Well-ordered and non-well-ordered lower and upper solutions for periodic planar systems

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Abstract

The aim of this paper is to extend the theory of lower and upper solutions to the periodic problem associated with planar systems of differential equations. We generalize previously given definitions and we are able to treat both the well-ordered case and the non-well-ordered case. The proofs involve topological degree arguments, together with a detailed analysis of the solutions in the phase plane.

1 Introduction

The method of lower and upper solutions for scalar second order differential equations of the type

$$x'' = g(t, x, x')$$

can be dated back to the pioneering papers by Picard [14], Scorza Dragoni [15] and Nagumo [12], dealing with separated boundary conditions. Its full extension to the periodic problem is due to Knobloch [9]. Further extensions to partial differential equations of elliptic or parabolic type have also been proposed, and there is nowadays a huge literature on this subject. For a rather complete historical and bibliographical account, we refer to the book [5].

Recently Toader [8], jointly with the first author, extended the main idea in the definition of lower and upper solutions to planar systems of ordinary differential equations, with the aim of finding bounded solutions through the method of Ważewski [16]. As a by-product, the theorem of Massera [11] provided also the existence of periodic solutions. It is the aim of this paper to further develop this theory, concentrating on the periodic problem, by the use of topological degree methods.

We consider the periodic problem

$$(P) \qquad \begin{cases} x' = f(t, x, y)\,, \quad y' = g(t, x, y)\,, \\ x(0) = x(T)\,, \quad y(0) = y(T)\,, \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, T-periodic in their first variable. Our purpose is to give a general definition of a lower and an upper solution with the aim of obtaining the existence of a solution to problem (P). In order to do this, let us first recall the definition of lower solution given in [8].

In [8], a continuously differentiable function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to be a *lower solution* for problem (P) if it is T-periodic and the following properties hold:

(i) there exists a unique function $y_\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} y < y_\alpha(t) & \Rightarrow \quad f(t, \alpha(t), y) < \alpha'(t)\,, \\ y > y_\alpha(t) & \Rightarrow \quad f(t, \alpha(t), y) > \alpha'(t)\,; \end{cases}$$

(ii) y_α is continuously differentiable, and

$$y'_\alpha(t) \geq g(t, \alpha(t), y_\alpha(t))\,, \qquad \text{for every } t \in \mathbb{R}\,;$$

(iii) there are two positive constants δ, m such that, when $|y - y_\alpha(t)| \leq \delta$,

$$\begin{cases} y < y_\alpha(t) - m|x - \alpha(t)| & \Rightarrow \quad f(t, x, y) < \alpha'(t)\,, \\ y > y_\alpha(t) + m|x - \alpha(t)| & \Rightarrow \quad f(t, x, y) > \alpha'(t)\,. \end{cases}$$

An analogous definition was provided for an *upper solution* $\beta : \mathbb{R} \to \mathbb{R}$, and an existence result was proved for problem (P) assuming $\alpha \leq \beta$, the so called *well-ordered case*.

We will generalize the above definition in two directions. First of all, condition (iii) will be removed. Moreover, the function α will not need to be differentiable on all its domain, and the function y_α will be allowed to have some discontinuity points. The precise definition will be given in Section 2. Moreover, after having proved the existence of a solution of problem (P) in the well-ordered case, we will also deal with the *non-well-ordered case* $\alpha \not\leq \beta$. Assuming some growth conditions on f and g in order to avoid resonance, we will then be able to prove an existence result also in this case.

A natural application of our results is provided by the periodic problem associated with the scalar equation

$$(\phi(x'))' = h(t, x, x')\,, \tag{1}$$

which can be written in the form of problem (P), with $f(t, x, y) = \phi^{-1}(y)$ and $g(t, x, y) = h(t, x, \phi^{-1}(y))$. Here, $\phi : I \to J$ is an increasing homeomorphism between two intervals I and J containing 0, and $\phi(0) = 0$. Typical examples in the applications involve the choice $\phi(v) = |v|^{p-2}v$, leading to the so-called "scalar p-Laplacian" operator (cf. [3]), or $\phi(v) = v/\sqrt{1 + v^2}$, providing a "mean curvature" operator (cf. [13]), or $\phi(v) = v/\sqrt{1 - v^2}$, providing a "relativistic" operator (cf. [2]). (See [8] for a detailed discussion in this direction.) A lower solution for the periodic problem associated with (1) is usually defined as a continuously differentiable function $\alpha : [0, T] \to \mathbb{R}$ such that $\alpha'(t) \in I$ for every t, with $\alpha(0) = \alpha(T)$, $\alpha'(0) \geq \alpha'(T)$ and

$$(\phi(\alpha'))'(t) \geq h(t, \alpha(t), \alpha'(t))\,, \qquad \text{for every } t \in [0, T]\,.$$

Our definition to be given in Section 2 extends also this one, with the natural choice $y_\alpha(t) = \phi(\alpha'(t))$. Similarly for what concerns an upper solution.

Notice however that for our problem (P) we do not need any monotonicity assumption on $f(t, x, y)$. Indeed, even in the simpler case $f(t, x, y) = f(y)$, the inequalities in (i) resemble some sign condition, which may be satisfied also if f is not an increasing function.

The paper is organized as follows. In Section 2 we introduce our main definitions and provide some remarks and preliminaries needed in the sequel.

In Section 3 we prove an existence result in the well-ordered case $\alpha \leq \beta$, assuming (like in [8]) the existence of some bounding curves, in order to control the solutions in the phase plane. The construction of these curves can be easily carried out in concrete examples, assuming a Nagumo-type condition (see [8] or Lemma 15 below).

In Section 4 we deal with the non-well-ordered case. Here we need to ask an extra technical condition on the lower and upper solutions; it remains an open question if it could possibly be avoided. Moreover, we assume the existence of a whole family of bounding curves. This assumption is again verified under some type of Nagumo conditions.

In Section 5 we present some variants of our main theorems and discuss on the possibility of further extending the theory to higher dimensional systems.

2 Main definitions and preliminaries

For any function $\nu : \mathbb{R} \to \mathbb{R}$ we use the notation

$$\nu(\tau^-) = \lim_{t\to\tau^-} \nu(t)\,, \qquad \nu(\tau^+) = \lim_{t\to\tau^+} \nu(t)\,.$$

Definition 1. *A continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to be a* lower solution *for problem (P) if it is T-periodic and there exist a T-periodic function $y_\alpha : \mathbb{R} \to \mathbb{R}$ and a finite number of points $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$ such that the following properties hold:*

1. *the restriction of α [resp. y_α] to each open interval $]\tau_{k-1}, \tau_k[$, with $k \in \{1, \ldots, n\}$, is continuously differentiable [resp. differentiable];*
2. *$\alpha'(\tau_k^\pm)$ and $y_\alpha(\tau_k^\pm)$ exist in $\mathbb{R}$ for every $k \in \{1, \ldots, n\}$, with*
$$\alpha'(\tau_k^-) \leq \alpha'(\tau_k^+) \quad \text{and} \quad y_\alpha(\tau_k^-) \leq y_\alpha(\tau_k^+)\,; \tag{2}$$
3. *for every $t \in \cup_{k=1}^n]\tau_{k-1}, \tau_k[$,*
$$\begin{cases} y < y_\alpha(t) & \Rightarrow \quad f(t, \alpha(t), y) < \alpha'(t)\,, \\ y > y_\alpha(t) & \Rightarrow \quad f(t, \alpha(t), y) > \alpha'(t)\,, \end{cases} \tag{3}$$
and
$$y_\alpha'(t) \geq g(t, \alpha(t), y_\alpha(t))\,. \tag{4}$$

Definition 2. *A continuous function $\beta : \mathbb{R} \to \mathbb{R}$ is said to be an* upper solution *for problem (P) if it is T-periodic and there exist a T-periodic function $y_\beta : \mathbb{R} \to \mathbb{R}$ and a finite number of points $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$ such that the following properties hold:*

1. *the restriction of β [resp. y_β] to each open interval $]\tau_{k-1}, \tau_k[$, with $k \in \{1, \ldots, n\}$, is continuously differentiable [resp. differentiable];*

2. $\beta'(\tau_k^\pm)$ *and* $y_\beta(\tau_k^\pm)$ *exist in* $\mathbb{R}$ *for every* $k \in \{1, \dots, n\}$*, with*

$$\beta'(\tau_k^-) \geq \beta'(\tau_k^+) \quad \text{and} \quad y_\beta(\tau_k^-) \geq y_\beta(\tau_k^+)\,; \tag{5}$$

3. for every $t \in \cup_{k=1}^n \,]\tau_{k-1}, \tau_k[\,$,

$$\begin{cases} y < y_\beta(t) & \Rightarrow \quad f(t, \beta(t), y) < \beta'(t)\,, \\ y > y_\beta(t) & \Rightarrow \quad f(t, \beta(t), y) > \beta'(t)\,, \end{cases} \tag{6}$$

and

$$y_\beta'(t) \leq g(t, \beta(t), y_\beta(t))\,. \tag{7}$$

In what follows, when dealing with a couple (α, β) of a lower and an upper solution, we will assume, without loss of generality, that the points $\{\tau_0, \tau_1, \dots, \tau_n\}$ provided in the previous definitions are the same, both for α and β. Moreover, since we are dealing with T-periodic functions, it is worth defining the sets

$$\mathcal{J} := \big\{t = \tau_k + \iota T \mid k \in \{1, \dots, n\}\,,\, \iota \in \mathbb{Z}\big\}\,, \qquad \mathcal{I} := \mathbb{R} \setminus \mathcal{J}\,.$$

Therefore, (2), (5) hold with τ_k replaced by any $\tau \in \mathcal{J}$, and (3), (4), (6), (7) hold for every $t \in \mathcal{I}$.

Remark 3. *When dealing with the periodic problem associated with the scalar equation* (1)*, the usual definitions of lower/upper solutions are contained in the above ones, taking* $f(t, x, y) = \phi^{-1}(y)$*,* $g(t, x, y) = h(t, x, \phi^{-1}(y))$*, and defining* $y_\alpha(t) = \phi(\alpha'(t))$*,* $y_\beta(t) = \phi(\beta'(t))$*. Indeed, the conditions* $\alpha(0) = \alpha(T)$*,* $\beta(0) = \beta(T)$ *permit to continuously extend the functions* $\alpha, \beta : [0, T] \to \mathbb{R}$ *to the whole real line* $\mathbb{R}$*, and the conditions* $\alpha'(0) \geq \alpha'(T)$*,* $\beta'(0) \leq \beta'(T)$ *are included in* (2)*,* (5)*. The possibility of having some discontinuity points* τ_k *can be useful in the applications, e.g., when taking as a lower solution the maximum of two or more smooth lower solutions, and as an upper solution the minimum of two or more smooth upper solutions.*

From (3) we have that

$$\alpha'(t) = f(t, \alpha(t), y_\alpha(t))\,, \qquad \text{for every } t \in \mathcal{I}\,, \tag{8}$$

and $y_\alpha(t)$ is the only value for which this identity holds. Similarly, from (6) we have

$$\beta'(t) = f(t, \beta(t), y_\beta(t))\,, \qquad \text{for every } t \in \mathcal{I}\,, \tag{9}$$

and $y_\beta(t)$ is uniquely defined on $\mathcal{I}$ by this identity.

It is well known in the case of scalar second order equations that if a function is at the same time a lower and an upper solution, then it is a solution. Let us write the analogous statement in our situation.

Proposition 4. *Let* $x : \mathbb{R} \to \mathbb{R}$ *be at the same time a lower and an upper solution for problem* (P)*. Then, there exists a function* $y : \mathbb{R} \to \mathbb{R}$ *such that* (x, y) *is a solution of problem* (P)*.*

Proof. Denote by y_α and y_β the functions provided by Definitions 1 and 2 taking $x = \alpha$ and $x = \beta$, respectively. From (8) and (9) we deduce that

$$x'(t) = f(t, x(t), y_\alpha(t)) \quad \text{ and } \quad y_\alpha(t) = y_\beta(t)\,, \quad \text{ for every } t \in \mathcal{I}\,.$$

Then, from (2) and (5) we first see that $x'(\tau_k^-) = x'(\tau_k^+)$, thus implying that $x : \mathbb{R} \to \mathbb{R}$ is continuously differentiable; moreover, on one hand we have $y_\alpha(\tau_k^-) \leq y_\alpha(\tau_k^+)$, and on the other hand

$$y_\alpha(\tau_k^-) = y_\beta(\tau_k^-) \geq y_\beta(\tau_k^+) = y_\alpha(\tau_k^+)\,,$$

showing that $y_\alpha(\tau_k^\pm) = y_\beta(\tau_k^\pm)$ for every k. We can thus define

$$y(t) = \begin{cases} y_\alpha(t)\,, & \text{if } t \in \mathcal{I}\,, \\ y_\alpha(t^\pm)\,, & \text{if } t \in \mathcal{J}\,, \end{cases}$$

a continuous function.

