t_0), v(t_0)) = 0$ and $(v(t_0) - \widehat{d}\,)^+ > 0$. On the other hand, if $d < v(t_0) < \widehat{d} + 1$, then $\partial_v \chi(u(t_0), v(t_0)) > 0$ and $(v(t_0) - \widehat{d}\,)^+ \geq 0$; moreover, $\partial_v H(t_0, \tilde{z}(t_0)) > 0$ by Assumption 3, thus proving that $\dot{u}(t_0) > 0$ in both cases. □

Proof of Lemma 3. Let z be a T-periodic solution of (16). We begin proving that $\alpha \leq u(t) \leq \beta$ for every t. By contradiction, assume that there exists $t_0 \in \mathbb{R}$ such that $u(t_0) > \beta$. If $z(t_0) \in A_{NE}$, by Proposition 6, since the function is periodic, $z(t) \in A_{NE}$ for every $t \in \mathbb{R}$. Then, we get a contradiction using Proposition 5. Similarly we cannot have $z(t_0) \in A_{SE}$. Finally, if $v(t_0) = 0$, Proposition 4 takes us to the previous contradicting situations. We have thus proved that $u(t) \leq \beta$ for every $t \in \mathbb{R}$. A similar argument proves that $u(t) \geq \alpha$ for every $t \in \mathbb{R}$.

Now we show that $v(t) < \gamma_+(u(t))$ for every $t \in \mathbb{R}$. Let us define the function $G_+(t) = v(t) - \gamma_+(u(t))$. By Proposition 7, it cannot be that $G_+(t) > 0$ for every $t \in [0, T]$. Assume by contradiction the existence of $t_0 \in \mathbb{R}$ such that $G_+(t_0) = 0$. Then, since $\nabla H(t, z) = \nabla \widehat{H}(t, z)$ for z in the closure of $\mathcal{V}$, by (10) we have

$$G'_+(t_0) = -\partial_u H(t_0, z(t_0)) - \gamma'_+(u(t_0))\, \partial_v H(t_0, z(t_0)) > 0\,,$$

where $z(t_0) = \big(q(t_0), p(t_0), u(t_0), \gamma_+(u(t_0))\big)$. This implies that $G_+(t) > 0$ for every $t > t_0$, which is in contradiction with the periodicity of z. We have thus proved that $G_+(t) < 0$ for every $t \in [0, T]$.

We can similarly prove that $\gamma_-(u(t)) < v(t)$ for every $t \in \mathbb{R}$, using (11). □

3.2 Working with the (q, p) coordinates

Let us fix $K > 0$ such that

$$|\chi(\zeta(u), v)| < K\,, \text{ for every } (u, v) \in \mathbb{R}^2\,.$$

Let $z(t) = (q(t), p(t), u(t), v(t))$ be a solution of system (16) starting at time $t = 0$ with $p(0) \in [a - \rho, b + \rho]$. This solution is defined on a maximal interval of future existence $[0, \omega[$. By Assumption 4, recalling (17),

$$|\dot{p}(t)| \leq C_K(|p(t)| + 1)\,, \qquad \text{for every } t \in [0, \omega[\,, \tag{22}$$

hence, setting

$$c = (\max\{|a|, |b|\} + \rho + 1)\; e^{C_K T}, \tag{23}$$

by Gronwall Lemma we have that

$$|p(t)| \leq c\,, \qquad \text{for every } t \in [0, T] \cap [0, \omega[\,. \tag{24}$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth nonincreasing function such that

$$\eta(s) = \begin{cases} 1\,, & \text{if } s \leq c\,, \\ 0\,, & \text{if } s \geq c + 1\,. \end{cases} \tag{25}$$

We can rewrite the Hamiltonian $\widehat{H}$ as

$$\widehat{H}(t, q, p, u, v) = \frac{1}{2}(v^2 - u^2) + \widehat{h}(t, q, p, u, v)\,, \tag{26}$$

and define the new Hamiltonian function

$$\widetilde{H}(t, q, p, u, v) = \frac{1}{2}(v^2 - u^2) + \widetilde{h}(t, q, p, u, v)\,, \tag{27}$$

where

$$\widetilde{h}(t,q,p,u,v) = \eta(|p|)\,\widehat{h}(t,q,p,u,v)\,.$$

Notice that

$$|p| \geq c+1 \quad \Rightarrow \quad \widetilde{h}(t,q,p,u,v) = 0 \quad \text{for every } (t,q,u,v) \in \mathbb{R}\times\mathbb{R}^3\,.$$

The Hamiltonian $\widetilde{H}$ induces the system

$$\begin{cases} \dot q = \partial_p \widetilde{h}(t,q,p,u,v)\,, & \dot p = -\partial_q \widetilde{h}(t,q,p,u,v)\,, \\ \dot u = v + \partial_v \widetilde{h}(t,q,p,u,v)\,, & \dot v = u - \partial_u \widetilde{h}(t,q,p,u,v)\,. \end{cases} \tag{28}$$

Since the Hamiltonian function is periodic in q and the functions ζ, ζ', χ, $\nabla\chi$, η and η' are all bounded, there exists a constant $\widetilde{C} > 0$ such that

$$\big|\partial_q \widetilde{h}(t,z)\big| + \big|\partial_p \widetilde{h}(t,z)\big| + \big|\partial_u \widetilde{h}(t,z)\big| + \big|\partial_v \widetilde{h}(t,z)\big| \leq \widetilde{C}\,,$$

for every $(t,z) \in \mathbb{R}\times\mathbb{R}^4$. Hence, the Hamiltonian function $\widetilde{H}$ is the sum of a nonresonant quadratic term and a function with bounded gradient $\nabla\widetilde{h}(t,z)$. In particular, all solutions of (28) are globally defined on $[0,T]$.

We now verify the twist condition for system (28). Let $z = (q,p,u,v)$ be a solution of (28) such that $p(0) \in [a-\rho, b+\rho]$. As long as $|p(t)| \leq c$, we have

$$|\dot p(t)| = |\partial_q H(t, q(t), p(t), \zeta(u(t)), \chi(\zeta(u(t)), v(t)))| \leq C_K(|p(t)|+1)\,.$$

Hence, by Gronwall Lemma, we conclude that $|p(t)| \leq c$ for every $t \in [0,T]$. So, $z = (q,p,u,v)$ is a solution of (16). Then, (q,p) is a solution of (6) with $U(t) = \zeta(u(t))$ and $V(t) = \chi(\zeta(u(t)), v(t))$, condition (5) is verified and, by (7),

$$\begin{cases} p(0) \in [a-\rho, a] & \Rightarrow \quad q(T) < q(0)\,, \\ p(0) \in [b, b+\rho] & \Rightarrow \quad q(T) > q(0)\,. \end{cases}$$

Hence, we can apply [18, Corollary 2.4] so to find two T-periodic solutions $z = (q,p,u,v)$ of (28) such that $p(0) \in \,]a,b[\,$. By the above estimates, these are indeed solutions of (16) and, recalling Lemma 3, we conclude that they are the T-periodic solutions of the original system (2) we were looking for. □

4 Some variants of Theorem 2

Let us start with two observations.

Remark 8. *Assumption 2 can be generalized by asking that there further exist* v_α, v_β *such that*

$$\begin{cases} (v - v_\alpha)\,\partial_v H(t,q,p,u,v) > 0\,, & \textit{when } u \in [\alpha-\delta, \alpha] \textit{ and } v \neq v_\alpha\,, \\ (v - v_\beta)\,\partial_v H(t,q,p,u,v) > 0\,, & \textit{when } u \in [\beta, \beta+\delta] \textit{ and } v \neq v_\beta\,, \end{cases}$$

and

$$\begin{cases} \partial_u H(t,q,p,u,v_\alpha) \geq 0\,, & \textit{when } u \in [\alpha-\delta, \alpha]\,, \\ \partial_u H(t,q,p,u,v_\beta) \leq 0\,, & \textit{when } u \in [\beta, \beta+\delta]\,. \end{cases}$$

Remark 9. *Instead of a fixed interval* $[a,b]$*, we could have a varying interval* $[a(q),b(q)]$*, where* $a,b:\mathbb{R}\to\mathbb{R}$ *are continuous* 2π*-periodic functions. Indeed, if* a *and* b *are continuously differentiable, this case can be reduced to the previous one by the symplectic change of variables*

$$\Psi(q,p,u,v) = \left(\int_0^q \frac{b(s)-a(s)}{2}\,ds\,,\ \frac{2p-b(q)-a(q)}{b(q)-a(q)},u,v\right),$$

cf. [43, Exercise 1, p. 132]. If a *and* b *are just continuous, they can be replaced by smooth functions by the use of Fejer Theorem. Notice that the new Hamiltonian* $\widetilde{H}(t,\tilde{q},\tilde{p},u,v) = H(t,\Psi^{-1}(\tilde{q},\tilde{p},u,v))$ *is periodic in* $\tilde{q}$*, with period* $\tau := \frac{1}{2}\int_0^{2\pi}(b(s)-a(s))\,ds$*.*

We now propose a variant of our result which is more in the spirit of the Poincaré–Birkhoff Theorem as originally stated by Poincaré [46]. We first recall the definition of rotation number. For $t_1 < t_2$, let $\eta : [t_1,t_2] \to \mathbb{R}^2$ be a continuous curve such that $\eta(t) \neq (0,0)$ for every $t \in [t_1,t_2]$. Writing $\eta(t) = \rho(t)(\cos\theta(t),\sin\theta(t))$, with $\rho:\mathbb{R}\to\,]0,+\infty[$ and $\theta:\mathbb{R}\to\mathbb{R}$ continuous, we define

$$\mathrm{Rot}\,(\eta;[t_1,t_2]) = -\frac{\theta(t_2)-\theta(t_1)}{2\pi}\,.$$

We need to suitably modify Assumption 4, in the following way

Assumption 5 (Energy growth)**.** *For every* $K>0$ *there is a constant* $C_K>0$ *such that*

$$\begin{aligned}\big|q\,\partial_p H(t,q,p,u,v) - &\ p\,\partial_q H(t,q,p,u,v)\big| \le C_K(q^2+p^2+1)\,,\\ &\text{when } u\in[\alpha-\delta,\beta+\delta] \text{ and } |v|\le K\,.\end{aligned}$$

In the sequel, we denote by $\mathcal{D}(\Gamma)$ the open bounded region delimited by a planar Jordan curve Γ.

