Non-well-ordered lower and upper solutions for semilinear systems of PDEs

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Abstract. We prove existence results for systems of boundary value problems involving elliptic second order differential operators. The assumptions involve lower and upper solutions, which may be either well-ordered, or not at all. The results are stated in an abstract framework, and can be translated also for systems of parabolic type.

1 Introduction

We consider a boundary value problem for a system of the type

$$\begin{cases} \mathcal{L}u_n = F_n(x, u_1, \dots, u_M) & \text{in } \Omega\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad n = 1, \dots, M\,.$$

Here, Ω is a regular bounded domain in $\mathbb{R}^N$, and the differential operator $\mathcal{L} : W^{2,r}(\Omega) \to L^r(\Omega)$ is of elliptic type:

$$(\mathcal{L}w)(x) = -\sum_{l,m=1}^{N} a_{lm}(x)\partial^2_{x_l x_m} w(x) + \sum_{i=1}^{N} a_i(x)\partial_{x_i} w(x) + a_0(x)w(x)\,,$$

with $a_i \in L^\infty(\Omega)$, for $i = 0, \dots, N$ and $a_{lm} \in C(\overline{\Omega})$, $a_{lm} = a_{ml}$, for $l, m = 1, \dots, N$, with the assumption that there exists $\bar{a} > 0$ such that

$$\sum_{l,m=1}^{N} a_{lm}(x)\xi_l\xi_m \ge \bar{a}\|\xi\|^2\,, \quad \text{for every } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^N.$$

We may assume without loss of generality that $a_0 \ge 0$. We take $r > N$, so that $W^{2,r}(\Omega)$ is compactly imbedded into $C^1(\overline{\Omega})$. The function $F : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ is assumed to be L^r-Carathéodory. Concerning the boundary operator $\mathcal{B} : C^1(\overline{\Omega}) \to C(\partial\Omega)$, assume that $\partial\Omega$ is the disjoint union of two closed sets Γ_1 and Γ_2 (the cases $\Gamma_1 = \varnothing$ or $\Gamma_2 = \varnothing$ are admitted), and take

$$\mathcal{B}w := \begin{cases} w & \text{on } \Gamma_1\,, \\ \displaystyle\sum_{i=1}^{N} b_i(x)\partial_{x_i} w + b_0(x)w & \text{on } \Gamma_2\,. \end{cases}$$

Here $b_i \in C^1(\partial\Omega)$, for $i = 0, \dots, N$, and there exists $\bar{b} > 0$ such that

$$b_0(x) \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} b_i(x)\nu_i(x) \geq \bar{b}\,, \quad \text{for every } x \in \partial\Omega\,.$$

The vector $\nu(x) = (\nu_1(x), \dots \nu_N(x))$ is the unit outer normal to Ω at $x \in \partial\Omega$. The boundary condition on Γ_1 is the (homogeneous) Dirichlet condition and the one on Γ_2 is the (non-homogeneous) regular oblique derivative condition.

We want to ensure the existence of a solution of problem (P), i.e., a function $u \in W^{2,r}(\Omega)$ satisfying the differential equation almost everywhere in Ω and the boundary condition pointwise. A function with these properties is usually called "strong solution" in the literature. This will be done by introducing the concepts of lower and upper solutions in this setting.

For simplicity we do not consider nonlinearities depending on ∇u, but our results can be adapted to such a situation, by adding a Nagumo type assumption. Moreover, our choice of taking the same differential operator and boundary conditions for all components has the same aim of simplifying the exposition, and our arguments are also suited to a more general setting.

We will follow a semi-abstract approach like the one in [10], which has the advantage of clarifying the main features needed in order to obtain the existence result. In this way, slight modifications lead to similar results for different problems. For example, differential operators of parabolic type may also be considered, assuming different types of boundary conditions, like in [8, 10].

The theory of lower and upper solutions for scalar equations has a long history (see [5] and the large bibliography therein). In particular, concerning the problem of non-well-ordered lower and upper solutions, we refer to [3, 6, 12, 13, 14]. An abstract approach to the theory of lower and upper solutions has also been proposed in [1, 2]. Fewer results are known for systems. We refer to [15, Chapter 8] for systems of elliptic or parabolic equations, where some type of monotonicity is assumed in order to get the existence results.