Since $x : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $y : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^3 \to \mathbb{R}$ are continuous, from (8) we deduce that $x'(t) = f(t, x(t), y(t))$ for every $t \in \mathbb{R}$. Moreover, by (4) and (7) we get $y'(t) = g(t, x(t), y(t))$ for every $t \in \mathcal{I}$; since $y : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous, we first see that $y : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and then also that $y'(t) = g(t, x(t), y(t))$ for every $t \in \mathbb{R}$, thus completing the proof. □

We will need the following estimates involving our lower and upper solutions, where we adopt the usual definition of the Dini derivatives:

$$D_\pm F(t_0) = \liminf_{t \to t_0^\pm} \frac{F(t) - F(t_0)}{t - t_0}\,, \qquad D^\pm F(t_0) = \limsup_{t \to t_0^\pm} \frac{F(t) - F(t_0)}{t - t_0}\,.$$

Proposition 5. *If α is a lower solution for problem (P), then*

$$D_\pm y_\alpha(\tau) \geq g(\tau, \alpha(\tau), y_\alpha(\tau^\pm))\,, \quad \textit{for every } \tau \in \mathcal{J}\,.$$

If β is an upper solution for problem (P), then

$$D^\pm y_\beta(\tau) \leq g(\tau, \beta(\tau), y_\beta(\tau^\pm))\,, \quad \textit{for every } \tau \in \mathcal{J}\,.$$

Proof. Let us fix k and consider the restrictions of the functions y_α and y_β to the interval $[\tau_k, \tau_{k+1}]$, redefining the two functions at the extremes in such a way to make them continuous. Then, since both y_α and y_β are differentiable in the interval $]\tau_k, \tau_{k+1}[$, by [6, Corollary 3.7] we have

$$\begin{aligned} D_- y_\alpha(\tau_{k+1}) &\geq \liminf_{t \to \tau_{k+1}^-} D^+ y_\alpha(t) = \liminf_{t \to \tau_{k+1}^-} y'_\alpha(t) \\ &\geq \liminf_{t \to \tau_{k+1}^-} g(t, \alpha(t), y_\alpha(t)) = g(\tau_{k+1}, \alpha(\tau_{k+1}), y_\alpha(\tau_{k+1}^-))\,, \end{aligned}$$

and

$$\begin{aligned} D^+ y_\beta(\tau_k) &\leq \limsup_{t \to \tau_k^+} D_- y_\beta(t) = \limsup_{t \to \tau_k^+} y'_\beta(t) \\ &\leq \limsup_{t \to \tau_k^+} g(t, \beta(t), y_\beta(t)) = g(\tau_k, \beta(\tau_k), y_\beta(\tau_k^+))\,. \end{aligned}$$

Similarly, we have

$$
\begin{aligned}
D_+ y_\alpha(\tau_k) &\geq \liminf_{t \to \tau_k^+} D^- y_\alpha(t) = \liminf_{t \to \tau_k^+} y'_\alpha(t) \\
&\geq \liminf_{t \to \tau_k^+} g(t, \alpha(t), y_\alpha(t)) = g(\tau_k, \alpha(\tau_k), y_\alpha(\tau_k^+)) \,,
\end{aligned}
$$

and

$$
\begin{aligned}
D^- y_\beta(\tau_{k+1}) &\leq \limsup_{t \to \tau_{k+1}^-} D_+ y_\beta(t) = \limsup_{t \to \tau_{k+1}^-} y'_\beta(t) \\
&\leq \limsup_{t \to \tau_{k+1}^-} g(t, \beta(t), y_\beta(t)) = g(\tau_{k+1}, \beta(\tau_{k+1}), y_\beta(\tau_{k+1}^-)) \,,
\end{aligned}
$$

thus ending the proof. □

3 Well-ordered lower and upper solutions

We will say that (α, β) is a well-ordered couple of lower/upper solutions of problem (P) if α and β are respectively a lower and an upper solution of problem (P), and $\alpha(t) \leq \beta(t)$ for every $t \in \mathbb{R}$. The following result generalizes that part of [8, Theorem 2.5] concerning the existence of periodic solutions.

Theorem 6. *Assume the existence of a well-ordered couple (α, β) of lower/upper solutions of problem (P). Set $A = \min \alpha$ and $B = \max \beta$, with $A < B$. Let there exist two continuously differentiable functions $\gamma_\pm : [A, B] \to \mathbb{R}$ such that, for every $t \in \mathbb{R}$ and $x \in [\alpha(t), \beta(t)]$,*

$$
\gamma_-(x) < \min\{y_\alpha(t^-), y_\beta(t^+)\} \leq \max\{y_\alpha(t^+), y_\beta(t^-)\} < \gamma_+(x) \,,
$$

and

$$
g(t, x, \gamma_-(x)) < f(t, x, \gamma_-(x))\gamma'_-(x) \,, \tag{10}
$$

$$
g(t, x, \gamma_+(x)) > f(t, x, \gamma_+(x))\gamma'_+(x) \,. \tag{11}
$$

Then there exists at least one solution of problem (P) such that

$$
\alpha(t) \leq x(t) \leq \beta(t) \quad \textit{and} \quad \gamma_-(x(t)) < y(t) < \gamma_+(x(t)) \,,
$$

for every $t \in \mathbb{R}$.

Some remarks are in order.

1) We will discuss in Section 5 on the possibility of reversing the inequalities in (10) and (11).

2) We will provide in Lemma 15 some Nagumo-type conditions which guarantee the existence of the curves $\gamma_\pm$.

3) The assumption $A < B$ is inessential, since if $A = B$ we have that $\alpha = \beta$, hence by Proposition 4 we immediately get a solution.

3.1 Proof of Theorem 6

3.1.1 An auxiliary problem

Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as

$$\Phi(t,x,y) = \big(f(t,x,y), g(t,x,y)\big)\,.$$

Fix $D > 0$ such that

$$-D < \gamma_-(x) < \gamma_+(x) < D\,, \quad \text{for every } x \in [A,B]\,.$$

Define

$$\|\alpha'\|_\infty = \max_{t\in[0,T]} |\alpha'(t^\pm)|\,, \qquad \|\beta'\|_\infty = \max_{t\in[0,T]} |\beta'(t^\pm)|\,, \tag{12}$$
$$\mu_1 = \max_{t\in[0,T]} |f(t,\alpha(t),\gamma_\pm(\alpha(t)))|\,, \qquad \mu_2 = \max_{t\in[0,T]} |f(t,\beta(t),\gamma_\pm(\beta(t)))|\,,$$

choose

$$M_X > \max\{\mu_1,\mu_2,\|\alpha'\|_\infty,\|\beta'\|_\infty\}\,, \tag{13}$$

and

$$M_Y > \|\gamma'_\pm\|_\infty M_X\,. \tag{14}$$

We interpolate the vector field $\Phi(t,x,y)$ on $\{A \le x \le B\,,\ \gamma_-(x) \le y \le \gamma_+(x)\}$ with a constant vector field on $\{A \le x \le B\,,\ |y| \ge D\}$. Precisely, we define $\widehat{\Phi} : \mathbb{R}\times[A,B]\times\mathbb{R} \to \mathbb{R}^2$ as

$$\widehat{\Phi}(t,x,y) = \begin{cases} (M_X, M_Y)\,, & \text{if } y \ge D\,,\\ \Phi(t,x,\gamma_+(x)) + \dfrac{y-\gamma_+(x)}{D-\gamma_+(x)}\Big((M_X,M_Y) - \Phi(t,x,\gamma_+(x))\Big), & \\ & \text{if } \gamma_+(x) \le y \le D\,,\\ \Phi(t,x,y)\,, & \text{if } \gamma_-(x) \le y \le \gamma_+(x)\,,\\ \Phi(t,x,\gamma_-(x)) - \dfrac{y-\gamma_-(x)}{D+\gamma_-(x)}\Big((-M_X,-M_Y) - \Phi(t,x,\gamma_-(x))\Big), & \\ & \text{if } -D \le y \le \gamma_-(x)\,,\\ (-M_X,-M_Y)\,, & \text{if } y \le -D\,. \end{cases}$$

We will write $\widehat{\Phi}(t,x,y) = \big(\hat f(t,x,y), \hat g(t,x,y)\big)$.

By the use of the auxiliary functions

$$\zeta(s;\mu,\nu) = \begin{cases} \mu\,, & \text{if } s < \mu\,,\\ s\,, & \text{if } \mu \le s \le \nu\,,\\ \nu\,, & \text{if } s > \nu\,, \end{cases}$$

and

$$e(s;\mu,\nu) = s - \zeta(s;\mu,\nu) = \begin{cases} s-\mu\,, & \text{if } s < \mu\,,\\ 0\,, & \text{if } \mu \le s \le \nu\,,\\ s-\nu\,, & \text{if } s > \nu\,, \end{cases}$$

we define, for every $(t,x,y) \in \mathbb{R}^3$,

$$\begin{aligned}\tilde f(t,x,y) =& \hat f\Big(t, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\, \zeta\big(y;-D,D\big)\Big) + e\big(y;-D,D\big)\,,\\ \tilde g(t,x,y) =& \hat g\Big(t\,,\, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\, \zeta\big(y;-D,D\big)\Big) + e\big(x;\alpha(t),\beta(t)\big)\,,\end{aligned}$$

so to introduce the modified problem

$$(\widetilde P) \qquad \begin{cases} x' = \tilde f(t,x,y)\,, \quad y' = \tilde g(t,x,y)\,,\\ x(0) = x(T)\,, \quad y(0) = y(T)\,. \end{cases}$$

We will write $\widetilde\Phi(t,x,y) = \big(\tilde f(t,x,y), \tilde g(t,x,y)\big)$. In the space

$$\mathcal{C}^0_T = \big\{v \in \mathcal{C}^0([0,T],\mathbb{R}^2) : v(0) = v(T)\big\}$$

we introduce the open set

$$\mathcal{V} = \{u \in \mathcal{C}^0_T \mid (t,u(t)) \in V \text{ for every } t \in [0,T]\}\,, \tag{15}$$

where, see Figure 1,

$$V = \{(t,x,y) \in \mathbb{R}^3 \mid \alpha(t) < x < \beta(t)\,,\, \gamma_-(x) < y < \gamma_+(x)\}\,.$$

Our aim is to prove that there exists a solution $u = (x,y)$ of problem $(\widetilde P)$ belonging to $\overline{\mathcal{V}}$. Since $\tilde f = f$ and $\tilde g = g$ on the set $\overline V$, then u will solve also (P).

3.1.2 No solutions of $(\widetilde P)$ outside $\overline{\mathcal{V}}$

We show that all the solutions $u = (x,y)$ of system $(\widetilde P)$ are such that $(t,u(t)) \in \overline V$, for every $t \in \mathbb{R}$.

Let us start proving a preliminary lemma.

Lemma 7. *For every $t \in \mathcal{I}$, the following inequalities hold:*

$$\begin{cases} \tilde f(t,x,y) < \alpha'(t)\,, & \textit{if } x \le \alpha(t) \textit{ and } y < y_\alpha(t)\,,\\ \tilde f(t,x,y) > \alpha'(t)\,, & \textit{if } x \le \alpha(t) \textit{ and } y > y_\alpha(t)\,; \end{cases} \tag{16}$$

$$\begin{cases} \tilde f(t,x,y) < \beta'(t)\,, & \textit{if } x \ge \beta(t) \textit{ and } y < y_\beta(t)\,,\\ \tilde f(t,x,y) > \beta'(t)\,, & \textit{if } x \ge \beta(t) \textit{ and } y > y_\beta(t)\,; \end{cases} \tag{17}$$

$$\begin{cases} \tilde g(t,x,y_\alpha(t)) < y_\alpha'(t)\,, & \textit{if } x < \alpha(t)\,,\\ \tilde g(t,x,y_\beta(t)) > y_\beta'(t)\,, & \textit{if } x > \beta(t)\,. \end{cases} \tag{18}$$

Moreover, for every $\tau \in \mathcal{J}$,

$$\begin{cases} \tilde g(\tau,x,y_\alpha(\tau^\pm)) < D_\pm y_\alpha(\tau)\,, & \textit{if } x < \alpha(\tau)\,,\\ \tilde g(\tau,x,y_\beta(\tau^\pm)) > D^\pm y_\beta(\tau)\,, & \textit{if } x > \beta(\tau)\,. \end{cases} \tag{19}$$

Proof. Let us prove the first inequality in (16). Suppose $t \in \mathcal{I}$, $x \le \alpha(t)$ and $y < y_\alpha(t)$. We have that

$$\begin{aligned}\tilde f(t,x,y) &= \hat f\Big(t, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\, \zeta\big(y;-D,D\big)\Big) + e(y;-D,D)\\ &= \hat f\Big(t, \alpha(t)\,,\, \zeta\big(y;-D,D\big)\Big) + e(y;-D,D)\,.\end{aligned}$$

We need to consider three different cases.