Theorem 10. *Let Assumptions 2, 3, and 5 hold. Let* k *be an integer and assume that there exist* $\rho>0$*,* $\tilde{\rho}>0$ *and two planar Jordan curves* Γ_1*,* Γ_2*, strictly star-shaped with respect to the origin, with*

$$0\in\mathcal{D}(\Gamma_1)\subseteq\overline{\mathcal{D}(\Gamma_1)}\subseteq\mathcal{D}(\Gamma_2)\,,$$

such that, for any two continuous functions $U,V:[0,T]\to\mathbb{R}$ *satisfying* (5)*, the solutions of system* (6) *with* $\mathrm{dist}\big((q(0),p(0)),\overline{\mathcal{D}(\Gamma_2)\setminus\mathcal{D}(\Gamma_1)}\big)\le\rho$ *which are defined on* $[0,T]$ *satisfy*

$$q(t)^2+p(t)^2\ge\tilde{\rho}\,,\quad \text{for every } t\in[0,T]\,,$$

and, if $(q(0),p(0))\notin\mathcal{D}(\Gamma_2)\setminus\overline{\mathcal{D}(\Gamma_1)}$*,*

$$\begin{aligned}&\mathrm{Rot}\,((q,p);[0,T])<k\,,\quad \text{when } \mathrm{dist}\big((q(0),p(0)),\Gamma_1\big)\le\rho\,,\\ &\mathrm{Rot}\,((q,p);[0,T])>k\,,\quad \text{when } \mathrm{dist}\big((q(0),p(0)),\Gamma_2\big)\le\rho\,.\end{aligned} \tag{29}$$

Then, the Hamiltonian system (2) *has at least two* T*-periodic solutions* $z_i(t)$*,* $i=1,2$*, satisfying* (8)*, with*

$$(q_i(0),p_i(0))\in\mathcal{D}(\Gamma_2)\setminus\overline{\mathcal{D}(\Gamma_1)}\,,$$

and

$$\mathrm{Rot}\,((q_i,p_i);[0,T]) = k.$$

The same is true if (29) is replaced by

$$\begin{aligned}\mathrm{Rot}\,((q,p);[0,T]) > k\,, &\quad \textit{when } \mathrm{dist}\big((q(0),p(0)),\Gamma_1\big) \le \rho\,,\\ \mathrm{Rot}\,((q,p);[0,T]) < k\,, &\quad \textit{when } \mathrm{dist}\big((q(0),p(0)),\Gamma_2\big) \le \rho\,.\end{aligned}$$

Proof. At first, let $R_0 > 0$ be such that

$$\mathrm{dist}\big((q(0),p(0)),\overline{\mathcal{D}(\Gamma_2)}\big) \le \rho \quad\Rightarrow\quad q(0)^2 + p(0)^2 \le R_0\,. \tag{30}$$

We modify the Hamiltonian H introducing the function $\widehat{H}$, arguing as in Section 3.1, thus finding the a priori bound in Lemma 3. Let $K > 0$ be such that

$$|\chi(\zeta(u),v)| < K\,,\ \text{for every } (u,v) \in \mathbb{R}^2\,.$$

Let $z(t) = (q(t),p(t),u(t),v(t))$ be any solution of system (16) starting at time $t = 0$ from a point $z(0)$ satisfying $q(0)^2 + p(0)^2 \le R_0$. This solution is defined on a maximal interval of future existence $[0,\omega[$. Defining $r(t) = q(t)^2 + p(t)^2$, by Assumption 5,

$$|\dot r(t)| \le C_K(r(t)+1)\,, \qquad \text{for every } t \in [0,\omega[\,,$$

hence, setting

$$c = (R_0+1)\,e^{C_K T},$$

by Gronwall Lemma we have that

$$r(t) \le c\,, \qquad \text{for every } t \in [0,T]\cap[0,\omega[\,.$$

Let η be the function introduced in (25). Arguing as above, we can write $\widehat{H}$ as in (26) and define $\widetilde{H}$ as in (27), where now

$$\widetilde{h}(t,q,p,u,v) = \eta(q^2+p^2)\,\widehat{h}(t,q,p,u,v)\,.$$

The Hamiltonian function $\widetilde{H}$ induces the system (28), whose solutions are globally defined on $[0,T]$.

Let $z(t) = (q(t),p(t),u(t),v(t))$ be any solution of system (28) with starting point $z(0)$ satisfying $\mathrm{dist}\big((q(0),p(0)),\overline{\mathcal{D}(\Gamma_2)}\setminus\mathcal{D}(\Gamma_1)\big) \le \rho$. Set $U(t) = \zeta(u(t))$ and $V(t) = \chi(\zeta(u(t),v(t))$. Notice that (5) is satisfied. It can be verified that $r(t) = q(t)^2 + p(t)^2 \le c$ for every $t \in [0,T]$, with the same constant c as above. Therefore, the couple $(q(t),p(t))$ is a solution of (6). So, by the assumption of the theorem, we get

$$r(t) \ge \tilde\rho\,, \qquad \text{for every } t \in [0,T]\,. \tag{31}$$

We now introduce a new smooth cut-off increasing function $\widetilde{\eta} : \mathbb{R} \to [0,1]$ such that

$$\widetilde{\eta}(s) = \begin{cases} 0\,, & \text{if } r \le \tilde\rho/2\,,\\ 1\,, & \text{if } r \ge \tilde\rho\,.\end{cases}$$

and the new Hamiltonian function

$$\overline{H}(t,q,p,u,v) = \frac{1}{2}(v^2-u^2) + \overline{h}(t,q,p,u,v)\,,$$

where $\overline{h}(t,q,p,u,v) = \widetilde{\eta}(q^2+p^2)\widetilde{h}(t,q,p,u,v)$.

For the new Hamiltonian system $J\dot{z} = \nabla \overline{H}(t,z)$, we introduce the symplectic change of variables

$$q(t) = \sqrt{2r(t)} \cos\left(\theta(t) - \tfrac{2\pi}{T}\, kt\right), \quad p(t) = \sqrt{2r(t)} \sin\left(\theta(t) - \tfrac{2\pi}{T}\, kt\right),$$

so to get a Hamiltonian system

$$\begin{cases} \dot\theta = \partial_r \mathcal{H}(t,\theta,r,u,v)\,, & \dot r = -\partial_\theta \mathcal{H}(t,\theta,r,u,v)\,, \\ \dot u = \partial_v \mathcal{H}(t,\theta,r,u,v)\,, & \dot v = -\partial_u \mathcal{H}(t,\theta,r,u,v)\,, \end{cases} \tag{32}$$

defined for $r > 0$. This system can now be extended also for $r \le 0$, preserving the regularity. Now, the periodicity Assumption 1 is recovered in the variable θ, while Assumption 5 implies that the linear growth Assumption 4 holds for the new variables (θ, r) instead of (q, p). Then Theorem 2 applies, in view of Remark 9, providing the existence of two geometrically distinct T-periodic solutions of (32), which are then translated, by the inverse change of variables $(\theta, r) \mapsto (q, p)$, into the T-periodic solutions of (2) we are looking for. □

5 Examples of applications

Both Theorem 2 and Theorem 10 open the way to a multitude of applications. We will just sketch here a few.

Let us consider for example a system of the type

$$\begin{cases} -\ddot q = g(t,q) - e(t) + \partial_q P(t,q,u)\,, \\ -\ddot u = -f(u) + \partial_u P(t,q,u)\,, \end{cases} \tag{33}$$

where all functions are continuous and T-periodic in t, with $\int_0^T e(t)\,dt = 0$. Let $P(t,q,u)$ be 2π-periodic in q and continuously differentiable in (q,u). Assume moreover $g(t,q)$ to be 2π-periodic in q, with $\int_0^{2\pi} g(t,s)\,ds = 0$.

Then system (33) is of the type (2), with

$$H(t,q,p,u,v) = \frac{1}{2}(p + E(t))^2 + \frac{1}{2}v^2 + \int_0^q g(t,s)\,ds - \int_0^u f(\sigma)\,d\sigma + P(t,q,u)\,,$$

where $E(t) = \int_0^t e(s)\,ds$. Precisely, we have

$$\begin{cases} \dot q = p + E(t)\,, & \dot p = -g(t,q) - \partial_q P(t,q,u)\,, \\ \dot u = v\,, & \dot v = f(u) - \partial_u P(t,q,u)\,. \end{cases}$$

Corollary 11. *In the above setting, assume moreover that there exist two constants $\alpha < \beta$ such that*

$$f(\alpha) < \partial_u P(t,q,\alpha), \qquad \partial_u P(t,q,\beta) < f(\beta), \tag{34}$$

for all $(t,q) \in [0,T] \times [0,2\pi]$. Then, system (33) has at least two geometrically distinct T-periodic solutions (q,u), with $\alpha \le u \le \beta$.

Proof. Notice that $H(t,q,p,u,v)$ is 2π-periodic in q. By continuity, there is a sufficiently small $\delta > 0$ such that Assumption 2 is satisfied. Also Assumption 3 is easily verified, taking $f(s) = s$ and $\varphi(s)$ constant. Moreover, there exist $R > 0$ with the following property: For any continuous function $U : [0,T] \to \mathbb{R}$ such that $\alpha - \delta \le U(t) \le \beta + \delta$ for every $t \in [0,T]$, the solutions of

$$-\ddot{q} = g(t,q) - e(t) + \partial_q P(t,q,U(t))$$

satisfy

$$\begin{cases} q(T) - q(0) < 0\,, & \text{when } \dot{q}(0) < -R\,, \\ q(T) - q(0) > 0\,, & \text{when } \dot{q}(0) > R\,. \end{cases}$$

Then, taking $b = -a = R + \|E\|_\infty$, Theorem 2 applies. $\square$

As an immediate consequence, we have the following.

Corollary 12. *The system of coupled pendulums of the form*

$$\begin{cases} \ddot{q} + a\sin q = e(t) - \partial_q P(t,q,u), \\ \ddot{u} + b\sin u = -\partial_u P(t,q,u), \end{cases} \tag{35}$$

where $P(t,q,u)$ and $e(t)$ are as above, has at least two geometrically distinct T-periodic solutions, for any $a > 0$, if $\|\partial_u P\|_\infty < b$.

We thus extend two classical results on the pendulum equation (cf. [42]). The first one states that, if $e(t)$ is T-periodic and $\int_0^T e(t)\,dt = 0$, then

$$\ddot{q} + a\sin q = e(t)$$

has at least two geometrically distinct T-periodic solutions, for any $a > 0$. The second one states that, if $\hat{e}(t)$ is T-periodic and $\|\hat{e}\|_\infty \le b$, then

$$\ddot{u} + b\sin u = \hat{e}(t)$$

has at least two geometrically distinct T-periodic solutions.

Here are some other possible examples of applications.