The paper is organized as follows.

In Section 2 we introduce the abstract setting, and we provide an existence result in the case of well-ordered lower and upper solutions. This is the analogue, in the setting of PDEs, of a result obtained in [4] for periodic systems of ODEs.

In Section 3 we consider non-well-ordered lower and upper solutions and we state our main theorem, whose proof is provided in Section 5. We emphasize that we do not need any monotonicity assumptions on our nonlinearities.

In Section 4 we give some illustrative examples of applications, and comment on some possible extensions of our result in different directions.

Finally, in Section 6 we show how to adapt our main theorem to systems involving differential operators of parabolic type.

2 Well-ordered lower and upper solutions

2.1 The abstract setting

Let Ω be a bounded domain in $\mathbb{R}^N$, and denote by $W(\Omega)$ a Banach space of real-valued functions which is continuously and compactly imbedded in $C^1(\overline{\Omega})$. Assume that $\mathcal{L} : W(\Omega) \to L^r(\Omega)$ is a linear operator, with $r > 1$, and $\mathcal{B} : C^1(\overline{\Omega}) \to C(\partial\Omega)$ is a linear and continuous operator. We are concerned with the boundary value problem

$$(P) \qquad \begin{cases} \mathcal{L}u_n = F_n(x, u_1, \dots, u_M) & \text{in } \Omega\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad n = 1, \dots, M\,.$$

The function $F : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ is L^r-Carathéodory, i.e.,

(i) $F(\cdot, u)$ is measurable in Ω, for every $u \in \mathbb{R}^M$;

(ii) $F(x, \cdot)$ is continuous in $\mathbb{R}^M$, for almost every $x \in \Omega$;

(iii) for every $\rho > 0$ there is a $h_\rho \in L^r(\Omega)$ such that, if $|u| \le \rho$, then

$$|F(x, u)| \le h_\rho(x) \quad \text{for a.e. } x \in \Omega\,.$$

We now introduce our abstract assumptions.

Assumption 1. *If $w \in W(\Omega)$ is such that*

$$\min_{\overline{\Omega}} w < 0\,, \quad \text{and} \quad \mathcal{B}w \ge 0\,,$$

then there is a point $x_0 \in \overline{\Omega}$ with the following properties:

a) $w(x_0) < 0$,

b) there is no neighborhood U of x_0 such that $(\mathcal{L}w)(x) > 0$, for almost every $x \in U \cap \Omega$.

Remark 1. *Concerning the elliptic operator, we take $W(\Omega) = W^{2,r}(\Omega)$ with $r > N$. Assumption 1 is a consequence of the Strong Maximum Principle (see, e.g., [10, 11, 16]).*

Let us introduce the subspaces

$$C^1_{\mathcal{B}}(\overline{\Omega}) = \{w \in C^1(\overline{\Omega}) : \mathcal{B}w = 0\}\,, \qquad W_{\mathcal{B}}(\Omega) = \{w \in W(\Omega) : \mathcal{B}w = 0\}\,,$$

endowed with the norms in $C^1(\overline{\Omega})$ and $W(\Omega)$, respectively. These are Banach spaces, since the operator $\mathcal{B}$ is assumed to be linear and continuous. We will denote by $\mathcal{L}_{\mathcal{B}} : W_{\mathcal{B}}(\Omega) \to L^r(\Omega)$ the restriction of $\mathcal{L}$ to $W_{\mathcal{B}}(\Omega)$.