Case 1. If $\gamma_-(\alpha(t)) \le y < y_\alpha(t)$, then

$$\tilde f(t,x,y) = \hat f(t,\alpha(t),y) = f(t,\alpha(t),y) < \alpha'(t)\,.$$

Case 2. If $-D \le y < \gamma_-(\alpha(t))$, then

$$\begin{aligned}
&\tilde f(t,x,y) = \hat f(t,\alpha(t),y)\\
&= f(t,\alpha(t),\gamma_-(\alpha(t))) - \frac{y-\gamma_-(\alpha(t))}{D+\gamma_-(\alpha(t))}\Big[-M_X - f(t,\alpha(t),\gamma_-(\alpha(t)))\Big]\\
&\le f(t,\alpha(t),\gamma_-(\alpha(t))) < \alpha'(t)\,.
\end{aligned}$$

Case 3. If $y < -D$ then, by (13),

$$\tilde f(t,x,y) = \hat f(t,\alpha(t),-D) + y + D = -M_X + y + D < -M_X < \alpha'(t)\,.$$

Hence, the first inequality in (16) is proved. The second one can be proved analogously, as well as the inequalities in (17).

We now prove the first inequality of (18). Let $x < \alpha(t)$. Since $-D < \gamma_-(\alpha(t)) \le y_\alpha(t) \le \gamma_+(\alpha(t)) < D$, we have

$$\begin{aligned}
\tilde g(t,x,y_\alpha(t)) &= \hat g\Big(t\,,\,\zeta\big(x;\alpha(t),\beta(t)\big)\,,\,\zeta\big(y_\alpha(t);-D,D\big)\Big) + e\big(x;\alpha(t),\beta(t)\big)\\
&= \hat g(t,\alpha(t),y_\alpha(t)) + x - \alpha(t)\\
&< \hat g(t,\alpha(t),y_\alpha(t))\\
&= g(t,\alpha(t),y_\alpha(t)) \le y_\alpha'(t)\,.
\end{aligned}$$

The second inequality in (18) follows analogously, and a similar computation proves the ones in (19). □

Let us define the sets

$$\begin{aligned}
A_{NW} &= \{(t,x,y)\in\mathbb{R}^3 \mid x<\alpha(t)\,,\ y>y_\alpha(t^+)\}\,,\\
A_{SW} &= \{(t,x,y)\in\mathbb{R}^3 \mid x<\alpha(t)\,,\ y<y_\alpha(t^-)\}\,,\\
A_{NE} &= \{(t,x,y)\in\mathbb{R}^3 \mid x>\beta(t)\,,\ y>y_\beta(t^-)\}\,,\\
A_{SE} &= \{(t,x,y)\in\mathbb{R}^3 \mid x>\beta(t)\,,\ y<y_\beta(t^+)\}
\end{aligned}$$

(see Figure 1).

Lemma 8. *For every solution $u=(x,y)$ of*

$$x' = \tilde f(t,x,y)\,, \qquad y' = \tilde g(t,x,y)\,, \tag{20}$$

the following assertions hold true:

$$\begin{aligned}
(t_0,u(t_0))\in A_{NW} &\Rightarrow (t,u(t))\in A_{NW} \text{ for every } t<t_0\,,\\
(t_0,u(t_0))\in A_{SE} &\Rightarrow (t,u(t))\in A_{SE} \text{ for every } t<t_0\,,\\
(t_0,u(t_0))\in A_{NE} &\Rightarrow (t,u(t))\in A_{NE} \text{ for every } t>t_0\,,\\
(t_0,u(t_0))\in A_{SW} &\Rightarrow (t,u(t))\in A_{SW} \text{ for every } t>t_0\,.
\end{aligned}$$

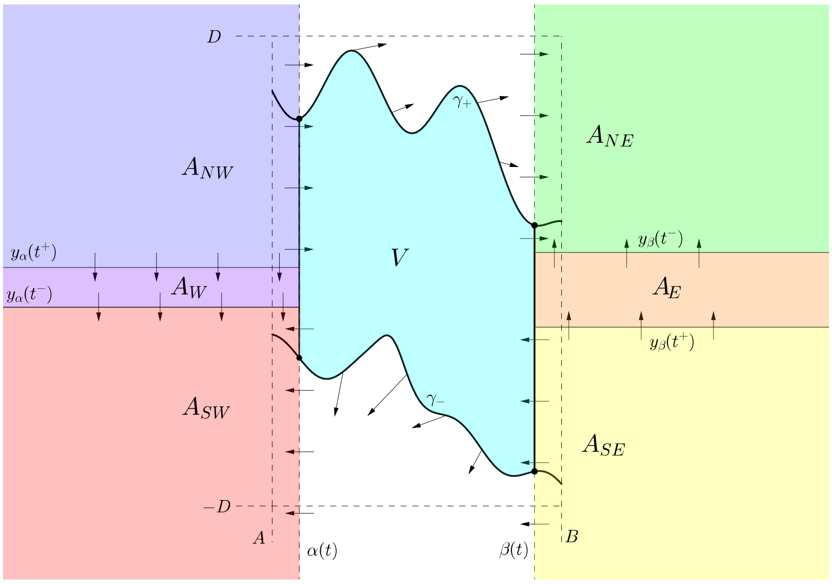

Figure 1: A sketch of the section at a fixed time t of the regions where to study the dynamics of $u' = \widetilde{\Phi}(t,u)$. Notice that the vertical lines $x = \alpha$, $x = \beta$ and the horizontal lines $y = y_\alpha$, $y = y_\beta$ move in time, while the curves $\gamma_\pm$ are fixed.

Proof. We will prove only the validity of the first assertion, since the others follow similarly. We argue by contradiction and assume the existence of $t_1 < t_0$ and of a solution $u = (x,y)$ of (20) such that $(t,u(t)) = (t,x(t),y(t)) \in A_{NW}$ for every $t \in]t_1,t_0]$ and $(t_1,u(t_1)) = (t_1,x(t_1),y(t_1)) \in \partial A_{NW}$, where (see Figure 2)

$$\begin{aligned}\partial A_{NW} =& \{(t,x,y) \in \mathbb{R}^3 \mid x = \alpha(t)\,,\, y \geq y_\alpha(t^+)\} \\ & \cup \{(t,x,y) \in \mathbb{R}^3 \mid x \leq \alpha(t)\,,\, y_\alpha(t^-) \leq y(t) \leq y_\alpha(t^+)\}\,. \end{aligned} \tag{21}$$

Without loss of generality we can assume the existence of $\delta > 0$ such that $]t_1, t_1+\delta] \subseteq \mathcal{I}$. We define $G(t) = x(t) - \alpha(t)$, for every $t \in [t_1, t_1+\delta]$. We have $G(t_1+\delta) < 0$ and, from (16),

$$G'(t) = x'(t) - \alpha'(t) = \tilde{f}(t,x(t),y(t)) - \alpha'(t) > 0\,,$$

for every $t \in]t_1, t_1+\delta]$. Hence, $G(t_1) < 0$. We conclude that $x(t) < \alpha(t)$ for every $t \in [t_1,t_0]$. So, being $x(t_1) < \alpha(t_1)$, recalling (21), we necessarily have $y_\alpha(t_1^-) \leq y(t_1) \leq y_\alpha(t_1^+)$.

If $y(t_1) = y_\alpha(t_1^+)$, then the function $H(t) = y(t) - y_\alpha(t^+)$ is continuous in the interval $[t_1,t_0]$ with $H(t_1) = 0$ and $H(t) > 0$ for all $t \in]t_1,t_0]$. Recalling that $x(t) < \alpha(t)$ for all $t \in [t_1,t_0]$, by (18) or (19) we have

$$D^+H(t_1) = y'(t_1) - D_+y_\alpha(t_1) = \tilde{g}(t_1,x(t_1),y_\alpha(t_1^+)) - D_+y_\alpha(t_1) < 0\,,$$

leading again to a contradiction.

The case $y_\alpha(t_1^-) \leq y(t_1) < y_\alpha(t_1^+)$ could arise only if $t_1 \in \mathcal{J}$. However, such a situation is not possible, indeed we would have the existence of $\delta > 0$ such that $H(t) < 0$ for every $t \in (t_1, t_1+\delta)$ which gives a contradiction, since we have assumed $(t,u(t)) \in A_{NW}$ for every $t \in]t_1,t_0[$. □

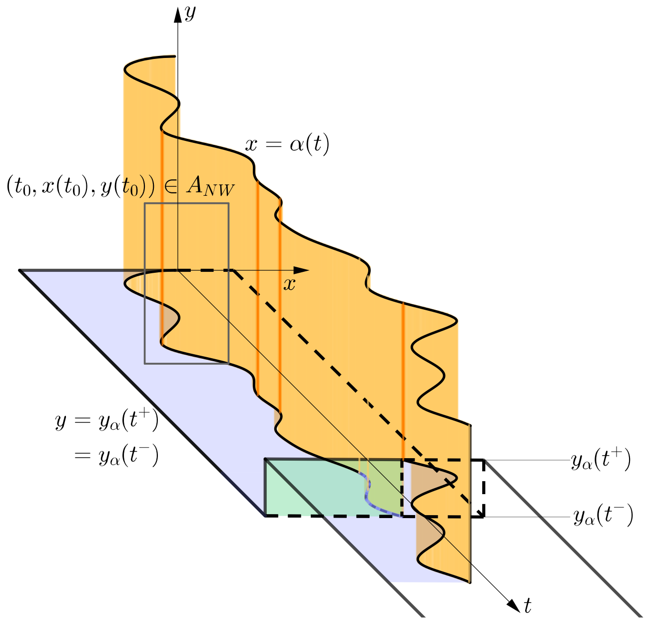

Figure 2: A sketch of the boundary of the set A_{NW}. It consists of a *wall* $x = \alpha(t)$, a *floor* $y = y_\alpha(t^+)$ and a possible *step* $y_\alpha(t^-) \le y < y_\alpha(t^+)$. For simplicity, the function y_α is drawn as being piecewise constant.

We have thus proved that the sets A_{NW}, A_{SE} are *invariant in the past*, while the sets A_{NE}, A_{SW} are *invariant in the future*. We also define the sets

$$A_W = \{(t,x,y) \in \mathbb{R}^3 \mid x < \alpha(t)\,,\ y_\alpha(t^-) \le y \le y_\alpha(t^+)\}\,,$$
$$A_E = \{(t,x,y) \in \mathbb{R}^3 \mid x > \beta(t)\,,\ y_\beta(t^+) \le y \le y_\beta(t^-)\}\,,$$

(see Figure 1).

Lemma 9. *If $u = (x,y)$ is a solution of* (20) *such that $(t_0, u(t_0)) \in A_W$, then there exists $\delta > 0$ such that*

$$t \in]t_0 - \delta, t_0[\quad \Rightarrow \quad (t, u(t)) \in A_{NW}\,,$$
$$t \in]t_0, t_0 + \delta[\quad \Rightarrow \quad (t, u(t)) \in A_{SW}\,.$$

Similarly, if $u = (x,y)$ is a solution of (20) *such that $(t_0, u(t_0)) \in A_E$, then there exists $\delta > 0$ such that*

$$t \in]t_0 - \delta, t_0[\quad \Rightarrow \quad (t, u(t)) \in A_{SE}\,,$$
$$t \in]t_0, t_0 + \delta[\quad \Rightarrow \quad (t, u(t)) \in A_{NE}\,.$$

Proof. We give the proof of the first part of the statement, the second one being similar. Let $u = (x,y)$ be a solution of (20) such that $(t_0, u(t_0)) \in A_W$. If $y(t_0) = y_\alpha(t_0^+)$ then, defining as above the function $H(t) = y(t) - y_\alpha(t^+)$,

$$D^+H(t_0) = y'(t_0) - D_+y_\alpha(t_0) = \tilde{g}(t_0, x(t_0), y_\alpha(t_0^+)) - D_+y_\alpha(t_0) < 0\,,$$

using (18) or (19). So, there exists $\delta > 0$ such that $y(t) < y_\alpha(t^+) = y_\alpha(t^-)$ and $x(t) < \alpha(t)$ for every $t \in]t_0, t_0 + \delta[$.