1. Let $f(u) = |u|^{\ell-1}u$, with $\ell > 0$, and let P be a function as above, and such that

$$\lim_{u\to\pm\infty} \frac{\partial_u P(t,q,u)}{|u|^\ell} = 0\,, \qquad \text{uniformly in } (t,q) \in [0,T] \times [0,2\pi]\,.$$

Then, there exist two constants $\beta = -\alpha > 0$ large enough such that (34) is satisfied. In particular, if $\partial_u P$ is bounded, it is enough to take

$$\alpha < -\|\partial_u P\|_\infty^{1/\ell}, \qquad \beta > \|\partial_u P\|_\infty^{1/\ell}.$$

Notice that this includes, e.g., the case of $\partial_u P$ being periodic in u. Hence, in this situation, there exist at least two geometrically distinct T-periodic solutions of system (33).

2. Let $f(u) = |u|^\ell \sin(u)$, with $\ell > 0$, and let P be as in the previous example. Then (34) is satisfied for an infinite number of positive and negative pairs (α_i, β_i). Hence, there exist infinitely many T-periodic solutions of system (33) with positive u component and infinitely many T-periodic solutions with negative u component.

3. Consider $f(u) = \arctan(u)$. Then, if $\partial_u P$ is bounded, with

$$\|\partial_u P\|_\infty < \frac{\pi}{2}\,,$$

there exist at least two geometrically distinct T-periodic solutions of system (33). It is sufficient to take $\beta = -\alpha > 0$ large enough.

When $P(t, q, u)$ is *not* periodic in q, we can appeal to Theorem 10. Consider for example the system

$$\begin{cases} -\ddot{q} = g(q) + \partial_q P(t, q, u)\,, \\ -\ddot{u} = -f(u) + \partial_u P(t, q, u)\,, \end{cases} \tag{36}$$

where $P(t, q, u)$ is continuously differentiable in (q, u). We can prove the following.

Corollary 13. *Assume that*

$$\lim_{|q|\to\infty} \frac{g(q)}{q} = +\infty\,, \tag{37}$$

and that $\partial_q P$ is bounded. If there exist two constants $\alpha < \beta$ such that (34) *holds, then system* (36) *has infinitely-many T-periodic solutions.*

Proof. We will briefly explain the main arguments. At first we modify system (36) by introducing the functions

$$\begin{aligned} \widehat{f}(u) &= (u - \beta)^+ - (u - \alpha)^- + f(\zeta(u))\zeta'(u)\,, \\ \widehat{P}(t, q, u) &= P(t, q, \zeta(u))\,, \end{aligned}$$

thus obtaining

$$\begin{cases} \dot{q} = p\,, \qquad \dot{p} = -g(q) - \partial_q \widehat{P}(t, q, u)\,, \\ \dot{u} = v\,, \qquad \dot{v} = \widehat{f}(u) - \partial_u \widehat{P}(t, q, u)\,, \end{cases} \tag{38}$$

and denoting by $\widehat{H}(t, z) = \frac{1}{2}(v^2 - u^2) + \widehat{h}(t, z)$ its Hamiltonian, for a suitable choice of $\widehat{h}$. Let $B(0, R) \subseteq \mathbb{R}^2$ be the ball of radius $R > 0$ centered at the origin, and set $\mathcal{B}_R := B(0, R) \times \mathbb{R}^2$. Then, following the arguments in [13, Section 2], we can prove that any solution of system (38) is globally defined in the interval $[0, T]$. Moreover, by the elastic property stated in [13, Lemma 1] we can find $r_3 > r_2 > r_1 > 0$ such that any solution $z(t) = (q(t), p(t), u(t), v(t))$ of (38) starting at $t = 0$ from a point belonging to $\partial\mathcal{B}_{r_2}$ satisfies $z(t) \in \mathcal{B}_{r_3} \setminus \mathcal{B}_{r_1}$ for every $t \in [0, T]$. For such solutions we can find $k > 0$ such that $\mathrm{Rot}\,((q, p); [0, T]) < k$.

Let us fix any $k' \geq k$. Assumption (37) provides the existence of $R_1 > 0$ such that every solution z of (38), with $z(t) \notin \mathcal{B}_{R_1}$ for any $t \in [0, T]$, satisfies $\mathrm{Rot}\,((q, p); [0, T]) > k'$. Then, using again [13, Lemma 1], we can find $R_3 > R_2 > R_1$ such that any solution z of (38) starting at $t = 0$ from a point belonging to $\partial\mathcal{B}_{R_2}$ satisfies $z(t) \in \mathcal{B}_{R_3} \setminus \mathcal{B}_{R_1}$ for every $t \in [0, T]$. Hence, we have for such solutions $\mathrm{Rot}\,((q, p); [0, T]) > k'$.

Now, we introduce a smooth cut-off function $\widehat{\eta} : [0, +\infty[\to [0, 1]$ such that

$$\widehat{\eta}(r) = \begin{cases} 0 & \text{if } r \in [0, r_1/2] \cup [2R_3, +\infty[\,, \\ 1 & \text{if } r \in [r_1, R_3]\,. \end{cases}$$

and the new Hamiltonian $\widetilde{H}(t, z) = \frac{1}{2}(v^2 - u^2) + \widetilde{h}(t, z)$ where now

$$\widetilde{h}(t, z) = \widehat{\eta}\big(\sqrt{q(t)^2 + p(t)^2}\big)\widehat{h}(t, z)\,.$$

Finally, we are able to apply Theorem 10 to system (38) thus obtaining two T-periodic solutions satisfying $\mathrm{Rot}\,((q, p); [0, T]) = k'$ with starting point $z(0) \in \mathcal{B}_{R_2} \setminus \overline{\mathcal{B}_{r_2}}$. Then, we can easily see that they are indeed solutions of (36).

Since the above construction can be made for every $k' \geq k$, we have proved the existence of infinitely many T-periodic solutions. □

We thus extend a result by Ding and Zanolin [13], stating that, if $g(q)$ satisfies (37) and $e(t)$ is T-periodic, the scalar equation

$$\ddot{q} + g(q) = e(t)$$

has infinitely-many T-periodic solutions. A similar result for system (33) remains an open problem, because global existence of the solutions is not guaranteed. However, one could follow the lines of [25, 34, 35, 36] by assuming that the first equation has 0 as an equilibrium point, and then prove that there are infinitely many T-periodic solutions.

Whenever the function g has a sublinear growth at infinity, the existence of periodic solutions whose minimal period is an arbitrarily large integer multiple of T has been investigated in [14]. These are called *subharmonic* solutions. We could also state such kind of result here, but we avoid the details, for briefness.

Another situation where our results can be applied is when the system in (q, p) has a different rotational behaviour at zero and at infinity. There are many papers dealing with such a problem (see, e.g., [39] and the references therein). This type of situation can be also exploited when dealing with some kind of asymmetric oscillators as, e.g., in [7], where the so-called *jumping nonlinearities* are treated. We do not enter into details, again, to be brief.

6 Going to higher dimensions

We consider system (1), assuming the Hamiltonian function $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ to be continuous, T-periodic in its first variable t, and continuously differentiable with respect to its second variable z, with corresponding gradient $\nabla H(t, z)$.

For $z \in \mathbb{R}^{2N}$, we write $z = (\phi, \psi, q, p, u, v)$, where, for some nonnegative integers L, M and D,

$$\begin{array}{ll} \phi = (\phi_1, \dots, \phi_L) \in \mathbb{R}^L, & \psi = (\psi_1, \dots, \psi_L) \in \mathbb{R}^L, \\ q = (q_1, \dots, q_M) \in \mathbb{R}^M, & p = (p_1, \dots, p_M) \in \mathbb{R}^M, \\ u = (u_1, \dots, u_D) \in \mathbb{R}^D, & v = (v_1, \dots, v_D) \in \mathbb{R}^D. \end{array}$$

Notice that one or more of these integers could be equal to zero, in which case the corresponding group will not be taken into account; for example, if $L = 0$, then ϕ and ψ will disappear from the list.

Our system (1) then reads as

$$\begin{cases} \dot{\phi} = \partial_\psi H(t,z)\,, & \dot{\psi} = -\partial_\phi H(t,z)\,, \\ \dot{q} = \partial_p H(t,z)\,, & \dot{p} = -\partial_q H(t,z)\,, \\ \dot{u} = \partial_v H(t,z)\,, & \dot{v} = -\partial_u H(t,z)\,. \end{cases} \tag{39}$$

Let us introduce our assumptions.

Assumption 6 (Periodicity)**.** *The function $H(t,z)$ is periodic in each of the variables included in ϕ, ψ, q.*

To fix the ideas, we assume that all periods are equal to 2π. The total number of variables in which our Hamiltonian function is 2π-periodic is thus $2L+M$. Under this setting, T-periodic solutions $z(t)$ of (1) appear in equivalence classes made of those solutions whose components in $\phi(t)$, $\psi(t)$, $q(t)$, differ by an integer multiple of 2π. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

In the sequel, inequalities $\leq$ involving vectors are to be interpreted componentwise. Moreover, for any $\sigma \in \mathbb{R}$, we use the notation $\bar{\sigma} = (\sigma, \dots, \sigma) \in \mathbb{R}^D$.

Assumption 7 (Lower and upper solutions)**.** *There exist some constants $\delta > 0$, $\alpha = (\alpha_1, \dots, \alpha_D)$ and $\beta = (\beta_1, \dots, \beta_D)$ with $\alpha \leq \beta$, having the following property. If $\alpha - \bar{\delta} \leq u \leq \beta + \bar{\delta}$, then, for every $j \in \{1, \dots, D\}$,*

$$v_j\, \partial_{v_j} H(t,z) > 0\,, \text{ when } u_j \in [\alpha_j - \delta, \alpha_j] \cup [\beta_j, \beta_j + \delta] \text{ and } v_j \neq 0\,,$$

and

$$\begin{cases} \partial_{u_j} H(t,z) \geq 0\,, & \text{when } u_j \in [\alpha_j - \delta, \alpha_j] \text{ and } v_j = 0\,, \\ \partial_{u_j} H(t,z) \leq 0\,, & \text{when } u_j \in [\beta_j, \beta_j + \delta] \text{ and } v_j = 0\,. \end{cases}$$

In the sequel, the constant $\delta > 0$ provided by Assumption 7 will be used without further mention.