Assumption 2. *There is a* $\sigma < 0$ *such that* $\mathcal{L}_{\mathcal{B}} - \sigma I : W_{\mathcal{B}}(\Omega) \to L^r(\Omega)$ *is invertible and the operator*

$$(\mathcal{L}_{\mathcal{B}} - \sigma I)^{-1} : L^r(\Omega) \to W_{\mathcal{B}}(\Omega)$$

is continuous. Here, I *denotes the identity operator.*

Remark 2. *For the elliptic operator, any constant* $\sigma < 0$ *can be taken.*

We will write any $u \in [W_{\mathcal{B}}(\Omega)]^M$ as $u = (u_1, \dots, u_M)$. Let $L : [W_{\mathcal{B}}(\Omega)]^M \to [L^r(\Omega)]^M$ be defined as

$$(Lu)(x) = \big((\mathcal{L}_{\mathcal{B}}u_1)(x), \dots, (\mathcal{L}_{\mathcal{B}}u_M)(x)\big) ,$$

and let us introduce the nonlinear operator $N : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [L^r(\Omega)]^M$, defined by

$$(Nu)(x) = F(x, u(x)) .$$

It is readily seen that N is continuous and maps bounded sets into bounded sets. Our problem (P) can then be rewritten as

$$Lu = Nu .$$

A solution of problem (P) will be a function $u \in [W_{\mathcal{B}}(\Omega)]^M$ which satisfies this equality in $[L^r(\Omega)]^M$, hence almost everywhere.

If σ is the number given by Assumption 2, problem (P) is equivalent to the fixed point problem

$$u = \mathcal{S}u ,$$

where the operator

$$\mathcal{S} : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [C^1_{\mathcal{B}}(\overline{\Omega})]^M$$

is defined by

$$\mathcal{S}u = (L - \sigma I)^{-1}(Nu - \sigma u) .$$

Since $(L - \sigma I)^{-1} : [L^r(\Omega)]^M \to [W(\Omega)]^M$ is continuous and $[W(\Omega)]^M$ is compactly imbedded in $[C^1(\overline{\Omega})]^M$, we have that $\mathcal{S}$ is completely continuous, so that we can use Leray-Schauder degree theory.

2.2 An existence result

Let us introduce the concept of lower and upper solutions.

Definition 3. *Given two functions* $\alpha, \beta \in [W(\Omega)]^M$*, we say that* (α, β) *is a* well-ordered pair of lower/upper solutions *of (P) if* $\alpha \leq \beta$ *and there exists a negligible set* $\mathcal{N} \subseteq \Omega$ *such that*

$$\begin{cases} \mathcal{L}\alpha_j(x) \leq F_j(x, u_1, \dots, u_{j-1}, \alpha_j(x), u_{j+1}, \dots, u_M) , \\ \mathcal{L}\beta_j(x) \geq F_j(x, u_1, \dots, u_{j-1}, \beta_j(x), u_{j+1}, \dots, u_M) , \\ \mathcal{B}\alpha_j \leq 0 \leq \mathcal{B}\beta_j , \end{cases}$$

for every $j \in \{1, \dots, M\}$ *and* $(x, u) \in (\Omega \setminus \mathcal{N}) \times \prod_{m=1}^M [\alpha_m(x), \beta_m(x)]$.

Here is our result, in the well-ordered case; it generalizes [4, Theorem 4.1].

Theorem 4. *Let Assumptions 1 and 2 hold true. If there exists a well-ordered pair of lower/upper solutions (α, β), then problem (P) has a solution u such that $\alpha \leq u \leq \beta$.*

Proof Let us define the functions

$$\gamma_j(x,s) = \begin{cases} \alpha_j(x) & \text{if } s \leq \alpha_j(x)\,, \\ s & \text{if } \alpha_j(x) < s < \beta_j(x)\,, \\ \beta_j(x) & \text{if } u \geq \beta_j(x)\,, \end{cases}$$

and the function

$$\Gamma(x,u) = \big(\gamma_1(x,u_1), \dots, \gamma_M(x,u_M)\big)\,.$$

Consider the auxiliary problem

$$(\bar{P}) \quad \begin{cases} \mathcal{L}u_j - \sigma u_j = F_j(x, \Gamma(x,u)) - \sigma\gamma_j(x,u_j) & \text{in } \Omega\,, \\ \mathcal{B}u_j = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad j = 1, \dots, M\,.$$

The remaining part of the proof is divided in two steps.