On the other hand, if $y_\alpha(t_0^-) \le y(t_0) < y_\alpha(t_0^+)$, then $t_0 \in \mathcal{J}$ and the strict inequalities $y(t_0) < y_\alpha(t_0^+)$ and $x(t_0) < \alpha(t_0)$ provide the same conclusion as before by a continuity argument.

We now give the proof for $t < t_0$. If $y(t_0) = y_\alpha(t_0^-)$ then

$$D^- H(t_0) = y'(t_0) - D_- y_\alpha(t_0) = \tilde{g}(t_0, x(t_0), y_\alpha(t_0^-)) - D_- y_\alpha(t_0) < 0\,,$$

and we get the existence of $\delta > 0$ such that $y(t) > y_\alpha(t^-) = y_\alpha(t^+)$ and $x(t) < \alpha(t)$, for every $t \in]t_0 - \delta, t_0[$. On the other hand, if $y_\alpha(t_0^-) < y(t_0) \le y_\alpha(t_0^+)$, we reach the same conclusion, by continuity. □

Lemma 10. *If $u = (x, y)$ is a solution of $(\widetilde{P})$, then*

$$\alpha(t) \le x(t) \le \beta(t)\,, \quad \text{for every } t \in \mathbb{R}\,. \tag{22}$$

Proof. Suppose that there exists a solution $u = (x, y)$ of $(\widetilde{P})$ such that $x(t_0) < \alpha(t_0)$ for a certain $t_0 \in [0, T]$. If $(t_0, u(t_0)) \in A_{NW}$, then, from Lemma 8, we have that $(t, u(t)) \in A_{NW}$ for every $t \in \mathbb{R}$. Moreover, from (16) we get

$$t \in \mathcal{I} \quad \Rightarrow \quad (x - \alpha)'(t) = \tilde{f}(t, x(t), y(t)) - \alpha'(t) > 0\,,$$

a contradiction, since $x - \alpha$ is a periodic function.

The same reasoning can be adopted if $(t_0, u(t_0)) \in A_{SW}$. Finally, if $(t_0, u(t_0))$ belongs to A_W, Lemma 9 brings us to the previous contradicting situations.

A similar argument can be adopted in order to show that there are no solutions $u = (x, y)$ of $(\widetilde{P})$ such that $\max_{[0,T]}(x - \beta) > 0$. □

Lemma 11. *If $u = (x, y)$ is a solution of $(\widetilde{P})$, then*

$$\gamma_-(x(t)) < y(t) < \gamma_+(x(t))\,, \quad \text{for every } t \in \mathbb{R}\,. \tag{23}$$

Proof. We already know from Lemma 10 that any solution of $(\widetilde{P})$ is such that $\alpha(t) \le x(t) \le \beta(t)$ for every $t \in [0, T]$. We claim that $|y(t)| < D$, for every $t \in [0, T]$. Indeed, if the function y has minimum at $t = t_m$ such that $y(t_m) < -D$, then we would have

$$y'(t_m) = \tilde{g}(t_m, x(t_m), y(t_m)) = -M_Y < 0\,,$$

a contradiction. Similarly, $\max_{[0,T]} y < D$ must hold.

We now define the periodic function $F_-(t) = y(t) - \gamma_-(x(t))$. Let $s_m \in [0, T]$ such that $F_-(s_m) = \min_{[0,T]} F_-$. If $F_-(s_m) \le 0$, we get the following contradiction:

$$\begin{aligned}
F_-'(s_m) &= y'(s_m) - \gamma_-'(x(s_m))x'(s_m)\\
&= \tilde{g}(s_m, x(s_m), y(s_m)) - \gamma_-'(x(s_m))\tilde{f}(t, x(s_m), y(s_m))\\
&= \hat{g}(s_m, x(s_m), y(s_m)) - \gamma_-'(x(s_m))\hat{f}(t, x(s_m), y(s_m))\\
&= \langle \hat{\Phi}(s_m, x(s_m), y(s_m)), (-\gamma_-'(x(s_m)), 1)\rangle\\
&= \left(1 - \frac{\gamma_-(x(s_m)) - y(s_m)}{D + \gamma_-(x(s_m))}\right) \Big\langle \Phi(s_m, x(s_m), \gamma_-(x(s_m)))\,,\, (-\gamma_-'(x(s_m)), 1)\Big\rangle\\
&\quad - \frac{\gamma_-(x(s_m)) - y(s_m)}{D + \gamma_-(x(s_m))}\Big\langle (M_X, M_Y), (-\gamma_-'(x(s_m)), 1)\Big\rangle < 0\,,
\end{aligned}$$

where we have used both (10) and (14). So, $\min_{[0,T]} F_- > 0$. Similarly we can prove that $\max_{[0,T]} F_+ < 0$, where $F_+(t) = y(t) - \gamma_+(x(t))$, thus concluding the proof. $\square$

3.1.3 A topological degree argument

We define the operators

$$\mathcal{L} : \mathcal{C}^1_T \to \mathcal{C}^0_T\,, \qquad \mathcal{L}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix},$$

where $\mathcal{C}^1_T = \big\{v \in \mathcal{C}^1([0,T],\mathbb{R}^2) : v(0) = v(T)\big\}$ and

$$\widetilde{\mathcal{N}} : \mathcal{C}^0_T \to \mathcal{C}^0_T\,, \qquad \widetilde{\mathcal{N}}\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \tilde{f}(t, x(t), y(t)) \\ \tilde{g}(t, x(t), y(t)) \end{pmatrix}. \tag{24}$$

So, a solution $u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of problem $(\widetilde{P})$ corresponds to a solution of

$$\mathcal{L}u - \widetilde{\mathcal{N}}u = 0\,. \tag{25}$$

In the previous section we have found the a priori bound $\overline{\mathcal{V}}$ for all the possible solutions of problem $(\widetilde{P})$. In order to apply the degree theory we need to consider an open ball $\mathcal{B}_R$ containing $\overline{\mathcal{V}}$. By the above arguments, we can deduce that if u solves (25), then $u \notin \partial\mathcal{B}_R$, so that the coincidence degree $d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{B}_R)$ is well defined. We refer to [10] for more details on this topic.

Since (22) and (23) hold, we can rewrite system (20) as

$$x' = y + p(t,x,y)\,, \qquad y' = x + q(t,x,y)\,,$$

where

$$p(t,x,y) = \hat{f}\Big(t, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\, \zeta\big(y;-D,D\big)\Big) - \zeta\big(y;-D,D\big)\,,$$
$$q(t,x,y) = \hat{g}\Big(t\,,\, \zeta\big(x;\alpha(t),\beta(t)\big)\,,\, \zeta\big(y;-D,D\big)\Big) - \zeta\big(x;\alpha(t),\beta(t)\big)\,,$$

are bounded functions. We now introduce the functions

$$\mathcal{F}_\lambda(t,u) = \mathcal{F}_\lambda(t,x,y) = \Big(y + \lambda p(t,x,y)\,,\, x + \lambda q(t,x,y)\Big),$$

and the problems

$$(Q_\lambda) \qquad \begin{cases} u' = \mathcal{F}_\lambda(t,u)\,, \\ u(0) = u(T)\,. \end{cases}$$

We define the Nemytskii operator related to the family of problem (Q_λ) as

$$(\mathcal{M}_\lambda u)(t) = \mathcal{F}_\lambda(t, u(t))\,,$$

Since the function $(p,q) : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is bounded, by a classical argument we can find a sufficiently large $R > 0$, such that, for every $\lambda \in [0,1]$, all the periodic solutions of (Q_λ) satisfy

$$\|u\|^2_\infty = \sup_{t\in[0,T]} [x^2(t) + y^2(t)] < R^2\,.$$

Since for $\lambda = 0$ we have an autonomous linear problem ruled by the function $\mathcal{G}(u) = \mathcal{G}(x, y) = \big(y, x\big)$, by [4, Lemma 1] we can conclude that

$$d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{B}_R) = d_{\mathcal{L}}(\mathcal{L} - \mathcal{M}_1, \mathcal{B}_R) = d_{\mathcal{L}}(\mathcal{L} - \mathcal{M}_0, \mathcal{B}_R) = \deg(\mathcal{G}, B_R) = -1\,,$$

where $\deg(\mathcal{G}, B_R)$ denotes the Brouwer degree of the function $\mathcal{G}$ on the ball $B_R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\}$ and $\mathcal{B}_R$ is the set of continuous functions having image in B_R. We have so found a solution of problem $(\widetilde{P})$ belonging to the set $\mathcal{B}_R$. However, such a solution belongs indeed to the a priori bound $\overline{\mathcal{V}}$, and so it is also a solution of problem (P), thus concluding the proof of Theorem 6.

3.2 An important consequence of the proof

We first recall the definition (15) of the open set

$$\mathcal{V} = \{u \in \mathcal{C}^0_T \mid (t, u(t)) \in V \text{ for every } t \in [0, T]\}\,, \tag{26}$$

where

$$V = \{(t, x, y) \in \mathbb{R}^3 \mid \alpha(t) < x < \beta(t)\,,\ \gamma_-(x) < y < \gamma_+(x)\}\,.$$

Let us introduce the Nemytskii operator related to problem (P) as

$$\mathcal{N} : \mathcal{C}^0_T \to \mathcal{C}^0_T \qquad \mathcal{N}\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} f(t, x(t), y(t)) \\ g(t, x(t), y(t)) \end{pmatrix}.$$

Corollary 12. *Under the assumptions of Theorem 6, if there are no solutions of (P) in $\partial\mathcal{V}$, then*

$$d_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \mathcal{V}) = -1\,.$$

Proof. Since $\Phi = \widetilde{\Phi}$ on $\overline{V}$, and so $\mathcal{N} = \widetilde{\mathcal{N}}$ on $\overline{\mathcal{V}}$, the additional assumption permits us to evaluate the coincidence degree also on the set $\mathcal{V}$. Recalling that all the solutions of problem $(\widetilde{P})$ satisfy the a priori bounds (22) and (23), by the excision property we have

$$-1 = d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{B}_R) = d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{V}) = d_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \mathcal{V})\,,$$

and the proof is completed. □

Remark 13. *The set $\mathcal{V}$ introduced in* (26) *depends on the well-ordered couple (α, β) of lower/upper solutions of problem (P) and the functions $\gamma_\pm$ given in the assumptions of Theorem 6. In the following section, we will denote this set by $\mathcal{V}(\alpha, \beta, \gamma_\pm)$ when we need to underline such a dependence.*

4 Non-well-ordered lower and upper solutions

We still consider the periodic problem

$$(P) \qquad \begin{cases} x' = f(t, x, y)\,, \quad y' = g(t, x, y)\,, \\ x(0) = x(T)\,, \quad y(0) = y(T)\,, \end{cases}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions, T-periodic in their first variable.

We will say that (α, β) is a non-well-ordered couple of lower/upper solutions of problem (P) if α and β are respectively a lower and an upper solution of problem (P), such that there exists $\hat{t}_0 \in [0, T]$ satisfying

$$\alpha(\hat{t}_0) > \beta(\hat{t}_0)\,. \tag{27}$$

Let us set

$$a(t) := \min\{\alpha(t), \beta(t)\}\,, \qquad b(t) := \max\{\alpha(t), \beta(t)\}\,,$$

$$A := \min a\,, \qquad B := \max b\,.$$

Notice that $A < B$, by (27).

Let us introduce our assumptions.