Assumption 8 (Nagumo condition)**.** *For every $j \in \{1, \dots, D\}$ there exist $d_j > 0$ and two continuous functions $f_j, \varphi_j : [d_j, +\infty[\to]0, +\infty[$, with*

$$\int_{d_j}^{+\infty} \frac{f_j(s)}{\varphi_j(s)}\, ds = +\infty\,,$$

satisfying the following property. If $\alpha - \bar{\delta} \leq u \leq \beta + \bar{\delta}$, then

$$\begin{cases} \partial_{v_j} H(t,z) \geq f_j(v_j)\,, & \text{when } v_j \geq d_j\,, \\ \partial_{v_j} H(t,z) \leq -f_j(-v_j)\,, & \text{when } v_j \leq -d_j\,, \end{cases}$$

and

$$\big|\partial_{u_j} H(t,z)\big| \leq \varphi_j(|v_j|)\,, \quad \text{when } |v_j| \geq d_j\,.$$

Assumption 9 (Linear growth)**.** *For every $K > 0$ there is a constant $C_K > 0$ with the following property. If $\alpha - \bar{\delta} \leq u \leq \beta + \bar{\delta}$ and $|v| \leq K$, then*

$$|\partial_q H(t,z)| \leq C_K(|p| + 1)\,.$$

Adapting the argument in Remark 1, under the above assumption, for any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$ with

$$\alpha - \bar{\delta} \le U(t) \le \beta + \bar{\delta}\,, \quad \text{for every } t \in [0,T]\,, \tag{40}$$

the solutions of the system

$$\begin{cases} \dot{\phi} = \partial_\psi H(t,\phi,\psi,q,p,U(t),V(t))\,, \\ \dot{\psi} = -\partial_\phi H(t,\phi,\psi,q,p,U(t),V(t))\,, \\ \dot{q} = \partial_p H(t,\phi,\psi,q,p,U(t),V(t))\,, \\ \dot{p} = -\partial_q H(t,\phi,\psi,q,p,U(t),V(t))\,, \end{cases} \tag{41}$$

are defined on $[0,T]$ and, setting $K = \|V\|_\infty$,

$$|p(t)| \le (|p(0)|+1)\, e^{C_K T}, \qquad \text{for every } t \in [0,T]\,. \tag{42}$$

In the following, we consider a convex body $\mathcal{D}$ of $\mathbb{R}^M$, i.e., a closed convex bounded set with nonempty interior. We denote by $\pi_{\mathcal{D}} \colon \mathbb{R}^M \setminus \mathring{\mathcal{D}} \to \partial\mathcal{D}$ the projection on the convex set $\mathcal{D}$ and by $\nu_{\mathcal{D}}(\zeta)$ the unit outward normal at $\zeta \in \partial\mathcal{D}$, assuming that $\mathcal{D}$ has a smooth boundary. We say that $\mathcal{D}$ is *strongly convex* if, for any $p \in \partial\mathcal{D}$, the function $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ defined as $\mathcal{F}(\eta) = \langle \eta - p, \nu_{\mathcal{D}}(p)\rangle$ has a unique maximum point at $\eta = p$.

Here is our first result in this higher dimensional setting, generalizing Theorem 2.

Theorem 14. *Let Assumptions 6, 7, 8, and 9 hold. Assume that there exist $\rho > 0$, a symmetric regular $M \times M$ matrix $\mathbb{A}$ and a strongly convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, with the following property: For any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$ satisfying* (40)*, the solutions of* (41) *with $p(0) \notin \mathring{\mathcal{D}}$ and $\operatorname{dist}(p(0), \partial\mathcal{D}) \le \rho$ are such that*

$$\big\langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big)\big\rangle > 0\,.$$

Then, system (39) *has at least $2L + M + 1$ geometrically distinct T-periodic solutions, satisfying*

$$p(0) \in \mathring{\mathcal{D}}\,,$$

and

$$\alpha \le u(t) \le \beta\,, \quad \textit{for every } t \in \mathbb{R}\,. \tag{43}$$

7 Proof of Theorem 14

Since the arguments will be similar to those provided in Section 3, we will try to be brief.

At first we need to suitably modify the Hamiltonian system working componentwise in the (u,v) variables.

For every $j \in \{1,\dots,D\}$, from Assumption 8 we can find some continuously differentiable functions $\gamma_j^\pm : \mathbb{R} \to \mathbb{R}$ and then $\widehat{d}_j > 0$ such that

$$-\widehat{d}_j < \gamma_j^-(s) < -d_j\,, \quad d_j < \gamma_j^+(s) < \widehat{d}_j\,,$$

for every $s \in [\alpha_j - \delta, \beta_j + \delta]$, satisfying

$$-\partial_{u_j} H\big(t, z\big) > \partial_{v_j} H\big(t, z\big)(\gamma_j^+)'(u_j)\,, \quad \text{when } v_j = \gamma_j^+(u_j)\,,$$
$$-\partial_{u_j} H\big(t, z\big) < \partial_{v_j} H\big(t, z\big)(\gamma_j^-)'(u_j)\,, \quad \text{when } v_j = \gamma_j^-(u_j)\,,$$

for every $(t, z) \in [0, T] \times \mathbb{R}^{2N}$ with $u_j \in [\alpha_j - \delta, \beta_j + \delta]$.

We then define $\gamma^\pm : \mathbb{R}^D \to \mathbb{R}^D$ as

$$\gamma^\pm(u) = (\gamma_1^\pm(u_1), \dots, \gamma_D^\pm(u_D))\,,$$

and introduce the set

$$\mathcal{V} = \{z = (\phi, \psi, q, p, u, v) \mid \alpha \le u \le \beta\,,\, \gamma^-(u) < v < \gamma^+(u)\}\,,$$

recalling that we need to check the inequalities componentwise. We can then introduce some functions $\zeta_j : \mathbb{R} \to \mathbb{R}$ and $\chi_j : [\alpha_j - \delta, \beta_j + \delta] \times \mathbb{R} \to \mathbb{R}$ similarly as in (12) and (14), define

$$\zeta(u) = (\zeta_1(u_1), \dots, \zeta_D(u_D))\,,$$

$$\chi(u, v) = (\chi_1(u_1, v_1), \dots, \chi_D(u_D, v_D))\,,$$

and consider the modified Hamiltonian $\widehat{H} : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ as

$$\widehat{H}(t, \phi, \psi, q, p, u, v) = H\big(t, \phi, \psi, q, p, \zeta(u), \chi(\zeta(u), v)\big) + \mathscr{H}(u, v)\,,$$

with

$$\mathscr{H}(u, v) = \frac{1}{2}\sum_{j=1}^{D}\bigg[\Big[(v_j - \widehat{d}_j)^+\Big]^2 + \Big[(v_j + \widehat{d}_j)^-\Big]^2 - \big[(u_j - \beta_j)^+\big]^2 - \big[(u_j - \alpha_j)^-\big]^2\bigg].$$

The modified Hamiltonian system

$$J\dot{z} = \nabla\widehat{H}(t, z) \tag{44}$$

complies the following a priori bound, whose proof can be given adapting the one of Lemma 3, arguing separately on every couple of variables (u_j, v_j), and verifying the validity of some analogues of Propositions 4, 5, 6, and 7.

Lemma 15. *If $z = (\phi, \psi, q, p, u, v)$ is a T-periodic solution of* (44)*, then $z(t) \in \mathcal{V}$ for every $t \in [0, T]$, hence it solves* (39).

The next step involves the (q, p) variables. The reasoning in Section 3.2 can be adapted to higher dimension. We fix $K > 0$ such that

$$|\chi(\zeta(u), v)| < K\,, \text{ for every } (u, v) \in \mathbb{R}^{2D}, \tag{45}$$

and, given the convex body $\mathcal{D}$ and the constant $\rho > 0$ as in the statement of the theorem, we set

$$P_0 = \max\{|p| : p \in \mathcal{D}\}\,, \quad \text{and} \quad c = (P_0 + \rho + 1)\mathrm{e}^{C_K T}\,.$$

Let $z(t)$ be a solution of system (44) having $[0,\omega[$ as its maximal interval of future existence, starting with $p(0)$ satisfying $\mathrm{dist}(p(0),\mathcal{D}) \le \rho$. Arguing as at the beginning of Section 3.2, from Assumption 9 and (42), we have that

$$|p(t)| \le c\,, \qquad \text{for every } t \in [0,T] \cap [0,\omega[\,. \tag{46}$$

So, we can introduce the cut-off function η as in (25) in order to change the Hamiltonian function

$$\widehat{H}(t,z) = \frac{1}{2}(|v|^2 - |u|^2) + \widehat{h}(t,z)\,.$$

into the new Hamiltonian function

$$\widetilde{H}(t,z) = \frac{1}{2}(|v|^2 - |u|^2) + \widetilde{h}(t,z)\,.$$

with $\widetilde{h}(t,z) = \eta(|p|)\,\widehat{h}(t,z)$. Notice that $|\nabla\widetilde{h}(t,z)| \le \widetilde{C}$ for a certain positive constant $\widetilde{C}$.

Our aim is now to apply [18, Corollary 2.3] to the modified system

$$J\dot{z} = \nabla\widetilde{H}(t,z)\,. \tag{47}$$

Any solution $z = (\phi,\psi,q,p,u,v)$ of (47) is defined on $[0,T]$. As in the proof of Theorem 2, we can prove that, if $\mathrm{dist}(p(0),\mathcal{D}) \le \rho$, then $|p(t)| \le c$ for every $t \in [0,T]$. In particular it is a solution of (44). Then, if we set $U(t) = \zeta(u(t))$ and $V(t) = \chi(\zeta(u(t)),v(t))$, we see that (ϕ,ψ,q,p) is a solution of (41). By the assumption of the theorem, if the solution starts with $\mathrm{dist}(p(0),\partial\mathcal{D}) \le \rho$ and $p(0) \notin \mathring{\mathcal{D}}$, then

$$\big\langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big)\big\rangle > 0\,. \tag{48}$$

The application of [18, Corollary 2.3] provides us $2L+M+1$ geometrically distinct T-periodic solutions of (47) satisfying $p(0) \in \mathring{\mathcal{D}}$. Then, by the above estimates, they are solutions of (44). Finally, from Lemma 15, these are indeed solutions of the original Hamiltonian system (39), and they satisfy $\alpha \le u(t) \le \beta$, for every $t \in \mathbb{R}$. □

8 Variants in higher dimensions

Here are two variants of Theorem 14. In the first one, the twist is formulated as an *avoiding rays condition.*

Theorem 16. *Let Assumptions 6, 7, 8, and 9 hold. Assume that there exist $\rho > 0$ and a convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, with the following property: For any two continuous functions $U,V : [0,T] \to \mathbb{R}^D$ satisfying* (40)*, the solutions of system* (41) *with $p(0) \notin \mathring{\mathcal{D}}$ and $\mathrm{dist}(p(0),\partial\mathcal{D}) \le \rho$ are such that*

$$q(T) - q(0) \notin \{\lambda\nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big) : \lambda \ge 0\}\,. \tag{49}$$

Then, the same conclusion of Theorem 14 holds.