Step 1: Problem $(\bar{P})$ admits a solution.

Let us introduce the operator $\bar{N} : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [L^r(\Omega)]^M$ defined by

$$(\bar{N}u)(x) = F(x, \Gamma(x, u(x))) - \sigma\Gamma(x, u(x))\,.$$

One can see that $\bar{N}$ is continuous and has a bounded image. Problem $(\bar{P})$ is equivalent to the fixed point problem

$$u = \overline{\mathcal{S}}u\,,$$

where the operator $\overline{\mathcal{S}} : [C^1_{\mathcal{B}}(\overline{\Omega})]^M \to [C^1_{\mathcal{B}}(\overline{\Omega})]^M$ is defined by

$$\overline{\mathcal{S}}u = (L - \sigma I)^{-1}\bar{N}u\,.$$

We have that $\overline{\mathcal{S}}$ is completely continuous, and its image is bounded. By Schauder Theorem, it has a fixed point, hence problem $(\bar{P})$ has a solution.

Step 2: Every solution u of $(\bar{P})$ is such that $\alpha \leq u \leq \beta$.

Let us prove that $\alpha \leq u$. Set $v = u - \alpha$, and assume by contradiction that $\min v_j < 0$, for some $j \in \{1, \dots, M\}$. Since $\mathcal{B}v_j = \mathcal{B}u_j - \mathcal{B}\alpha_j = -\mathcal{B}\alpha_j \geq 0$, by Assumption 1 there is a point $x_0 \in \overline{\Omega}$ such that $v_j(x_0) < 0$, and there is no neighborhood U of x_0 such that $\mathcal{L}v_j > 0$, almost everywhere on $U \cap \Omega$. On the other hand, as $v_j(x_0) < 0$, there is a neighborhood V of x_0 such that $v_j < 0$ on $V \cap \Omega$, i.e., $u_j < \alpha_j$ on $V \cap \Omega$. Hence,

$$\begin{aligned}
\mathcal{L}v_j &= \mathcal{L}u_j - \mathcal{L}\alpha_j \\
&= F_j(x, \Gamma(x,u)) - \sigma(\gamma_j(x,u_j) - u_j) - \mathcal{L}\alpha_j \\
&= F_j(x, \big(\gamma_1(x,u_1), \dots, \alpha_j(x), \dots, \gamma_M(x,u_M)\big) - \sigma(\alpha_j - u_j) - \mathcal{L}\alpha_j \\
&\geq \sigma v_j > 0\,,
\end{aligned}$$

almost everywhere on $V \cap \Omega$, a contradiction. In a similar way it can be shown that $u \leq \beta$.

Hence, every solution u of $(\bar{P})$ solves (P), and the proof is completed. $\square$

2.3 Computation of the degree

In the following, for any two continuous real-valued functions v, w, we write

$$v < w \quad \Leftrightarrow \quad v(x) < w(x)\,, \quad \text{for every } x \in \overline{\Omega}\,.$$

Let us consider the sets

$$C^1_{\mathcal{B}^-}(\overline{\Omega}) = \{w \in C^1(\overline{\Omega}) : \mathcal{B}w \leq 0\}\,, \quad C^1_{\mathcal{B}^+}(\overline{\Omega}) = \{w \in C^1(\overline{\Omega}) : \mathcal{B}w \geq 0\}\,,$$

endowed with the norm in $C^1(\overline{\Omega})$.

We now introduce a further assumption.

Assumption 3. *A relation $v \ll w$ is defined in $C^1(\overline{\Omega})$, with the following properties:*

$$\begin{array}{l}
v < w \quad \Rightarrow \quad v \ll w \quad \Rightarrow \quad v \leq w\,, \\
[\, v \leq w \quad \textit{and} \quad w \ll z \,] \quad \Rightarrow \quad v \ll z\,, \\
[\, v \ll w \quad \textit{and} \quad w \leq z \,] \quad \Rightarrow \quad v \ll z\,, \\
v \ll w \quad \Rightarrow \quad v + z \ll w + z\,, \\
[\, c > 0 \quad \textit{and} \quad v \ll w \,] \quad \Rightarrow \quad cv \ll cw\,,
\end{array}$$