(H1) There is a continuous function $\chi : \mathbb{R} \to [0, +\infty[$ and a constant $M > 0$ such that

$$|f(t, x, y)| \le \chi(y)(1 + |x|)\,, \qquad \text{for every } (t, x, y) \in \mathbb{R}^3\,, \tag{28}$$

$$|g(t, x, y)| \le M(1 + |y|)\,, \qquad \text{for every } (t, x, y) \in \mathbb{R}^3\,. \tag{29}$$

(H2) There exist two continuous functions $\gamma_\pm : [A, B] \times [1, +\infty[\to \mathbb{R}$, continuously differentiable with respect to the first variable, such that

$$\lim_{\lambda \to +\infty} \gamma_\pm(x; \lambda) = \pm\infty\,, \quad \text{uniformly with respect to } x \in [A, B]\,,$$

and

$$g\big(t, x, \gamma_-(x; \lambda)\big) < f\big(t, x, \gamma_-(x; \lambda)\big)\gamma_-'(x; \lambda)\,, \tag{30}$$

$$g\big(t, x, \gamma_+(x; \lambda)\big) > f\big(t, x, \gamma_+(x; \lambda)\big)\gamma_+'(x; \lambda)\,, \tag{31}$$

for every $t \in \mathbb{R}$, $x \in [a(t), b(t)]$ and $\lambda \in [1, +\infty[$. (Here we denote by $\gamma_\pm'$ the derivative with respect to the first variable.)

Theorem 14. *Assume the existence of a non-well-ordered couple (α, β) of lower/upper solutions of problem (P) with the additional property that there exists a constant $\hat{c} > 0$ such that, for every $k \in \{1, \dots, n\}$,*

$$\begin{cases} y \le -\hat{c} \quad \Rightarrow \quad f(\tau_k, \alpha(\tau_k^-), y) < \alpha'(\tau_k^-)\,, \\ y \ge \hat{c} \quad \Rightarrow \quad f(\tau_k, \alpha(\tau_k^+), y) > \alpha'(\tau_k^+)\,, \end{cases} \tag{32}$$

$$\begin{cases} y \le -\hat{c} \quad \Rightarrow \quad f(\tau_k, \beta(\tau_k^+), y) < \beta'(\tau_k^+)\,, \\ y \ge \hat{c} \quad \Rightarrow \quad f(\tau_k, \beta(\tau_k^-), y) > \beta'(\tau_k^-)\,. \end{cases} \tag{33}$$

If (H1) and (H2) hold, there exists at least one solution of problem (P) such that, for some $t_1, t_2 \in [0, T]$, one has $x(t_1) \le \alpha(t_1)$ and $x(t_2) \ge \beta(t_2)$.

This theorem extends some classical results for scalar second order differential equations of the type (1). We will show below two examples of applications. Conditions (H1) and (H2) will be necessary in order to avoid resonance phenomena, and to obtain a priori bounds. Notice that (3) and (6) imply a weaker form of (32) and (33), i.e., with only weak inequalities. It remains an open problem if these additional assumptions can be omitted.

We will discuss in Section 5 on the possibility of reversing the inequalities in (30) and (31). Concerning the existence of the functions $\gamma_\pm$, let us prove the following lemma.

Lemma 15. *Let the following assumptions hold:*

(G1) *there are a constant* $d > 0$ *and two continuous functions* $f_+ : [d, +\infty[\to \mathbb{R}$ *and* $f_- :]-\infty, -d] \to \mathbb{R}$ *such that*

$$\begin{cases} y \geq d & \Rightarrow \quad f(t,x,y) \geq f_+(y) > 0\,, \\ y \leq -d & \Rightarrow \quad f(t,x,y) \leq f_-(y) < 0\,, \end{cases}$$
$$\text{for every } (t,x) \in [0,T] \times [A,B]\,;$$

(G2) *there is a positive continuous function* $\varphi : [0, +\infty[\to \mathbb{R}$ *such that*

$$|g(t,x,y)| \leq \varphi(|y|)\,, \qquad \text{for every } (t,x,y) \in [0,T] \times [A,B] \times \mathbb{R}\,;$$

(G3) *the above functions are such that*

$$\int_d^{+\infty} \frac{f_+(s)}{\varphi(s)}\, ds = +\infty\,, \qquad \int_{-\infty}^{-d} \frac{f_-(s)}{\varphi(|s|)}\, ds = -\infty\,. \tag{34}$$

Then, there exist four continuous functions $\gamma_{\pm,1}$, $\gamma_{\pm,2} : [A,B] \times [1, +\infty[\to \mathbb{R}$, *continuously differentiable with respect to the first variable, such that*

$$\lim_{\lambda \to +\infty} \gamma_{\pm,1}(x;\lambda) = \pm\infty \quad \text{and} \quad \lim_{\lambda \to +\infty} \gamma_{\pm,2}(x;\lambda) = \pm\infty\,,$$
$$\text{uniformly with respect to } x \in [A,B]\,, \tag{35}$$

and

$$g\big(t,x,\gamma_{+,1}(x;\lambda)\big) > f\big(t,x,\gamma_{+,1}(x;\lambda)\big)\gamma'_{+,1}(x;\lambda)\,, \tag{36}$$
$$g\big(t,x,\gamma_{+,2}(x;\lambda)\big) < f\big(t,x,\gamma_{+,2}(x;\lambda)\big)\gamma'_{+,2}(x;\lambda)\,, \tag{37}$$
$$g\big(t,x,\gamma_{-,1}(x;\lambda)\big) < f\big(t,x,\gamma_{-,1}(x;\lambda)\big)\gamma'_{-,1}(x;\lambda)\,, \tag{38}$$
$$g\big(t,x,\gamma_{-,2}(x;\lambda)\big) > f\big(t,x,\gamma_{-,2}(x;\lambda)\big)\gamma'_{-,2}(x;\lambda)\,, \tag{39}$$

for every $t \in [0,T]$, $x \in [A,B]$ *and* $\lambda \in [1, +\infty[$.

Proof. For every $y_0 \geq d$, we introduce the continuous strictly increasing function $\mathcal{F}_{y_0} : [d, +\infty[\to \mathbb{R}$ defined as

$$\mathcal{F}_{y_0}(\xi) = \int_{y_0}^{\xi} \frac{f_+(s)}{\varphi(s)}\, ds\,.$$

We can easily verify that $\mathcal{F}_{y_0}(y_0) = 0$ and, from (34),

$$\lim_{\xi \to +\infty} \mathcal{F}_{y_0}(\xi) = +\infty\,.$$

Construction of $\gamma_{+,1}$. For every $y_0 \geq d$ and for every $x \in [A,B]$ there exists a unique $\xi \geq y_0$ such that $\mathcal{F}_{y_0}(\xi) = 2(B-x)$. Hence, we can define $\gamma_{+,1}(x;\lambda)$, for $\lambda \geq 1$, as the unique solution of equation

$$\mathcal{F}_{\lambda-1+d}\big(\gamma_{+,1}(x;\lambda)\big) = 2(B-x)\,. \tag{40}$$

In particular, since $\mathcal{F}_{\lambda-1+d}\big(\gamma_{+,1}(B;\lambda)\big)=0$, we get

$$\gamma_{+,1}(x;\lambda)\geq\gamma_{+,1}(B;\lambda)=\lambda-1+d\,,$$

which provides the validity of (35) for the function $\gamma_{+,1}$. Differentiating in (40) we see that $\gamma'_{+,1}(x;\lambda)<0$ for every $x\in[A,B]$, and

$$\begin{aligned}f(t,x,\gamma_{+,1}(x;\lambda))\gamma'_{+,1}(x;\lambda)&\leq f_+(\gamma_{+,1}(x;\lambda))\gamma'_{+,1}(x;\lambda)\\&=-2\varphi(\gamma_{+,1}(x;\lambda))<-\varphi(\gamma_{+,1}(x;\lambda))\\&<g(t,x,\gamma_{+,1}(x;\lambda))\,,\end{aligned}$$

thus proving (36).

Construction of $\gamma_{+,2}$. Arguing similarly as above, for every $y_0\geq d$ and for every $x\in[A,B]$ there exists a unique $\xi\geq y_0$ such that $\mathcal{F}_{y_0}(\xi)=2(x-A)$. Hence we can define $\gamma_{+,2}(x;\lambda)$ by

$$\mathcal{F}_{\lambda-1+d}\big(\gamma_{+,2}(x;\lambda)\big)=2(x-A)\,. \tag{41}$$

In particular, since $\mathcal{F}_{\lambda-1+d}\big(\gamma_{+,2}(A;\lambda)\big)=0$, we get

$$\gamma_{+,2}(x;\lambda)\geq\gamma_{+,2}(A;\lambda)=\lambda-1+d\,,$$

so that (35) holds for the function $\gamma_{+,2}$. Differentiating in (41),

$$\begin{aligned}f(t,x,\gamma_{+,2}(x;\lambda))\gamma'_{+,2}(x;\lambda)&\geq f_+(\gamma_{+,2}(x;\lambda))\gamma'_{+,2}(x;\lambda)\\&=2\varphi(\gamma_{+,2}(x;\lambda))>\varphi(\gamma_{+,2}(x;\lambda))\\&>g(t,x,\gamma_{+,2}(x;\lambda))\,,\end{aligned}$$

thus proving (37).

The construction of the functions $\gamma_{-,1}$ and $\gamma_{-,2}$ satisfying (38) and (39) is similar. □

Let us illustrate how our result applies to two classical scalar second order differential equations of the type (1), involving a *scalar p-Laplacian* and a *mean curvature* operator, with $\phi(s)=|s|^{p-2}s$ and $\phi(s)=s/\sqrt{1+s^2}$, respectively.

Consider first the problem

$$\begin{cases}(|x'|^{p-2}x')'=h(t,x,x')\,,\\x(0)=x(T)\,,\quad x'(0)=x'(T)\,,\end{cases} \tag{42}$$

with $p>1$, which is equivalent to problem (P), taking $f(t,x,y)=f(y)=|y|^{q-2}y$, with $(1/p)+(1/q)=1$, and $g(t,x,y)=h(t,x,|y|^{q-2}y)$.

Corollary 16. *Assume the existence of a non-well-ordered couple* (α,β) *of lower/upper solutions of problem* (42)*, and of a constant* $M>0$ *for which*

$$|h(t,x,z)|\leq M(1+|z|^{p-1})\,,\qquad \text{for every } (t,x,z)\in\mathbb{R}^3. \tag{43}$$

Then, there exists at least one solution of problem (42) *such that, for some* $t_1,t_2\in[0,T]$*, one has* $x(t_1)\leq\alpha(t_1)$ *and* $x(t_2)\geq\beta(t_2)$.

Proof. Notice that (43) implies (29). We can use Lemma 15 with $\varphi(s) = M(1+|y|)$ to construct the curves $\gamma_\pm$. Then, Theorem 14 applies. $\square$

Consider now the problem

$$\begin{cases} \left(\dfrac{x'}{\sqrt{1+(x')^2}} \right)' = h(t,x,x')\,, \\ x(0) = x(T)\,, \quad x'(0) = x'(T)\,, \end{cases} \tag{44}$$

which is equivalent to problem (P), taking $f(t,x,y) = \phi^{-1}(y) = y/\sqrt{1-y^2}$ and $g(t,x,y) = h\big(t,x,y/\sqrt{1-y^2}\,\big)$. Notice that these functions are now only defined on $\mathbb{R}\times\mathbb{R}\times\,]-1,1[\,$.

Corollary 17. *Assume the existence of a non-well-ordered couple (α,β) of lower/upper solutions of problem (44), and of a positive continuous function $\zeta : [0,+\infty[\,\to\mathbb{R}$ such that*

$$|h(t,x,z)| \le \zeta(|z|)\,, \qquad \textit{for every } (t,x,z)\in\mathbb{R}^3, \tag{45}$$

and

$$\int_0^{+\infty} \frac{ds}{(1+s^2)^{3/2}\,\zeta(s)} > \frac{T}{2}\,. \tag{46}$$

Then, there exists at least one solution of problem (44) such that, for some $t_1,t_2\in[0,T]$, one has $x(t_1)\le\alpha(t_1)$ and $x(t_2)\ge\beta(t_2)$.

Proof. Recalling that $\phi(s) = s/\sqrt{1+s^2}$, by (46) there is a $c\in\,]0,1[$ such that

$$\int_0^{\phi^{-1}(c)} \frac{\phi'(s)}{\zeta(s)}\,ds > \frac{T}{2}\,. \tag{47}$$

We define the functions $f_c:\mathbb{R}\to\mathbb{R}$ and $g_c:\mathbb{R}^3\to\mathbb{R}$ as

$$f_c(y) = \begin{cases} \phi^{-1}(-c)+y+c\,, & \text{if } y<-c\,, \\ \phi^{-1}(y)\,, & \text{if } |y|\le c\,, \\ \phi^{-1}(c)+y-c\,, & \text{if } y>c\,, \end{cases}$$

and

$$g_c(t,x,y) = \begin{cases} g(t,x,-c)\,, & \text{if } y<-c\,, \\ g(t,x,y)\,, & \text{if } |y|\le c\,, \\ g(t,x,c)\,, & \text{if } y>c\,, \end{cases}$$

and we consider the system

$$x' = f_c(y)\,, \qquad y' = g_c(t,x,y)\,. \tag{48}$$

Using Lemma 15, we see that all the assumptions of Theorem 14 hold, so that problem (48) has a T-periodic solution (x,y).