Proof. The argument follows the lines of the proof of Theorem 14, with the only difference of applying [18, Corollary 2.1] instead of [18, Corollary 2.3]. □

Notice that the twist condition (49) may as well be replaced by

$$q(T) - q(0) \notin \{\lambda \nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big) : \lambda \le 0\},$$

and the same conclusion of Theorem 14 still holds.

We now consider the case when $\mathcal{D}$ is a M-cell, namely

$$\mathcal{D} = [a_1, b_1] \times \cdots \times [a_M, b_M].$$

Theorem 17. *Let Assumptions 6, 7, 8, and 9 hold. Assume that there exist $\rho > 0$ and a M-tuple $\sigma = (\sigma_1, \dots, \sigma_M) \in \{-1, 1\}^M$, with the following property: For any two continuous functions $U, V : [0, T] \to \mathbb{R}^D$ satisfying* (40)*, the solutions of* (41) *with $p(0) \notin \mathring{\mathcal{D}}$ and $\mathrm{dist}(p(0), \partial\mathcal{D}) \le \rho$ are such that, for every $i \in \{1, \dots, M\}$,*

$$\begin{cases} \sigma_i(q_i(T) - q_i(0)) < 0, & \text{when } p_i(0) \in [a_i - \rho, a_i], \\ \sigma_i(q_i(T) - q_i(0)) > 0, & \text{when } p_i(0) \in [b_i, b_i + \rho]. \end{cases}$$

Then, the same conclusion of Theorem 14 holds.

Proof. Again the proof is similar to the one of Theorem 14, just applying [18, Corollary 2.4] instead of [18, Corollary 2.3]. □

Remark 18. *As noticed in Remark 9, instead of the fixed intervals $[a_i, b_i]$ defining the M-cell $\mathcal{D}$, we could have varying intervals $[a_i(q_i), b_i(q_i)]$, where $a_i, b_i : \mathbb{R} \to \mathbb{R}$ are continuous 2π-periodic functions.*

We now generalize Theorem 10; we thus drop the periodicity in the q-variables, still maintaining it in the ϕ and ψ variables, as stated below.

Assumption 10 (Periodicity)**.** *The function $H(t, z)$ is periodic in each of the variables included in ϕ, ψ.*

We also need to suitably modify Assumption 4, in the following way

Assumption 11 (Energy growth)**.** *For every $K > 0$ there is a constant $C_K > 0$ such that, for every $i \in \{1, \dots, M\}$,*

$$\big|q_i\, \partial_{p_i} H(t, q, p, u, v) - p_i\, \partial_{q_i} H(t, q, p, u, v)\big| \le C_K(q_i^2 + p_i^2 + 1), \quad \text{when } u \in [\alpha - \delta, \beta + \delta] \text{ and } |v| \le K.$$

Here is the generalization of Theorem 10.

Theorem 19. *Let Assumptions 7, 8, 10 and 11 hold. Let $k_1, \dots, k_M$ be integers and assume that that there exist $\rho > 0$, $\tilde{\rho} > 0$ and, for each $i \in \{1, \dots, M\}$ there exist two planar Jordan curves Γ_1^i, Γ_2^i, strictly star-shaped with respect to the origin, with*

$$0 \in \mathcal{D}(\Gamma_1^i) \subseteq \overline{\mathcal{D}(\Gamma_1^i)} \subseteq \mathcal{D}(\Gamma_2^i),$$

such that, for any two continuous functions $U, V : [0, T] \to \mathbb{R}$ satisfying (40)*, the solutions of system* (41) *with*

$$\mathrm{dist}\big((q_i(0), p_i(0)), \overline{\mathcal{D}(\Gamma_2^i)} \setminus \mathcal{D}(\Gamma_1^i)\big) \le \rho \quad \text{for every } i \in \{1, \dots, M\}$$

which are defined on $[0,T]$*, satisfy*

$$q_i(t)^2 + p_i(t)^2 \geq \tilde{\rho}\,, \ \textit{for every } t \in [0,T]\,,$$

and, if $(q_i(0), p_i(0)) \notin \mathcal{D}(\Gamma_2^i) \setminus \overline{\mathcal{D}(\Gamma_1^i)}$,

$$\begin{aligned} &\mathrm{Rot}\,((q_i,p_i);[0,T]) < k_i\,, \ \ \textit{when } \mathrm{dist}\big((q_i(0),p_i(0)),\Gamma_1^i\big) \leq \rho\,,\\ &\mathrm{Rot}\,((q_i,p_i);[0,T]) > k_i\,, \ \ \textit{when } \mathrm{dist}\big((q_i(0),p_i(0)),\Gamma_2^i\big) \leq \rho\,. \end{aligned} \tag{50}$$

Then, system (39) *has at least* $2L+M+1$ *distinct* T*-periodic solutions*

$$z^{(1)}(t)\,, \ \ldots\,, \ z^{(2L+M+1)}(t)\,,$$

satisfying (43)*, with*

$$(q_i^{(n)}(0), p_i^{(n)}(0)) \in \mathcal{D}(\Gamma_2^i) \setminus \overline{\mathcal{D}(\Gamma_1^i)}\,,$$

and

$$\mathrm{Rot}\,((q_i^{(n)}, p_i^{(n)});[0,T]) = k_i\,,$$

for every $i = 1,\ldots,M$ *and* $n = 1,\ldots,2L+M+1$*. The same is true if, for some* $i \in \{1,\ldots,M\}$*, assumption (50) is replaced by*

$$\begin{aligned} &\mathrm{Rot}\,((q_i,p_i);[0,T]) > k_i\,, \ \ \textit{when } \mathrm{dist}\big((q_i(0),p_i(0)),\Gamma_1^i\big) \leq \rho\,,\\ &\mathrm{Rot}\,((q_i,p_i);[0,T]) < k_i\,, \ \ \textit{when } \mathrm{dist}\big((q_i(0),p_i(0)),\Gamma_2^i\big) \leq \rho\,. \end{aligned}$$

Proof. It is perfectly analogous to the one of Theorem 10, the only difference being the use of Theorem 17 (and Remark 18) instead of Theorem 2. □

Let us now add a further equation, and consider the more general system

$$\begin{cases} \dot{\phi} = \partial_\psi H(t,z)\,, & \dot{\psi} = -\partial_\phi H(t,z)\,,\\ \dot{q} = \partial_p H(t,z)\,, & \dot{p} = -\partial_q H(t,z)\,,\\ \dot{u} = \partial_v H(t,z)\,, & \dot{v} = -\partial_u H(t,z)\,,\\ J\dot{w} = \partial_w H(t,z)\,. \end{cases} \tag{51}$$

Here, and in the following, the symbol J is always used as the standard symplectic matrix, in different dimensions. Moreover, now $z = (\phi,\psi,q,p,u,v,w)$. We will generalize Theorem 14. Assumptions 6, 7 and 8 will remain the same, while Assumption 9 needs to be modified as follows.

Assumption 12 (Linear growth)**.** *Let*

$$H(t,z) = \tfrac{1}{2}\big\langle \mathbb{B}(t)w, w\big\rangle + \mathcal{H}(t,z)\,,$$

where the symmetric matrix $\mathbb{B}(t)$ *is continuous,* T*-periodic, and such that*

$$w(t) \equiv 0 \quad \textit{is the only } T\textit{-periodic solution of } J\dot{w} = \mathbb{B}(t)w\,.$$

The function $\mathcal{H}$ *is such that for every* $K > 0$ *there is a constant* $C_K > 0$ *with the following property: If* $\alpha - \bar{\delta} \leq u \leq \beta + \bar{\delta}$ *and* $|v| \leq K$*, then*

$$|\partial_q \mathcal{H}(t,z)| + |\partial_w \mathcal{H}(t,z)| \leq C_K(|p|+1)\,.$$

Under the above assumption, given any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$, with

$$\alpha - \bar{\delta} \leq U(t) \leq \beta + \bar{\delta}\,, \quad \text{for every } t \in [0,T]\,, \tag{52}$$

the solutions of the system

$$\begin{cases} \dot{\phi} = \partial_\psi H(t,\phi,\psi,q,p,U(t),V(t),w)\,, \\ \dot{\psi} = -\partial_\phi H(t,\phi,\psi,q,p,U(t),V(t),w)\,, \\ \dot{q} = \partial_p H(t,\phi,\psi,q,p,U(t),V(t),w)\,, \\ \dot{p} = -\partial_q H(t,\phi,\psi,q,p,U(t),V(t),w)\,, \\ J\dot{w} = \partial_w H(t,\phi,\psi,q,p,U(t),V(t),w)\,, \end{cases} \tag{53}$$

are defined on $[0,T]$, cf. Remark 1.

Theorem 20. *Let Assumptions 6, 7, 8 and 12 hold. Assume that there exist $\rho > 0$, a symmetric regular $M \times M$ matrix $\mathbb{A}$ and a strongly convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, with the following property: For any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$ satisfying* (52)*, the solutions of* (53) *with $p(0) \notin \mathring{\mathcal{D}}$ and $\operatorname{dist}(p(0), \partial\mathcal{D}) \leq \rho$ are such that*

$$\Big\langle q(T) - q(0)\,,\, \mathbb{A}\nu_{\mathcal{D}}\big(\pi_{\mathcal{D}}(p(0))\big)\Big\rangle > 0\,. \tag{54}$$

Then, system (51) *has at least $2L + M + 1$ geometrically distinct T-periodic solutions, such that $p(0) \in \mathring{\mathcal{D}}$, and $\alpha \leq u(t) \leq \beta$, for every $t \in \mathbb{R}$.*

Proof. Following the lines of the proof of Theorem 14, we can introduce a modified system ruled by a Hamiltonian function of the type

$$\widehat{H}(t,z) = \tfrac{1}{2}\big\langle \mathbb{B}(t)w, w\big\rangle + \frac{1}{2}(|v|^2 - |u|^2) + \widehat{h}(t,z)\,.$$

Moreover, $\widehat{H}$ can be introduced so to guarantee that every T-periodic solution of $J\dot{z} = \nabla\widehat{H}(t,z)$ satisfies an a priori bound as in Lemma 15, in particular $\alpha \leq u \leq \beta$. Then, from Assumption 12, we can introduce $K > 0$ as in (45) such that, if $\alpha - \bar{\delta} \leq u \leq \beta + \bar{\delta}$ and $|v| \leq K$, then

$$|\partial_q \widehat{h}(t,z)| + |\partial_w \widehat{h}(t,z)| \leq \widehat{C}_K(|p| + 1)\,.$$

Hence, we can find a constant $c > 0$ such that any solution of $J\dot{z} = \nabla\widehat{H}(t,z)$ starting with $z(0)$ satisfying $\operatorname{dist}(p(0), \mathcal{D}) \leq \rho$ is such that (46) holds.