for every $v, w, z \in C^1(\overline{\Omega})$) and every real constant c. Sometimes, we will write $w \gg v$ instead of $v \ll w$. Moreover, we assume that the set

$$\{w \in C^1_{\mathcal{B}^-}(\overline{\Omega}) : w \ll 0\}$$

is open in $C^1_{\mathcal{B}^-}(\overline{\Omega})$ or, equivalently, that the set $\{w \in C^1_{\mathcal{B}^+}(\overline{\Omega}) : w \gg 0\}$ is open in $C^1_{\mathcal{B}^+}(\overline{\Omega})$.

Notice indeed that $w \gg 0$ if and only if $-w \ll 0$. As a consequence of Assumption 3, one has that the sets

$$\{w \in C^1_{\mathcal{B}}(\overline{\Omega}) : w \ll 0\} \quad \text{and} \quad \{w \in C^1_{\mathcal{B}}(\overline{\Omega}) : w \gg 0\}$$

are open in $C^1_{\mathcal{B}}(\overline{\Omega})$. Notice also that the closures of these sets are contained in

$$\{w \in C^1_{\mathcal{B}}(\overline{\Omega}) : w \leq 0\} \quad \text{and} \quad \{w \in C^1_{\mathcal{B}}(\overline{\Omega}) : w \geq 0\},$$

respectively.

Remark 5. *For the system with the elliptic operator, we will write $v \ll w$ if the following two conditions hold:*

a) for every $x \in \Omega$, $v(x) < w(x)$,

b) for every $x \in \partial\Omega$, either $v(x) < w(x)$, or

$$v(x) = w(x) \quad \text{and} \quad \partial_\nu v(x) > \partial_\nu w(x).$$

Here, ν denotes the outer unit normal to $\partial\Omega$ at the point x.

If the two functions v, w have values in $\mathbb{R}^d$, for any dimension d, then we write

$$\begin{cases} v \leq w & \Leftrightarrow \quad v_m \leq w_m, \\ v \ll w & \Leftrightarrow \quad v_m \ll w_m, \end{cases} \qquad \text{for every } m \in \{1, \ldots, d\}.$$

Definition 6. *A well-ordered pair of lower/upper solutions (α, β) is said to be* strict *if $\alpha \ll \beta$, and any solution u of (P) satisfying $\alpha \leq u \leq \beta$ is such that*

$$\alpha \ll u \ll \beta.$$

If (α, β) is strict, then the set

$$\mathcal{U}_{(\alpha,\beta)} = \{u \in [C^1_{\mathcal{B}}(\overline{\Omega})]^M : \alpha \ll u \ll \beta\}$$

is open in $[C^1_{\mathcal{B}}(\overline{\Omega})]^M$, by Assumption 3. Moreover, if u is a fixed point of $\mathcal{S}$ in $\overline{\mathcal{U}}_{(\alpha,\beta)}$, then $\alpha \leq u \leq \beta$ and, by the strictness hypothesis, $u \in \mathcal{U}_{(\alpha,\beta)}$. So, there are no fixed points of $\mathcal{S}$ on the boundary of $\mathcal{U}_{(\alpha,\beta)}$, and we can define the Leray-Schauder degree

$$\deg\left(I - \mathcal{S}, \mathcal{U}_{(\alpha,\beta)}\right).$$

Theorem 7. *Let Assumptions 1, 2 and 3 hold true. If there exists a strict well-ordered pair of lower/upper solutions (α, β), then*

$$\deg\left(I - \mathcal{S}, \mathcal{U}_{(\alpha,\beta)}\right) = 1.$$

Proof Going back to the proof of Theorem 4, any fixed point u of $\overline{\mathcal{S}}$ is such that $\alpha \leq u \leq \beta$, and it is a fixed point of $\mathcal{S}$. Hence, all fixed points of $\overline{\mathcal{S}}$ belong to $\mathcal{U}_{(\alpha,\beta)}$, and since $\mathcal{S}$ and $\overline{\mathcal{S}}$ coincide on $\mathcal{U}_{(\alpha,\beta)}$, we have

$$\deg\left(I - \mathcal{S}, \mathcal{U}_{(\alpha,\beta)}\right) = \deg\left(I - \overline{\mathcal{S}}, \mathcal{U}_{(\alpha,\beta)}\right).$$