We now show that $|y(t)| \leq c$ for every t, implying that (x, y) is indeed a solution of problem (44). By contradiction, assume that $\max y > c$, or $\min y < -c$. Let us treat the first case, the other one being similar. By the periodicity, there exists a $\xi \in [0, T]$ such that $x'(\xi) = 0$ and, correspondingly, $y(\xi) = 0$. Let ξ_1 and ξ_2 be such that $|\xi_2 - \xi_1| \leq \frac{T}{2}$, $y(\xi_1) = 0$, $y(\xi_2) = c$, and $y(t) \in]0, c[$ for every $t \in]\xi_1, \xi_2[$. (When $\xi_1 > \xi_2$, we write $]\xi_1, \xi_2[=]\xi_2, \xi_1[$ and $[\xi_1, \xi_2] = [\xi_2, \xi_1]$.) For every $t \in [\xi_1, \xi_2]$, by (45) we have

$$|y'(t)| \leq \zeta(\phi^{-1}(y(t))) ,$$

so that, by (47),

$$|\xi_2 - \xi_1| \geq \left| \int_{\xi_1}^{\xi_2} \frac{y'(t)}{\zeta(\phi^{-1}(y(t)))} \, dt \right| = \int_0^{\phi^{-1}(c)} \frac{\phi'(s)}{\zeta(s)} \, ds > \frac{T}{2} ,$$

a contradiction. □

The above corollary generalizes [13, Proposition 3.7], where $\zeta(s)$ is a constant function with positive value $K < \frac{2}{T}$.

4.1 Proof of Theorem 14

4.1.1 An auxiliary problem

Let us set

$$d_y := \max\{\|y_\alpha\|_\infty \, , \, \|y_\beta\|_\infty \, , \, \|\alpha'\|_\infty \, , \, \|\beta'\|_\infty \, , \, \hat{c}\} , \tag{49}$$

where, for all these functions, the norm $\| \cdot \|_\infty$ can be defined as in (12).

We recall here a classical result, which is a straightforward consequence of the Gronwall Lemma, often mentioned as *elastic property.*

Lemma 18. *For every constant $\mathcal{K} > 0$ we can define a function $\mathcal{E}_{\mathcal{K}} : [0, +\infty[\to [0, +\infty[$ with the following property: given a differentiable function $z : \mathbb{R} \to \mathbb{R}$ satisfying*

$$|z'(t)| \leq \mathcal{K}(1 + |z(t)|) , \quad \text{for every } t \in \mathbb{R} ,$$

if $|z(\bar{t})| \leq Z$ for a certain $\bar{t} \in \mathbb{R}$, then $|z(t)| \leq \mathcal{E}_{\mathcal{K}}(Z)$ for every $t \in [\bar{t} - T, \bar{t} + T]$.

For example, we can take $\mathcal{E}_{\mathcal{K}}(Z) = (Z + \mathcal{K}T)e^{\mathcal{K}T}$.

Using the notation introduced in the previous lemma, let us now set

$$D := \mathcal{E}_M(d_y) ,$$

where M and d_y have been introduced respectively in (29) and (49).

By assumption (H2), we can find a sufficiently large constant $\Lambda > 1$ such that

$$|\gamma_\pm(x; \lambda)| > D , \quad \text{for every } x \in [A, B] \text{ and } \lambda \geq \Lambda . \tag{50}$$

Let us introduce the sets

$$N_\Lambda := \{(t, x, y) \in \mathbb{R}^3 \, : \, a(t) \leq x \leq b(t) \, , \, y > \gamma_+(x; \Lambda)\} ,$$
$$C_\Lambda := \{(t, x, y) \in \mathbb{R}^3 \, : \, a(t) \leq x \leq b(t) \, , \, \gamma_-(x; \Lambda) \leq y \leq \gamma_+(x; \Lambda)\} ,$$
$$S_\Lambda := \{(t, x, y) \in \mathbb{R}^3 \, : \, a(t) \leq x \leq b(t) \, , \, y < \gamma_-(x; \Lambda)\} ,$$

(see Figure 3).

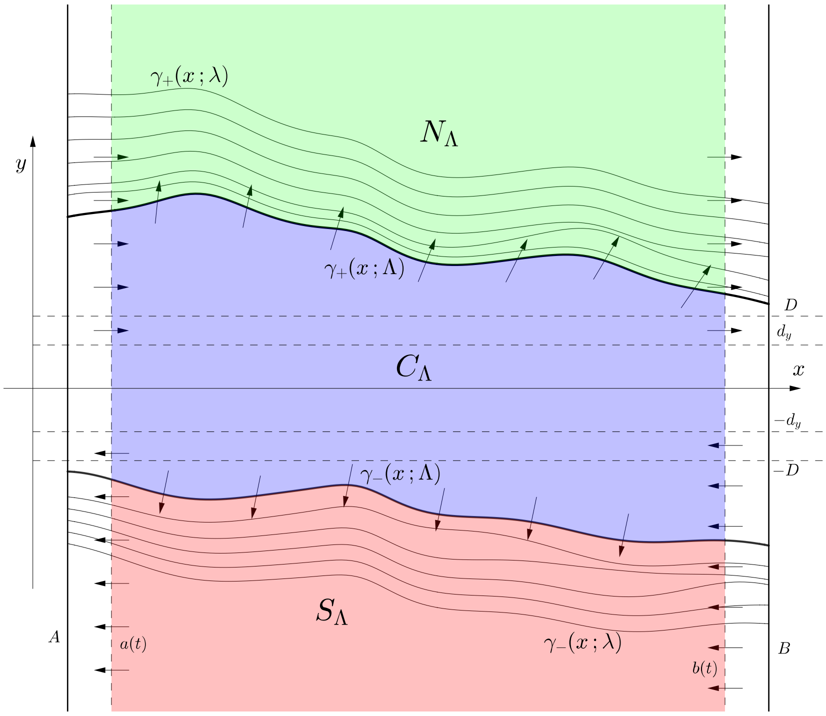

Figure 3: A sketch of the section at a fixed time t of the regions N_Λ, C_Λ, and S_Λ. Notice that the vertical lines $x = \alpha$ and $x = \beta$ move in time, while the curves $\gamma_\pm(\cdot, \Lambda)$ are fixed.

Lemma 19. *There are two constants ℓ_x and ℓ_y with the following property: if $u = (x, y)$ is a solution of*

$$x' = f(t, x, y)\,, \quad y' = g(t, x, y) \tag{51}$$

such that $(t_0, u(t_0)) \in C_\Lambda$ for a certain $t_0 \in [0, T]$, then

$$|x(t)| \le \ell_x \quad \textit{and} \quad |y(t)| \le \ell_y\,, \quad \textit{for every } t \in [0, T]\,.$$

Proof. Since the set C_Λ is bounded, we can fix two constants $X > 0$ and $Y > 0$ such that

$$C_\Lambda \subseteq [-X, X] \times [-Y, Y]\,.$$

Hence, applying Lemma 18 in the setting $(z, \mathcal{K}, Z, \bar{t}) = (y, M, Y, t_0)$, we see that every solution $u = (x, y)$ of (51) such that $(t_0, u(t_0)) \in C_\Lambda$, for a certain $t_0 \in [0, T]$, satisfies

$$|y(t)| \le \ell_y := \mathcal{E}_M(Y)\,, \quad \text{for every } t \in [0, T]\,.$$

Now, recalling (28) and setting $M_\chi = \max_{[-\ell_y, \ell_y]} \chi$, applying Lemma 18 in the setting $(z, \mathcal{K}, Z, \bar{t}) = (x, M_\chi, X, t_0)$, we see that any such solution also satisfies

$$|x(t)| \le \ell_x := \mathcal{E}_{M_\chi}(X)\,, \quad \text{for every } t \in [0, T]\,.$$

The lemma is thus proved. □

We will now modify the functions f, g by a procedure which resembles the one in Section 3.1.1. From assumption (H2) we can find $\Lambda_1 > \Lambda$ such that

$$|\gamma_\pm(x;\lambda)| > \ell_y + 1\,, \qquad \text{for every } x \in [A,B] \text{ and } \lambda \geq \Lambda_1\,. \tag{52}$$

We introduce the constants

$$c_\gamma := \max\{|\gamma'_\pm(x;\lambda)| \,:\, x\in[A,B]\,,\, \lambda\in[1,\Lambda_1]\,\}\,, \tag{53}$$

$$\mathcal{M}_X := \max\{|f(t,x,y)| \,:\, t\in[0,T]\,, |x|\leq \ell_x\,,\, |y|\leq \ell_y+1\}\,, \tag{54}$$

and choose

$$\mathcal{M}_Y > c_\gamma\,\mathcal{M}_X\,. \tag{55}$$

Setting $\Phi(t,x,y) = \big(f(t,x,y), g(t,x,y)\big)$, we define $\widehat{\Phi} : \mathbb{R}\times[-\ell_x,\ell_x]\times\mathbb{R} \to \mathbb{R}^2$ as

$$\widehat{\Phi}(t,x,y) = \begin{cases} (\mathcal{M}_X,\mathcal{M}_Y)\,, & \text{if } y \geq \ell_y+1\,,\\ \Phi(t,x,y) + (y-\ell_y)\Big((\mathcal{M}_X,\mathcal{M}_Y) - \Phi(t,x,y)\Big), & \\ & \text{if } \ell_y \leq y \leq \ell_y+1\,,\\ \Phi(t,x,y)\,, & \text{if } -\ell_y \leq y \leq \ell_y\,,\\ \Phi(t,x,y) - (y+\ell_y)\Big((-\mathcal{M}_X,-\mathcal{M}_Y) - \Phi(t,x,y)\Big), & \\ & \text{if } -\ell_y - 1 \leq y \leq -\ell_y\,,\\ (-\mathcal{M}_X,-\mathcal{M}_Y)\,, & \text{if } y \leq -\ell_y - 1\,. \end{cases}$$

We will write $\widehat{\Phi}(t,x,y) = \big(\hat{f}(t,x,y), \hat{g}(t,x,y)\big)$. Finally, we define $\widetilde{\Phi} : \mathbb{R}^3 \to \mathbb{R}^2$ as

$$\widetilde{\Phi}(t,x,y) = \begin{cases} (y,1)\,, & \text{if } x \geq \ell_x+1\,,\\ \widehat{\Phi}(t,\ell_x,y) + (x-\ell_x)\Big((y,1) - \widehat{\Phi}(t,\ell_x,y)\Big), & \\ & \text{if } \ell_x \leq x \leq \ell_x+1\,,\\ \widehat{\Phi}(t,x,y)\,, & \text{if } -\ell_x \leq x \leq \ell_x\,,\\ \widehat{\Phi}(t,-\ell_x,y) - (x+\ell_x)\Big((y,-1) - \widehat{\Phi}(t,-\ell_x,y)\Big), & \\ & \text{if } -\ell_x - 1 \leq y \leq -\ell_x\,,\\ (y,-1)\,, & \text{if } x \leq -\ell_x - 1\,. \end{cases}$$

We will write $\widetilde{\Phi}(t,x,y) = \big(\tilde{f}(t,x,y), \tilde{g}(t,x,y)\big)$.

Remark 20. *The functions $\tilde{f}$, $\tilde{g}$ coincide with f, g on the rectangle $[-\ell_x,\ell_x]\times[-\ell_y,\ell_y]$, and $\tilde{f}(t,x,y) = y$ when $|x| \geq \ell_x + 1$. Moreover, the function $\tilde{g}$ is bounded, so we can find a constant $\widetilde{M} > 0$ such that*

$$|\tilde{g}(t,x,y)| \leq \widetilde{M}\,, \quad \textit{for every } (t,x,y)\in\mathbb{R}^3\,.$$

We will consider the modified problem

$$(\widetilde{P}) \qquad \begin{cases} x' = \tilde{f}(t,x,y)\,, & y' = \tilde{g}(t,x,y)\,,\\ x(0) = x(T)\,, & y(0) = y(T)\,. \end{cases}$$

We have the following a priori bound.