Again, we can introduce the cut-off function η, as in (25), and the Hamiltonian function

$$\widetilde{H}(t,z) = \tfrac{1}{2}\big\langle \mathbb{B}(t)w, w\big\rangle + \frac{1}{2}(|v|^2 - |u|^2) + \widetilde{h}(t,z)\,,$$

where $\widetilde{h}(t,z) = \eta(|p|)\widehat{h}(t,z)$. Using Assumption 12 we get

$$|\nabla\widetilde{h}(t,z)| \leq \widetilde{C}\,,$$

for a certain positive constant $\widetilde{C}$. We can rewrite the previous Hamiltonian as

$$\widetilde{H}(t,z) = \tfrac{1}{2}\Big\langle \mathbb{M}(t)\big(w,(u,v)\big), \big(w,(u,v)\big)\Big\rangle + \widetilde{h}(t,z)\,,$$

where

$$\mathbb{M}(t)\big(w,(u,v)\big) = \big(\mathbb{B}(t)w,(-u,v)\big)\,.$$

Our aim is to apply [18, Corollary 2.3] to the modified system $J\dot{z} = \nabla \widetilde{H}(t,z)$. We easily verify the nonresonance condition

$$\omega(t) \equiv 0 \quad \text{is the only } T\text{-periodic solution of } J\dot{\omega} = \mathbb{M}(t)\omega\,,$$

where $\omega = (w,(u,v))$.

We finally verify, with the usual argument, that any solution z of the previous system with $0 < \operatorname{dist}(p(0),\mathcal{D}) \le \rho$ is such that

$$\langle q(T) - q(0), \mathbb{A}p(0)\rangle > 0\,. \tag{55}$$

The application of [18, Corollary 2.3] provides us $2L + M + 1$ geometrically distinct T-periodic solutions of $J\dot{z} = \nabla \widetilde{H}(t,z)$ satisfying $p(0) \in \mathring{\mathcal{D}}$. Then, as in the previous proofs we end showing that they are solutions of (51), too, and they satisfy $\alpha \le u(t) \le \beta$, for every $t \in \mathbb{R}$. □

Clearly enough, the twist condition (54) could be replaced by an avoiding rays condition, as in Theorem 16, or by a sign condition on the edges of an M-cell, like in Theorem 17. Also, we could provide a statement on an annulus, similarly as in Theorem 19. Or even some combination of these could be considered. We avoid the details, for briefness.

9 Examples in higher dimensions

In this section we just briefly mention how the examples given in Section 5 generalize to higher dimensions applying the results of Sections 6 and 8. To this aim, consider a system of the form

$$\begin{cases} \dot{\phi} = \partial_\psi P(t,\phi,\psi,q,u)\,, \quad \dot{\psi} = -\partial_\phi P(t,\phi,\psi,q,u)\,, \\ -\ddot{q}_i = \ g_i(t,q_i) - e_i(t) + \partial_{q_i} P(t,\phi,\psi,q,u)\,, \quad i = 1,\dots,M, \\ -\ddot{u}_j = -h_j(u_j) + \partial_{u_j} P(t,\phi,\psi,q,u)\,, \quad j = 1,\dots,D, \end{cases} \tag{56}$$

where all functions are continuous and T-periodic in t. For simplicity, we assume that $P(t,\phi,\psi,q,u)$ is 2π-periodic in each of the variables included in ϕ, ψ and q, so, due to this, we fix an arbitrary cube in $\mathbb{R}^{2L+M}$ of length $[0,2\pi]$, denoting it by Θ. Moreover, let $P(t,\phi,\psi,q,u)$ be continuously differentiable in (ϕ,ψ,q,u) and assume that, for every $i \in \{1,\dots,M\}$,

$$\int_0^T e_i(t)dt = 0 \quad \text{and} \quad \int_0^{2\pi} g_i(t,s)ds = 0\,.$$

Then system (56) is of the type (39), with

$$\begin{aligned} H(t,z) = \frac{1}{2}|p + E(t)|^2 + \frac{1}{2}|v|^2 + \sum_{i=1}^{M} \int_0^{q_i} g(t,s)\,ds \\ - \sum_{j=1}^{D} \int_0^{u_j} h_j(\sigma)\,d\sigma + P(t,\phi,\psi,q,u)\,, \end{aligned}$$

where $z = (\phi,\psi,q,p,u,v)$ and $E = (E_1,\dots,E_M) : [0,T] \to \mathbb{R}^M$ is a primitive of the field $e(t) = (e_1(t),\dots,e_M(t))$.

Let us state the analogues of Corollaries 11 and 12.

Corollary 21. *In the above setting, assume moreover that there exist two vectors $\alpha, \beta \in \mathbb{R}^D$, with $\alpha \le \beta$, such that, for all $j \in \{1, \dots, D\}$,*

$$\begin{aligned} h_j(\alpha_j) &< \partial_{u_j} P(t, \phi, \psi, q, u), \qquad \text{when } u_j = \alpha_j\,, \\ h_j(\beta_j) &> \partial_{u_j} P(t, \phi, \psi, q, u), \qquad \text{when } u_j = \beta_j\,, \end{aligned} \tag{57}$$

for all $(t, \phi, \psi, q) \in [0, T] \times \Theta$ and all $u \in \mathbb{R}^D$ such that $\alpha \le u \le \beta$. Then, system (33) *has at least $2L + M + 1$ geometrically distinct T-periodic solutions, with $\alpha \le u \le \beta$.*

Proof. All the assumptions of Theorem 17 are easily verified, whence the conclusion. □

As an immediate consequence, we have the following.

Corollary 22. *The system*

$$\begin{cases} \dot\phi = \partial_\psi P(t, \phi, \psi, q, u)\,, \quad \dot\psi = -\partial_\phi P(t, \phi, \psi, q, u)\,, \\ \ddot q_i + a_i \sin q_i = e_i(t) - \partial_{q_i} P(t, \phi, \psi, q, u)\,, \quad i = 1, \dots, M, \\ \ddot u_j + b_j \sin u_j = -\partial_{u_j} P(t, \phi, \psi, q, u)\,, \quad j = 1, \dots, D, \end{cases} \tag{58}$$

where $P(t, \phi, \psi, q, u)$ and $e(t)$ are as above, has at least $2L+M+1$ geometrically distinct T-periodic solutions, for any $a_i > 0$, if $\|\partial_{u_j} P\|_\infty < b_j$ for every $j \in \{1, \dots, D\}$.

All the other examples presented in Section 5 can be displayed in this more general setting, also with a mixing of assumptions on each component. In particular, we thus generalize, e.g., the results in [7, 8, 10, 13, 14, 25, 28, 32, 35, 36, 40, 41]. Also equations with singularities could be considered, as in [22, 26]. We will avoid entering further into details, for briefness.

A further example of application is analyzed in the next section.

10 Periodic perturbations of completely integrable systems

In [3, 18], perturbations of completely integrable Hamiltonian systems were studied (see also [19], and the references therein). We will now add an extra term to the Hamiltonian function, involving lower and upper solutions.

We consider the system

$$\begin{cases} \dot\varphi = \nabla\mathcal{K}(I) + \varepsilon\partial_I P(t, \varphi, I, u)\,, \quad \dot I = -\varepsilon\partial_\varphi P(t, \varphi, I, u)\,, \\ -\ddot u = \partial_u G(t, u) + \varepsilon\partial_u P(t, \varphi, I, u)\,. \end{cases} \tag{59}$$

Here, we have $(\varphi, I) \in \mathbb{R}^{2M}$ and $u \in \mathbb{R}^D$. We assume that $\mathcal{K} : \mathbb{R}^M \to \mathbb{R}$ is continuously differentiable and the same for $G : \mathbb{R} \times \mathbb{R}^D \to \mathbb{R}$ with respect to the second variable. The perturbation function $P : \mathbb{R} \times \mathbb{R}^{2M+D} \to \mathbb{R}$ is continuous, T-periodic in t, and continuously differentiable with respect to all the other variables, with a bounded gradient. Moreover, the function P is τ_i-periodic in each φ_i, i.e., for every $i \in \{1, \dots, M\}$,

$$P(\dots, \varphi_i + \tau_i, \dots) = P(\dots, \varphi_i, \dots)\,.$$

We also assume that there exist some integers $m_1, \dots, m_M$ for which

$$T\nabla\mathcal{K}(I^0) = (m_1\tau_1, \dots, m_M\tau_M)\,.$$

The expert reader will recognize that we are dealing with a *completely resonant torus*. Here is our result.

Theorem 23. *In the above setting, assume that there exist $I^0 \in \mathbb{R}^M$, a symmetric invertible $M \times M$ matrix $\mathbb{A}$ and $\bar\rho > 0$ such that*

$$0 < |I - I^0| \le \bar\rho \quad \Rightarrow \quad \langle \nabla\mathcal{K}(I) - \nabla\mathcal{K}(I^0), \mathbb{A}(I - I^0)\rangle > 0\,. \tag{60}$$

Moreover, let there exist some constants $\delta > 0$, $\varsigma > 0$, $\alpha = (\alpha_1, \dots, \alpha_D)$ and $\beta = (\beta_1, \dots, \beta_D)$ with $\alpha \le \beta$, having the following property: If $\alpha - \bar\delta \le u \le \beta + \bar\delta$, for every $j \in \{1, \dots, D\}$ one has

$$\begin{cases} \partial_{u_j} G(t,u) \ge \varsigma\,, & \textit{when } u_j \in [\alpha_j - \delta, \alpha_j]\,, \\ \partial_{u_j} G(t,u) \le -\varsigma\,, & \textit{when } u_j \in [\beta_j, \beta_j + \delta]\,. \end{cases}$$

Then, for every $\sigma > 0$ there exists $\bar\varepsilon > 0$ such that, if $|\varepsilon| \le \bar\varepsilon$, there are at least $M + 1$ geometrically distinct solutions of system (59) *satisfying*

$$\varphi(t+T) = \varphi(t) + T\nabla\mathcal{K}(I^0)\,, \quad I(t+T) = I(t)\,,$$

$$u(t+T) = u(t)\,, \quad v(t+T) = v(t)\,,$$

and such that

$$|\varphi(t) - \varphi(0) - t\,\nabla\mathcal{K}(I^0)| + |I(t) - I^0| < \sigma\,,$$

and

$$\alpha \le u(t) \le \beta\,,$$

for every $t \in \mathbb{R}$.