By Schauder Theorem and the excision property of the degree, taking $R > 0$ large enough, we have

$$\deg\left(I - \overline{\mathcal{S}}, \mathcal{U}_{(\alpha,\beta)}\right) = \deg\left(I - \overline{\mathcal{S}}, B(0, R)\right) = 1,$$

thus ending the proof. $\square$

3 Non-well-ordered lower and upper solutions

We will say that the couple $(\mathcal{J},\mathcal{K})$ is a partition of the set of indices $\{1,\dots,M\}$ if and only if $\mathcal{J}\cap\mathcal{K}=\varnothing$ and $\mathcal{J}\cup\mathcal{K}=\{1,\dots,M\}$. Correspondingly we can decompose a vector $u=(u_1,\dots,u_M)\in\mathbb{R}^M$ as $u=(u_{\mathcal{J}},u_{\mathcal{K}})$ where $u_{\mathcal{J}}=(u_j)_{j\in\mathcal{J}}\in\mathbb{R}^{\#\mathcal{J}}$ and $u_{\mathcal{K}}=(u_k)_{k\in\mathcal{K}}\in\mathbb{R}^{\#\mathcal{K}}$. Here $\#\mathcal{J}$ and $\#\mathcal{K}$ denote respectively the cardinality of the sets $\mathcal{J}$ and $\mathcal{K}$. Also every function $\mathcal{F}:\mathcal{A}\to\mathbb{R}^M$ can be written as $\mathcal{F}(x)=\big(\mathcal{F}_{\mathcal{J}}(x),\mathcal{F}_{\mathcal{K}}(x)\big)$ where $\mathcal{F}_{\mathcal{J}}:\mathcal{A}\to\mathbb{R}^{\#\mathcal{J}}$ and $\mathcal{F}_{\mathcal{K}}:\mathcal{A}\to\mathbb{R}^{\#\mathcal{K}}$.

Definition 8. *Given two functions $\alpha,\beta\in[W(\Omega)]^M$, we say that (α,β) is a* pair of lower/upper solutions of (P) related to the partition $(\mathcal{J},\mathcal{K})$ of $\{1,\dots,M\}$ *if the following four conditions hold:*

1. $\alpha_{\mathcal{J}}\le\beta_{\mathcal{J}}$;

2. $\alpha_k\not\le\beta_k$, *for every* $k\in\mathcal{K}$;

3. *there is a negligible set $\mathcal{N}\subseteq\Omega$ such that*

$$\begin{cases}\mathcal{L}\alpha_n(x)\le F_n(x,u_1,\dots,u_{n-1},\alpha_n(x),u_{n+1},\dots,u_M)\,,\\ \mathcal{L}\beta_n(x)\ge F_n(x,u_1,\dots,u_{n-1},\beta_n(x),u_{n+1},\dots,u_M)\,,\end{cases}$$

   for any $n\in\{1,\dots,M\}$ and every $(x,u)\in\mathcal{E}$, where

$$\mathcal{E}:=\Big\{(x,u)\in(\Omega\setminus\mathcal{N})\times\mathbb{R}^M\,:\,u=(u_{\mathcal{J}},u_{\mathcal{K}})\,,u_{\mathcal{J}}\in\prod_{j\in\mathcal{J}}[\alpha_j(x),\beta_j(x)]\Big\}\,.$$

4. $\mathcal{B}\alpha_n\le0\le\mathcal{B}\beta_n$, *for every* $n\in\{1,\dots,M\}$.