Lemma 21. *If $u = (x, y)$ is a solution of $(\widetilde{P})$ such that $(t_0, u(t_0)) \in C_\Lambda$ for a certain $t_0 \in [0, T]$, then*

$$|x(t)| \le \ell_x \quad \textit{and} \quad |y(t)| \le \ell_y\,, \quad \textit{for every } t \in \mathbb{R}\,.$$

Hence, u is also a solution of (P).

Proof. As long as the solution u of $(\widetilde{P})$ is such that $u(t) \in [-\ell_x, \ell_x] \times [-\ell_y, \ell_y]$, it is a solution of (51). Hence Lemma 19 applies, guaranteeing that indeed $u(t) \in [-\ell_x, \ell_x] \times [-\ell_y, \ell_y]$ for every $t \in [0, T]$. □

Remark 22. *Since we have assumed the validity of* (3), (6), (32) *and* (33), *thanks to the choice* (54) *we have the following assertions.*

If a solution (x, y) of $(\widetilde{P})$ is such that $y(t_0) > d_y$ and $x(t_0) = a(t_0)$ [resp. $x(t_0) = b(t_0)$], then we have $x > a$ [resp. $x > b$], in a right neighborhood of t_0.

If a solution (x, y) of $(\widetilde{P})$ is such that $y(t_0) < -d_y$ and $x(t_0) = a(t_0)$ [resp. $x(t_0) = b(t_0)$], then we have $x < a$ [resp. $x < b$], in a right neighborhood of t_0.

4.1.2 An a priori bound for the desired solutions

Our aim is to show the existence of a solution of $(\widetilde{P})$ belonging to the set

$$\mathcal{S} = \big\{u = (x, y) \in \mathcal{C}^0_T \,:\, \text{there exist } t_1, t_2 \in [0, T] \text{ such that} \\ x(t_1) \le \alpha(t_1) \text{ and } x(t_2) \ge \beta(t_2)\big\}\,. \quad (56)$$

In the following lemma we will prove that a solution belonging to $\mathcal{S}$ satisfies the hypotheses of Lemma 21, permitting us to conclude that it is a solution of the original problem (P).

Lemma 23. *If $u = (x, y) \in \mathcal{S}$ is a solution of $(\widetilde{P})$, then there exists a $t_0 \in [0, T]$ such that $(t_0, u(t_0)) \in C_\Lambda$, where Λ is given in* (50).

Proof. Let us first prove the following preliminary assertion.

Claim. For any solution u of $(\widetilde{P})$, it cannot be that $(t, u(t)) \in N_\Lambda$ for every $t \in [0, T]$.

By contradiction, assume this is true. We distinguish two cases.

If $y(t) \ge \ell_y + 1$ for every $t \in [0, T]$, then, since $N_\Lambda \subseteq [-\ell_x, \ell_x] \times \mathbb{R}$, we get

$$y'(t) = \tilde{g}(t, x(t), y(t)) = \hat{g}(t, x(t), y(t)) = \mathcal{M}_Y > 0\,,$$

which is in contradiction with the periodicity of the function y.

If $y(\bar t_0) < \ell_y + 1$ for some $\bar t_0 \in [0,T]$, recalling (52) and the continuity of the function γ_+ with respect to λ, we can find $\lambda_0 \in [1,\Lambda_1[$ such that $y(\bar t_0) = \gamma_+(x(\bar t_0);\lambda_0)$. By (53) and (55), we have

$$\big\langle \big(\mathcal{M}_X, \mathcal{M}_Y\big)\,,\,\big(-\gamma_+'(x;\lambda),1\big)\big\rangle \geq -c_\gamma \mathcal{M}_X + \mathcal{M}_Y > 0\,, \tag{57}$$

for every $x \in [A,B]$ and $\lambda \in [1,\Lambda_1]$. Moreover, we can rewrite (31) as

$$\big\langle \Phi\big(t,x,\gamma_+(x;\lambda)\big)\,,\,\big(-\gamma_+'(x;\lambda),1\big)\big\rangle > 0\,. \tag{58}$$

The function $F_+(t;\lambda_0) = y(t) - \gamma_+(x(t);\lambda_0)$ is T-periodic in t, and $F_+(\bar t_0;\lambda_0) = 0$. From the above estimates (57) and (58), since $a(\bar t_0) \leq x(\bar t_0) \leq b(\bar t_0)$,

$$\begin{aligned} F_+'(\bar t_0;\lambda_0) &= \tilde g\big(\bar t_0, x(\bar t_0), \gamma_+(x(\bar t_0);\lambda_0)\big) \\ &\qquad - \tilde f\big(\bar t_0, x(\bar t_0), \gamma_+(x(\bar t_0);\lambda_0)\big)\gamma_+'\big(x(\bar t_0);\lambda_0\big) \\ &= \Big\langle \widetilde\Phi\big(\bar t_0, x(\bar t_0), \gamma_+(x(\bar t_0);\lambda_0)\big)\,,\,\big(-\gamma_+'(x(\bar t_0);\lambda_0),1\big)\Big\rangle \\ &= \Big\langle \widehat\Phi\big(\bar t_0, x(\bar t_0), \gamma_+(x(\bar t_0);\lambda_0)\big)\,,\,\big(-\gamma_+'(x(\bar t_0);\lambda_0),1\big)\Big\rangle > 0\,. \end{aligned} \tag{59}$$

So, there exists $\varepsilon \in]0,T/2[$ such that

$$F_+(\bar t_0 + \varepsilon;\lambda_0) > 0 > F_+(\bar t_0 + T - \varepsilon;\lambda_0)\,,$$

providing the existence of a certain $\bar t_1 \in [\bar t_0 + \varepsilon\,,\,\bar t_0 + T - \varepsilon]$ such that $F_+(\bar t_1;\lambda_0) = 0$ and $F_+'(\bar t_1;\lambda_0) \leq 0$. However, similarly as in (59), we get the contradiction $F_+'(\bar t_1;\lambda_0) > 0$.

The proof of the Claim is thus completed. Similarly one proves that it cannot be that $(t,u(t)) \in S_\Lambda$ for every $t \in [0,T]$.

Now, let $u = (x,y)$ be a solution of $(\widetilde P)$ belonging to $\mathcal{S}$. Then, there exists a $t_0 \in [0,T]$ such that

$$A \leq a(t_0) \leq x(t_0) \leq b(t_0) \leq B\,. \tag{60}$$

We will prove that $(t_0,u(t_0)) \in C_\Lambda$.

Assume by contradiction that $(t_0,u(t_0)) \in N_\Lambda$. Recalling the Claim, let

$$t_1 := \inf\{t \in [t_0,t_0+T[\,:\,(t,u(t)) \notin N_\Lambda\}\,. \tag{61}$$

Since $(t_1,u(t_1)) \in \partial N_\Lambda$, we need to treat the following three cases (see Figure 3).

Case 1: $y(t_1) \geq \gamma_+(x(t_1);\Lambda)$ *and* $x(t_1) = b(t_1)$. Let

$$t_2 := \sup\{t \in [t_1,t_0+T]\,;\,x(s) \geq b(s)\,\forall s \in [t_1,t]\}\,. \tag{62}$$

Since $y(t_1) > D > d_y$, from Remark 22 we have that $t_2 > t_1$. By Lemma 18, since $y(t_1) > D = \mathcal{E}_M(d_y)$, we get $y(t_2) > d_y$. Again from Remark 22 we have $x - b > 0$ in a right neighborhood of t_2, in contradiction with its definition in (62).

Case 2: $y(t_1) = \gamma_+(x(t_1);\Lambda)$ *and* $a(t_1) \leq x(t_1) < b(t_1)$. The function $F_+(\cdot;\Lambda)$ is well defined and non-negative in the nontrivial interval $[t_0,t_1]$. Reasoning as in (59), we can show that $F_+'(t_1;\Lambda) > 0$, contradicting the definition of t_1 in (61).

Case 3: $y(t_1) > \gamma_+(x(t_1);\Lambda)$ and $x(t_1) = a(t_1) < b(t_1)$. This situation is forbidden, by Remark 22.

Hence, we can conclude that $(t_0, u(t_0)) \notin N_\Lambda$. Similarly one proves that $(t_0, u(t_0)) \notin S_\Lambda$, and the proof is thus completed. □

4.1.3 Creating well-ordered couples of lower/upper solutions of $(\widetilde{P})$

Lemma 24. *Both the constant $\hat{\alpha} \equiv -\ell_x - 2$ and α are lower solutions of problem $(\widetilde{P})$. At the same time, both the constant $\hat{\beta} \equiv \ell_x + 2$ and β are upper solutions of problem $(\widetilde{P})$.*

Proof. We first verify that the constant functions $\hat{\alpha} \equiv -\ell_x - 2$ and $\hat{\beta} \equiv \ell_x + 2$ are respectively a lower solution and an upper solution of $(\widetilde{P})$. Indeed, setting $y_{\hat{\alpha}} \equiv 0$ and $y_{\hat{\beta}} \equiv 0$, since $\tilde{f}(t, -\ell_x - 2, y) = \tilde{f}(t, \ell_x + 2, y) = y$, then (3) and (6) easily follow. Moreover, (4) and (7) are an immediate consequence of $\tilde{g}(t, -\ell_x - 2, 0) = -1 < 0$ and $\tilde{g}(t, \ell_x + 2, 0) = 1 > 0$.

In order to check that the functions α and β are respectively a lower solution and an upper solution also for problem $(\widetilde{P})$, we need to verify the validity of (3), (6), (32) and (33), where we replace the functions f with $\widetilde{f}$. This fact is guaranteed by the choice (54). The validity of both (4) and (7) with g replaced by $\tilde{g}$ is trivial since $g = \widetilde{g}$ at the points we have to deal with. □

Remark 25. *The couples $(\hat{\alpha}, \hat{\beta})$, $(\hat{\alpha}, \beta)$, and $(\alpha, \hat{\beta})$ are well-ordered couples of lower/upper solutions of problem $(\widetilde{P})$.*

Lemma 26. *There exist two continuously differentiable functions $\Gamma_\pm : [\hat{\alpha}, \hat{\beta}] \to \mathbb{R}$, such that*

$$\tilde{g}\big(t, x, \Gamma_-(x)\big) < \tilde{f}\big(t, x, \Gamma_-(x)\big)\Gamma'_-(x)\,,$$
$$\tilde{g}\big(t, x, \Gamma_+(x)\big) > \tilde{f}\big(t, x, \Gamma_+(x)\big)\Gamma'_+(x)\,,$$

for every $t \in \mathbb{R}$ and $x \in [\hat{\alpha}, \hat{\beta}]$.

Proof. From Remark 20 we deduce the validity of the hypotheses of Lemma 15 adopting the following choices

$$[A, B] = [\hat{\alpha}, \hat{\beta}]\,, \quad f_+(y) = -f_-(y) \equiv d = \max\{\mathcal{M}_X, \ell_y + 1\}\,, \quad \varphi \equiv \widetilde{M}\,.$$

We take $\Gamma_- = \gamma_{-,1}(\cdot;\lambda)$ and $\Gamma_+ = \gamma_{+,1}(\cdot;\lambda)$, for $\lambda > 0$ sufficiently large. □

4.1.4 Degree theory and conclusion of the proof of Theorem 14

We define the sets

$$U_1 = \big\{(t, x, y) \in \mathbb{R}^3 \,:\, \hat{\alpha} < x < \hat{\beta}\,,\, \Gamma_-(x) < y < \Gamma_+(x)\big\}\,,$$
$$U_2 = \big\{(t, x, y) \in \mathbb{R}^3 \,:\, \hat{\alpha} < x < \beta(t)\,,\, \Gamma_-(x) < y < \Gamma_+(x)\big\}\,,$$
$$U_3 = \big\{(t, x, y) \in \mathbb{R}^3 \,:\, \alpha(t) < x < \hat{\beta}\,,\, \Gamma_-(x) < y < \Gamma_+(x)\big\}\,.$$

Notice that $U_2 \cup U_3 \subseteq U_1$. Correspondingly, define the sets

$$\begin{aligned}
\mathcal{U}_1 &= \{u = (x,y) \in \mathcal{C}^0_T \,:\, (t,u(t)) \in U_1 \text{ for every } t \in [0,T]\}\,,\\
\mathcal{U}_2 &= \{u = (x,y) \in \mathcal{C}^0_T \,:\, (t,u(t)) \in U_2 \text{ for every } t \in [0,T]\}\,,\\
\mathcal{U}_3 &= \{u = (x,y) \in \mathcal{C}^0_T \,:\, (t,u(t)) \in U_3 \text{ for every } t \in [0,T]\}\,,\\
\mathcal{U}_4 &= \mathcal{U}_1 \setminus (\overline{\mathcal{U}}_2 \cup \overline{\mathcal{U}}_3)\,.
\end{aligned}$$

The last set can also be written as

$$\begin{aligned}
\mathcal{U}_4 = \{u = (x,y) \in \mathcal{C}^0_T \,:\, &(t,u(t)) \in U_1 \text{ for every } t \in [0,T] \text{ and}\\
&\text{there exist } t_1,t_2 \in [0,T] \text{ such that } x(t_1) < \alpha(t_1) \text{ and } x(t_2) > \beta(t_2)\}\,.
\end{aligned}$$

So, $\mathcal{U}_4 \subseteq \mathcal{S}$, the set $\mathcal{S}$ being defined in (56).