Proof. Since we are looking for solutions with $I(t)$ in the open ball $B(I^0, \sigma)$, we can suitably modify the function $\mathcal{K}$ outside this set and assume that it has a bounded gradient. We perform the change of variables

$$q(t) = \varphi(t) - t\,\nabla\mathcal{K}(I^0)\,, \quad p(t) = I(t) - I^0\,,$$

thus obtaining the new Hamiltonian function

$$\mathscr{H}(t,q,p,u,v) = \mathscr{K}(p) + \tfrac{1}{2}|v|^2 + G(t,u) + \varepsilon Q(t,q,p,u)\,,$$

where the functions

$$\mathscr{K}(p) = \mathcal{K}(I) - \big\langle \nabla\mathcal{K}(I^0)\,,\, I - I^0 \big\rangle\,, \qquad Q(t,q,p,u) = P(t,\varphi,I,u)$$

are implicitly defined. We notice that Q is periodic in $q_1, \dots, q_M$ and both Q and $\mathscr{K}$ have bounded gradient: so, there are $C_{\mathscr{K}}, C_Q > 0$ such that

$$|\nabla \mathscr{K}(p)| \leq C_{\mathscr{K}}\,, \qquad |\nabla Q(t,q,p,u)| \leq C_Q\,,$$

for every $(t,q,p,u) \in [0,T] \times \mathbb{R}^{2M+D}$. Our goal is now to prove that for every $\overline{\sigma} > 0$ there exists $\overline{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \overline{\varepsilon}$, there are $M+1$ geometrically distinct T-periodic solutions $z = (q,p,u,v)$ of the system

$$J\dot{z} = \nabla \mathscr{H}(t,z)\,, \tag{61}$$

satisfying $\max\{|q(t) - q(0)|\,, |p(t)|\} < \overline{\sigma}$ and $\alpha \leq u(t) \leq \beta$, for every $t \in \mathbb{R}$.

Let us fix $\overline{\sigma} \in \,]0, \overline{\rho}[\,$. We are in the setting of Theorem 20. Assumptions 6, 7, 8 and 12 can be easily checked. We need to verify the twist condition (54).

Since $\nabla \mathscr{K}(0) = 0$, we can choose $r < \overline{\sigma}/4$ such that

$$|p| \leq 4r \quad \Rightarrow \quad 2T|\nabla \mathscr{K}(p)| < \overline{\sigma}\,, \tag{62}$$

and by (60) there is $\ell > 0$ such that

$$r \leq |p| \leq 4r \quad \Rightarrow \quad \langle \nabla \mathscr{K}(p), \mathbb{A}p) \rangle > 4\ell\,. \tag{63}$$

Let us fix $\delta > 0$ satisfying

$$\delta < \min\left\{ r\,, \frac{2\ell}{C_{\mathscr{K}} \|\mathbb{A}\|} \right\}.$$

Reducing $\overline{\varepsilon}$ if necessary, every solution of (61), with $|\varepsilon| \leq \overline{\varepsilon}$, is such that $|p(t) - p(0)| < \delta$ for every $t \in [0,T]$.

Let us consider a solution z of (61) with $|\varepsilon| \leq \overline{\varepsilon}$ and initial condition satisfying $2r \leq |p(0)| \leq 3r$. Then, $r \leq |p(t)| \leq 4r$ for every $t \in [0,T]$, and so, by (63),

$$\langle \nabla \mathscr{K}(p(t)), \mathbb{A}p(t) \rangle > 4\ell\,, \quad \text{for every } t \in [0,T]\,.$$

Reducing $\overline{\varepsilon}$ if necessary, we have that

$$\begin{aligned}
&\langle \partial_p H(t,z(t)), \mathbb{A}p(0) \rangle = \\
&\quad = \langle \nabla \mathscr{K}(p(t)), \mathbb{A}p(t) \rangle - \langle \nabla \mathscr{K}(p(t)), \mathbb{A}(p(t) - p(0)) \rangle + \varepsilon \langle \partial_p Q(t,z(t)), \mathbb{A}p(0)) \rangle \\
&\quad > 4\ell - C_{\mathscr{K}} \|\mathbb{A}\| \delta - 3\varepsilon C_Q \|\mathbb{A}\| r > \ell\,.
\end{aligned}$$

Integrating the previous estimate in the interval $[0,T]$, we get

$$\langle q(T) - q(0), \mathbb{A}p(0) \rangle > \ell T > 0\,.$$

We can thus apply Theorem 20, choosing $\mathcal{D} = \overline{B}(0,2r)$ and $\rho = r$. We have that

$$|p(t)| \leq |p(0)| + \delta < 2r + \delta < 3r < \overline{\sigma}\,, \qquad \text{for every } t \in [0,T]\,.$$

Moreover, by (62), we deduce that $|q(t) - q(0)| < \overline{\sigma}$, for every $t \in [0,T]$.

The T-periodic solutions we have found are then translated, by the inverse change of variables, into the solutions of (59) we are looking for. $\square$

Remark 24. *The twist condition* (60) *is surely verified if* $\mathcal{K}$ *is twice continuously differentiable with* $\det \mathcal{K}''(I^0) \neq 0$*, by taking* $\mathbb{A} = \mathcal{K}''(I^0)$*.*

Remark 25. *A more general twist condition (see [16]) can be considered, e.g.,*

$$0 \in cl \left\{ r \in]0, +\infty[\, : \min_{|I - I^0| = r} \langle \nabla \mathcal{K}(I) - \nabla \mathcal{K}(I^0), \mathbb{A}(I - I^0) \rangle > 0 \right\}.$$

Remark 26. *The same type of result holds, with the due changes, for a more general system of the type*

$$\begin{cases} \dot\phi = \varepsilon \partial_\psi P(t, \phi, \psi, \varphi, I, u, w)\,, \quad \dot\psi = -\varepsilon \partial_\phi P(t, \phi, \psi, \varphi, I, u, w)\,, \\ \dot\varphi = \nabla \mathcal{K}(I) + \varepsilon \partial_I P(t, \phi, \psi, \varphi, I, u, w)\,, \quad \dot I = -\varepsilon \partial_\varphi P(t, \phi, \psi, \varphi, I, u, w)\,, \\ -\ddot u = \partial_u G(t, u) + \varepsilon \partial_u P(t, \phi, \psi, \varphi, I, u, w)\,, \\ J\dot w = \mathbb{B}(t) w + \varepsilon \partial_w P(t, \phi, \psi, \varphi, I, u, w)\,, \end{cases}$$

when P *is also periodic in* $\phi_1, \dots, \phi_L$ *and* $\psi_1, \dots, \psi_L$*, and* $\mathbb{B}(t)$ *is a symmetric matrix, continuous and* T*-periodic, such that* $w(t) \equiv 0$ *is the only* T*-periodic solution of* $J\dot w = \mathbb{B}(t) w$*.*

Remark 27. *It would be interesting to see how Theorem 23 could be extended to infinite dimensions, in the spirit of [19].*

11 The general result

We consider system (1), assuming the Hamiltonian function $H \colon \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ to be continuous, T-periodic in its first variable t, and continuously differentiable with respect to the variable z, with corresponding gradient $\nabla H(t, z)$.

For $z \in \mathbb{R}^{2N}$, we use the notation $z = (x, y)$, with $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \dots, y_N) \in \mathbb{R}^N$. Moreover, we gather into five groups the variables of x and y, respectively, thus writing

$$x = (x^a, x^b, x^c, x^d, x^e), \quad y = (y^a, y^b, y^c, y^d, y^e),$$

where, for some nonnegative integers N^a, N^b, N^c, N^d, N^e,

$$\begin{array}{ll} x^a = (x^a_1, \dots, x^a_{N^a}) \in \mathbb{R}^{N^a}, & y^a = (y^a_1, \dots, y^a_{N^a}) \in \mathbb{R}^{N^a}, \\ x^b = (x^b_1, \dots, x^b_{N^b}) \in \mathbb{R}^{N^b}, & y^b = (y^b_1, \dots, y^b_{N^b}) \in \mathbb{R}^{N^b}, \\ x^c = (x^c_1, \dots, x^c_{N^c}) \in \mathbb{R}^{N^c}, & y^c = (y^c_1, \dots, y^c_{N^c}) \in \mathbb{R}^{N^c}, \\ x^d = (x^d_1, \dots, x^d_{N^d}) \in \mathbb{R}^{N^d}, & y^d = (y^d_1, \dots, y^d_{N^d}) \in \mathbb{R}^{N^d}, \\ x^e = (x^e_1, \dots, x^e_{N^e}) \in \mathbb{R}^{N^e}, & y^e = (y^e_1, \dots, y^e_{N^c}) \in \mathbb{R}^{N^e}. \end{array}$$

and we also introduce the notation

$$z^a = (x^a, y^a)\,,\ z^b = (x^b, y^b)\,,\ z^c = (x^c, y^c)\,,\ z^d = (x^d, y^d)\,,\ z^e = (x^e, y^e)\,.$$

Notice that one or more of these integers could be equal to zero, in which case the corresponding group will not be taken into account; for example, if $N^a = 0$, then x^a, y^a and z^a will disappear from the list. In the following, for simplicity we sometimes write (u, v) instead of (x^e, y^e), and $D = N^e$.

Let us introduce our assumptions in this general setting.

Assumption 13 (Periodicity). *The function $H(t,z)$ is periodic in each of the variables included in x^a, x^b, y^a, y^c.*

The total number of variables in which our Hamiltonian function is periodic is thus $2N^a + N^b + N^c$. Under this setting, T-periodic solutions $z(t)$ of (1) appear in equivalence classes made of those solutions whose components in $x^a(t), x^b(t), y^a(t), y^c(t)$ differ by an integer multiple of the corresponding periods. We say that two T-periodic solutions are *geometrically distinct* if they do not belong to the same equivalence class.

Assumption 14 (Lower and upper solutions). *There exist some constants $\delta > 0$, $\alpha = (\alpha_1, \dots, \alpha_D)$ and $\beta = (\beta_1, \dots, \beta_D)$ with $\alpha \le \beta$, having the following property. If $\alpha - \bar{\delta} \le u \le \beta + \bar{\delta}$, then, for every $j \in \{1, \dots, D\}$,*

$$v_j\, \partial_{v_j} H(t,z) > 0\,, \text{ when } u_j \in [\alpha_j - \delta, \alpha_j] \cup [\beta_j, \beta_j + \delta] \text{ and } v_j \neq 0\,,$$

and

$$\begin{cases} \partial_{u_j} H(t,z) \ge 0\,, & \text{when } u_j \in [\alpha_j - \delta, \alpha_j] \text{ and } v_j = 0\,, \\ \partial_{u_j} H(t,z) \le 0\,, & \text{when } u_j \in [\beta_j, \beta_j + \delta] \text{ and } v_j = 0\,. \end{cases}$$

Assumption 15 (Nagumo condition). *For every $j \in \{1, \dots, D\}$ there exist $d_j > 0$ and two continuous functions $f_j, \varphi_j : [d_j, +\infty[\to]0, +\infty[$, with*

$$\int_{d_j}^{+\infty} \frac{f_j(s)}{\varphi_j(s)}\, ds = +\infty\,,$$

satisfying the following property. If $\alpha - \bar{\delta} \le u \le \beta + \bar{\delta}$, then

$$\begin{cases} \partial_{v_j} H(t,z) \ge f_j(v_j)\,, & \text{when } v_j \ge d_j\,, \\ \partial_{v_j} H(t,z) \le -f_j(-v_j)\,, & \text{when } v_j \le -d_j\,, \end{cases}$$

and

$$\left|\partial_{u_j} H(t,z)\right| \le \varphi_j(|v_j|)\,, \quad \text{when } |v_j| \ge d_j\,.$$

Assumption 16 (Linear growth). *There exists a symmetric $2N^d \times 2N^d$ matrix $\mathbb{B}(t)$, T-periodic and continuous in t, satisfying the nonresonance condition*

$$z^d(t) \equiv 0 \quad \text{is the only } T\text{-periodic solution of } J\dot{z}^d(t) = \mathbb{B}(t) z^d(t)\,,$$

and such that, writing

$$H(t,z) = \tfrac{1}{2}\langle \mathbb{B}(t) z^d, z^d \rangle + \mathcal{H}(t,z)\,,$$

for every $K > 0$ there is a $C_K > 0$ with the following property: If $\alpha - \bar{\delta} \le u \le \beta + \bar{\delta}$ and $|v| \le K$, then

$$|\partial_{x^b} \mathcal{H}(t,z)| + |\partial_{y^c} \mathcal{H}(t,z)| + |\partial_{x^d} \mathcal{H}(t,z)| + |\partial_{y^d} \mathcal{H}(t,z)| \le C_K(|y^b| + |x^c| + 1)\,.$$

The above assumption guarantees that, for any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$, with

$$\alpha - \bar{\delta} \le U(t) \le \beta + \bar{\delta}\,, \quad \text{for every } t \in [0,T]\,, \tag{64}$$

setting $W(t) = (U(t), V(t))$ and

$$\overline{H}(t,z) = H(t, z^a, z^b, z^c, z^d, W(t)),$$

the solutions of

$$J\dot{z} = \nabla \overline{H}(t,z) \tag{65}$$

are defined on $[0,T]$, cf. Remark 1.

Let us also introduce a C^1-function $h\colon \mathbb{R}^{N^b+N^c} \to \mathbb{R}$ and a regular symmetric $(N^b+N^c)\times(N^b+N^c)$ matrix $\mathbb{S}$ such that, for some positive constants C_1, C_2,

$$|h(\upsilon) - \tfrac{1}{2}\langle \mathbb{S}\upsilon, \upsilon\rangle| \le C_1 \quad \text{and} \quad |\nabla h(\upsilon) - \mathbb{S}\upsilon| \le C_2\,, \qquad \text{for every } \upsilon \in \mathbb{R}^{N^b+N^c},$$

and let

$$\mathcal{D} = \{\upsilon \in \mathbb{R}^{N^b+N^c} : \nabla h(\upsilon) = 0\}\,.$$

We assume that such a set is compact. Our main result is the following.

Theorem 28. *Let Assumptions 13, 14, 15, and 16 hold. Assume moreover that there exists $\rho > 0$ such that, for any two continuous functions $U, V : [0,T] \to \mathbb{R}^D$ satisfying (64), the solutions of (65) with $(y^b(0), x^c(0)) \notin \mathring{\mathcal{D}}$ and $\operatorname{dist}((y^b(0), x^c(0)), \mathcal{D}) \le \rho$ are such that*

$$(x^b(T) - x^b(0), y^c(T) - y^c(0)) \notin \{\lambda J \nabla h((y^b(0), x^c(0))) : \lambda \ge 0\}\,.$$

Then, system (1) has at least $2N^a+N^b+N^c+1$ geometrically distinct T-periodic solutions $z(t)$, such that

$$(y^b(0), x^c(0)) \in \mathring{\mathcal{D}}\,,$$

and

$$\alpha \le u(t) \le \beta\,, \quad \textit{for every } t \in \mathbb{R}\,.$$

Proof. With the same procedure adopted in the proof of Theorem 16 provided in Section 7, dealing with the $z^e = (u,v)$ coordinates, we can introduce a modified problem ruled by a Hamiltonian of the type

$$\widehat{H}(t,z) = \tfrac{1}{2}\langle \mathbb{B}(t) z^d, z^d\rangle + \frac{1}{2}(|v|^2 - |u|^2) + \widehat{h}(t,z)\,.$$

Moreover, $\widehat{H}$ can be introduced so to guarantee that every T-periodic solution of $J\dot{z} = \nabla\widehat{H}(t,z)$ satisfies an a priori bound as in Lemma 15, in particular $\alpha \le u \le \beta$. Then, from Assumption 16, we can introduce $K > 0$ as in (45) so to obtain

$$|\partial_{x^b}\widehat{h}(t,z)| + |\partial_{y^c}\widehat{h}(t,z)| \le C_K(|y^b| + |x^c| + 1)\,.$$

when $\alpha - \bar{\delta} \le u \le \beta + \bar{\delta}$ and $|v| \le K$. Hence, as in (46), we can find a constant $c > 0$ such that, if $z(t)$ is a solution of $J\dot{z} = \nabla\widehat{H}(t,z)$ starting with $\operatorname{dist}((y^b(0), x^c(0), \mathcal{D}) \le \rho$, with maximal interval of future existence $[0,\omega[$, then

$$\max\{|y^b(t)|\,,\, |x^c(t)|\} \le c\,, \qquad \text{for every } t \in [0,T] \cap [0,\omega[\,.$$

Again, we can introduce the cut-off function η as in (25), and the Hamiltonian

$$\widetilde{H}(t,z) = \tfrac{1}{2}\langle \mathbb{B}(t) z^d, z^d\rangle + \frac{1}{2}(|v|^2 - |u|^2) + \widetilde{h}(t,z)\,,$$

where $\widetilde{h}(t,z) = \eta(|y^b|)\eta(|x^c|)\widehat{h}(t,z)$. Using Assumption 16 we get

$$|\nabla\widetilde{h}(t,z)| \leq \widetilde{C}\,,$$

for a certain positive constant $\widetilde{C}$.

We can rewrite the previous Hamiltonian as

$$\widetilde{H}(t,z) = \tfrac{1}{2}\big\langle \mathbb{M}(t)(z^d,z^e),(z^d,z^e)\big\rangle + \widetilde{h}(t,z)\,,$$

where

$$\mathbb{M}(t)\big(z^d,z^e\big) = \mathbb{M}(t)\big(z^d,(u,v)\big) = \big(\mathbb{B}(t)z^d,(-u,v)\big)\,.$$

In particular, we can verify the nonresonance condition

$$z^{d,e}(t)\equiv 0 \;\text{ is the only } T\text{-periodic solution of }\; J\dot z^{d,e}(t) = \mathbb{M}(t)z^{d,e}(t)\,,$$

where $z^{d,e} = (z^d,z^e)$. Our aim is to apply [18, Theorem 1.1] to the modified system $J\dot z = \nabla\widetilde{H}(t,z)$. Its solutions are globally defined on $[0,T]$.

Let $z = (z^a,z^b,z^c,z^d,z^e)$ be a solution of $J\dot z = \nabla\widetilde{H}(t,z)$. As in the above proofs we can show that if $\mathrm{dist}((y^b(0),x^c(0),\mathcal{D})\leq\rho$, then $\max\{|y^b(t)|\,,\,|x^c(t)|\}\leq c$ for every $t\in[0,T]$. Hence, z solves $J\dot z = \nabla\widehat{H}(t,z)$ and, setting $W(t) = \big(\zeta(u(t)),\chi(\zeta(u(t)),v(t))\big)$, we see that (z^a,z^b,z^c,z^d) is a solution of (65).

If $0<\mathrm{dist}((y^b(0),x^c(0)),\mathcal{D})\leq\rho$, then

$$(x^b(T)-x^b(0),y^c(T)-y^c(0))\notin\{\lambda J\nabla h((y^b(0),x^c(0))):\lambda\geq 0\}\,,\qquad(66)$$

by the hypothesis of the theorem.

The application of [18, Theorem 1.1] provides us $2N^a+N^b+N^c+1$ geometrically distinct T-periodic solutions of $J\dot z = \nabla\widetilde{H}(t,z)$ satisfying $(y^b(0),x^c(0))\in\mathring{\mathcal{D}}$. Then, following the argument of the previous proofs, it can be seen that they are solutions of $J\dot z = \nabla H(t,z)$, as well, and $\alpha\leq u\leq\beta$. □

Remark 29. *Theorem 28 generalizes all three Theorems 14, 16 and 17 previously stated. For example, let the assumptions of Theorems 14 hold. We consider a smooth function* $\sigma\colon\mathbb{R}\to\mathbb{R}$ *such that*

$$\sigma(s) = \begin{cases}0\,, & \text{if } s\leq 0\,,\\ 1\,, & \text{if } s\geq 1\,,\end{cases}\qquad\text{and}\qquad \sigma'(s)>0\;\text{ if } s\in\,]0,1[\,,$$

and, adapting the notations, we define the function $h\colon\mathbb{R}^{N^b+N^c}\to\mathbb{R}$ *by*

$$h(p) = -\xi(p)\langle\mathbb{A}(p-\pi_{\mathcal{D}}(p)),p-\pi_{\mathcal{D}}(p)\rangle\,,$$

where

$$\xi(p) = \begin{cases}0\,, & \text{if } p\in\mathcal{D}\,,\\ \tfrac{1}{2}\sigma(|p-\pi_{\mathcal{D}}(p)|)\,, & \text{if } p\notin\mathcal{D}\,.\end{cases}$$

Following the proof of [18, Corollary 2.3] one can verify that h satisfies the assumptions in Theorem 28.

Having extended with Theorem 28 the main theorem in [18], we thus have generalized, e.g., the results in [9, 11, 17, 30, 33, 37, 38, 40].

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