With the notation introduced in Remark 13, the sets $\mathcal{U}_i$, with $i \in \{1,2,3\}$, can be written as

$$\mathcal{U}_1 = \mathcal{V}(\hat\alpha,\hat\beta,\Gamma_\pm)\,,\quad \mathcal{U}_2 = \mathcal{V}(\hat\alpha,\beta,\Gamma_\pm)\,,\quad \mathcal{U}_3 = \mathcal{V}(\alpha,\hat\beta,\Gamma_\pm)\,.$$

Remark 27. *The validity of Lemma 26 forbids the possibility of finding a solution $u = (x,y)$ of $(\widetilde{P})$ belonging to $\overline{\mathcal{U}}_j$, with $j \in \{1,2,3\}$, satisfying $y(t_0) = \Gamma_\pm(x(t_0))$ at a certain time $t_0 \in [0,T]$. Indeed, we would have $(t,u(t)) \notin U_j$ in a right neighborhood of t_0, since $\pm\frac{d}{dt}(y - \Gamma_\pm(x))(t_0) > 0$.*

We now prove that there are no solutions of $(\widetilde{P})$ in $\partial\mathcal{U}_1$, i.e., if $u \in \overline{\mathcal{U}}_1$ solves $(\widetilde{P})$ then $u \in \mathcal{U}_1$. Assume that $x(t) \geq \hat\alpha$ for every $t \in [0,T]$ and there exists t_0 such that $x(t_0) = \hat\alpha$. Then $\hat\alpha \leq x(t) < -\ell_x - 1$ in a neighborhood of t_0, where $x'(t) = \tilde f(t,x(t),y(t)) = y(t)$, so that

$$x''(t_0) = y'(t_0) = \tilde g(t_0,\hat\alpha,y(t_0)) = -1 < 0\,,$$

providing a contradiction. Similarly, the situation when $x(t) \leq \hat\beta$ for every $t \in [0,T]$ and there exists t_0 such that $x(t_0) = \hat\beta$ cannot arise. Remark 27 completes the argument.

Since there are no solutions of $(\widetilde{P})$ in $\partial\mathcal{U}_1$, we can apply Corollary 12 and get

$$d_{\mathcal{L}}(\mathcal{L} - \widetilde{\mathcal{N}}, \mathcal{U}_1) = -1\,, \tag{63}$$

where $\widetilde{\mathcal{N}}$ is the Nemytskii operator associated to problem $(\widetilde{P})$, defined as in (24).

Assume the existence of a solution belonging to $\partial\mathcal{U}_2$. Then, recalling the above argument and Remark 27, we have

$$\hat\alpha < x(t) \leq \beta(t)\,,\quad \Gamma_-(x) < y(t) < \Gamma_+(x)\,,\quad \text{for every } t \in [0,T]\,,$$

and there exists a $t_0 \in [0,T]$ such that $x(t_0) = \beta(t_0)$. So, such a solution belongs to $\mathcal{S}$, with $t_2 = t_0$ and $t_1 = \hat t_0$, where $\hat t_0$ was defined in (27).

Similarly, if we assume the existence of a solution belonging to $\partial\mathcal{U}_3$, then we have necessarily

$$\alpha(t) \leq x(t) < \hat\beta\,,\quad \Gamma_-(x) < y(t) < \Gamma_+(x)\,,\quad \text{for every } t \in [0,T]\,,$$

and there exists a $t_0 \in [0,T]$ such that $x(t_0) = \alpha(t_0)$. So, such a solution belongs to $\mathcal{S}$, with $t_1 = t_0$ and $t_2 = \hat t_0$.

If at least one of the previous two situations arises, then we have found the solution we are looking for, and the proof of Theorem 14 is concluded. Otherwise, we are in the hypotheses of Corollary 12, which provides

$$d_{\mathcal{L}}(\mathcal{L}-\widetilde{\mathcal{N}},\mathcal{U}_2) = -1 \quad \text{and} \quad d_{\mathcal{L}}(\mathcal{L}-\widetilde{\mathcal{N}},\mathcal{U}_3) = -1\,. \tag{64}$$

Then, from (63) and (64), by the excision property,

$$d_{\mathcal{L}}(\mathcal{L}-\widetilde{\mathcal{N}},\mathcal{U}_4) = d_{\mathcal{L}}(\mathcal{L}-\widetilde{\mathcal{N}},\mathcal{U}_1) - \Big(d_{\mathcal{L}}(\mathcal{L}-\widetilde{\mathcal{N}},\mathcal{U}_2) + d_{\mathcal{L}}(\mathcal{L}-\widetilde{\mathcal{N}},\mathcal{U}_3)\Big) = 1\,,$$

and we thus find a solution of $(\widetilde{P})$ belonging to $\mathcal{U}_4 \subseteq \mathcal{S}$. The proof of Theorem 14 is thus completed, recalling Lemmas 23 and 21, in this order.

5 Further generalizations and applications

The inequalities in (10) and (11) can be reversed, and we can restate Theorem 6 as follows.

Theorem 28. *Assume the existence of a well-ordered couple (α, β) of lower/upper solutions of problem (P). Set $A = \min\alpha\,, B = \max\beta$, with $A < B$. Let there exist two continuously differentiable functions $\gamma_\pm : [A, B] \to \mathbb{R}$ such that*

$$\gamma_-(x) < \inf_{[0,T]}\{y_\alpha(t^-), y_\beta(t^+)\} \le \sup_{[0,T]}\{y_\alpha(t^+), y_\beta(t^-)\} < \gamma_+(x)\,,$$

with the following property:

$$\text{either} \quad g(t,x,\gamma_-(x)) < f(t,x,\gamma_-(x))\gamma_-'(x)\,, \quad \forall t\in\mathbb{R}\,, \forall x \in [\alpha(t),\beta(t)]\,, \tag{65}$$

$$\text{or} \quad g(t,x,\gamma_-(x)) > f(t,x,\gamma_-(x))\gamma_-'(x)\,, \quad \forall t\in\mathbb{R}\,, \forall x \in [\alpha(t),\beta(t)]\,; \tag{66}$$

and

$$\text{either} \quad g(t,x,\gamma_+(x)) > f(t,x,\gamma_+(x))\gamma_+'(x)\,, \quad \forall t\in\mathbb{R}\,, \forall x \in [\alpha(t),\beta(t)]\,, \tag{67}$$

$$\text{or} \quad g(t,x,\gamma_+(x)) < f(t,x,\gamma_+(x))\gamma_+'(x)\,, \quad \forall t\in\mathbb{R}\,, \forall x \in [\alpha(t),\beta(t)]\,. \tag{68}$$

Then there exists at least one solution of problem (P) such that

$$\alpha(t) \le x(t) \le \beta(t) \qquad \text{and} \qquad \gamma_-(x(t)) < y(t) < \gamma_+(x(t))\,,$$

for every $t \in \mathbb{R}$.

In this statement we allow the additional situations (66) and (68). Similar conditions were given, e.g., in [1]. The proof of Theorem 28 needs minor changes with respect to the one of Theorem 6. For example, if we assume the validity of (66) and (68) instead of (65) and (67), in the proof of Theorem 6 we simply need to modify the function $\widehat{\Phi}$ as follows:

$$\widehat{\Phi}(t,x,y)=\begin{cases}(M_X,-M_Y)\,, & \text{if } y\ge D\,,\\ \Phi(t,x,\gamma_+(x))+\dfrac{y-\gamma_+(x)}{D-\gamma_+(x)}\Big((M_X,-M_Y)-\Phi(t,x,\gamma_+(x))\Big), & \\ & \text{if } \gamma_+(x)\le y\le D\,,\\ \Phi(t,x,y)\,, & \text{if } \gamma_-(x)\le y\le\gamma_+(x)\,,\\ \Phi(t,x,\gamma_-(x))-\dfrac{y-\gamma_-(x)}{D+\gamma_-(x)}\Big((-M_X,M_Y)-\Phi(t,x,\gamma_-(x))\Big), & \\ & \text{if } -D\le y\le\gamma_-(x)\,,\\ (-M_X,M_Y)\,, & \text{if } y\le -D\,.\end{cases}$$

In general, the definition of $\widehat{\Phi}$ for $y \geq D$ is related to the choice (65) vs. (66), while its definition for $y \leq -D$ is related to the choice (67) vs. (68). The proof of Lemma 11 can be adapted to all the possible settings of Theorem 28.

Concerning Theorem 14, hypothesis (H2) can be similarly modified as follows.

(H2′) There exist two continuous functions $\gamma_\pm : [A,B]\times[1,+\infty[\,\to\mathbb{R}$, continuously differentiable with respect to the first variable, such that

$$\lim_{\lambda\to+\infty}\gamma_\pm(x;\lambda)=\pm\infty\,,\text{uniformly with respect to } x\in[A,B]\,,$$

with the following property

$$\begin{aligned}\text{either}\quad & g\big(t,x,\gamma_-(x;\lambda)\big) < f\big(t,x,\gamma_-(x;\lambda)\big)\gamma_-'(x;\lambda)\,,\\ & \qquad\qquad \forall t\in\mathbb{R}\,,\ \forall x\in[A,B]\,,\ \forall\lambda\in[1,+\infty[\,,\\ \text{or}\quad & g\big(t,x,\gamma_-(x;\lambda)\big) > f\big(t,x,\gamma_-(x;\lambda)\big)\gamma_-'(x;\lambda)\,,\\ & \qquad\qquad \forall t\in\mathbb{R}\,,\ \forall x\in[A,B]\,,\ \forall\lambda\in[1,+\infty[\,,\end{aligned}$$

and

$$\begin{aligned}\text{either}\quad & g\big(t,x,\gamma_+(x;\lambda)\big) > f\big(t,x,\gamma_+(x;\lambda)\big)\gamma_+'(x;\lambda)\,,\\ & \qquad\qquad \forall t\in\mathbb{R}\,,\ \forall x\in[A,B]\,,\ \forall\lambda\in[1,+\infty[\,,\\ \text{or}\quad & g\big(t,x,\gamma_+(x;\lambda)\big) > f\big(t,x,\gamma_+(x;\lambda)\big)\gamma_+'(x;\lambda)\,,\\ & \qquad\qquad \forall t\in\mathbb{R}\,,\ \forall x\in[A,B]\,,\ \forall\lambda\in[1,+\infty[\,.\end{aligned}$$

Assuming (H2′) instead of (H2) in Theorem 14 we get the same conclusion. However, some steps of the proof need some wise adjustments. In particular, the small changes due in the definition of the function $\widehat{\Phi}$, some lines above, in the setting of Theorem 28, can be proposed again similarly for the function $\widehat{\Phi}$ introduced in the proof of Theorem 14. Indeed, in the proof of Lemma 23, the estimates in (59) must provide a different sign. Then, in the second part of the same proof we need to *go back in time*: instead of (61), the following definition is in order

$$t_1 := \sup\{t\in\,]t_0-T,t_0[\ :\ (t,u(t))\notin N_\Lambda\}\,.$$

A similar reasoning in the interval $[t_0-T,t_0]$ can be performed. We omit to enter in major details for briefness.

A further extension of our results could lead to systems in $\mathbb{R}^{2N}$ of the type (P), with $f, g : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$. The case $f(t,x,y) = y$ and $g(t,x,y) = g(t,x)$ has been treated in [7], where also an infinite-dimensional system of the type $x'' = g(t,x)$ has been proposed. However, in the non-well-ordered case, the existence of *strict* lower and upper solutions was needed there. We believe that a similar procedure could be undertaken also in the more general framework of system (P). The notion of *strict* lower and upper solutions would probably be the one introduced in [8]; going back to the Introduction, one would need the strict inequality in (ii) and condition (iii) would also be necessary.

We prefer not to enter further into this discussion here.

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