Definition 9. *The pair (α,β) of lower/upper solutions of (P) is said to be* strict with respect to the $\mathcal{J}$-th component *if $\alpha_{\mathcal{J}}\ll\beta_{\mathcal{J}}$ and, for every solution u of (P) we have*

$$\alpha_{\mathcal{J}}\le u_{\mathcal{J}}\le\beta_{\mathcal{J}}\quad\Rightarrow\quad\alpha_{\mathcal{J}}\ll u_{\mathcal{J}}\ll\beta_{\mathcal{J}}\,;$$

it is said to be strict with respect to the k-th component, *with $k\in\mathcal{K}$, if for every solution u of (P) we have*

$$\begin{aligned}u_k\ge\alpha_k\quad&\Rightarrow\quad u_k\gg\alpha_k\,,\\ u_k\le\beta_k\quad&\Rightarrow\quad u_k\ll\beta_k\,.\end{aligned}$$

We need to introduce some further assumptions.

Assumption 4. *There is a number $\lambda_1\ge0$ and a function $\varphi_1\in W_{\mathcal{B}}(\Omega)$, with $\varphi_1\gg0$, such that*

$$\ker(\mathcal{L}_{\mathcal{B}}-\lambda_1 I)=\{c\varphi_1:c\in\mathbb{R}\}\,.$$

We will assume that $\max_{\overline{\Omega}}\varphi_1=1$.

Remark 10. *The existence of a "first" eigenvalue* λ_1 *with the required properties is standard in the elliptic case, where the spectrum is made of isolated eigenvalues* $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$, *all contained in* $[0 + \infty[$, *cf. [11, 16].*

Lemma 11. *Let Assumptions 3 and 4 hold. Given a bounded set* $\mathcal{A}$ *in* $W(\Omega)$, *there is a constant* $C_{\mathcal{A}} \geq 0$ *such that, if* $w \in \mathcal{A}$ *satisfies* $\mathcal{B}w \leq 0$, *then* $w \leq C_{\mathcal{A}}\varphi_1$, *and if* $w \in \mathcal{A}$ *satisfies* $\mathcal{B}w \geq 0$, *then* $w \geq -C_{\mathcal{A}}\varphi_1$.

Proof See [10, Lemma 4.1]. □

Definition 12. *A pair of functions* $(\psi, \Psi) \in L^r(\Omega) \times L^r(\Omega)$ *is said to be* admissible *if it satisfies* $\psi \leq \lambda_1 \leq \Psi$ *almost everywhere in* Ω *and, for every* $q \in L^r(\Omega)$, *with* $\psi \leq q \leq \Psi$ *almost everywhere in* Ω, *if* w *is a solution of*

$$(P_{lin}) \qquad \begin{cases} \mathcal{L}w = q(z)w & \text{in } \Omega\,, \\ \mathcal{B}w = 0 & \text{on } \partial\Omega\,, \end{cases}$$

then, either $w = 0$, *or* $w \ll 0$, *or* $w \gg 0$.

Remark 13. *For a self-adjoint elliptic problem, the above property of the couple* ψ, Ψ *is satisfied, e.g., if* $\Psi \leq \lambda_2$ *(the second eigenvalue), with strict inequality on a subset of positive measure (cf. [13]).*

Lemma 14. *Let Assumptions 2, 3 and 4 hold. Given an admissible pair of functions* (ψ, Ψ), *there are two positive constants* $c_{\psi,\Psi}$ *and* $C_{\psi,\Psi}$ *such that, for every* $q \in L^r(\Omega)$, *with* $\psi \leq q \leq \Psi$ *almost everywhere in* Ω, *if* u *is a solution of* (P_{lin}), *then*

$$c_{\psi,\Psi}\|u\|_{L^\infty}\varphi_1 \leq |u| \leq C_{\psi,\Psi}\|u\|_{L^\infty}\varphi_1\,.$$

Proof See [10, Lemma 4.3]. □

Assumption 5. *There is a function* $\varphi_0 \in W(\Omega)$ *such that*

$$\mu := \min_{\overline{\Omega}} \varphi_0 > 0\,, \qquad \mathcal{L}\varphi_0 \geq 0 \quad \text{and} \quad \mathcal{B}\varphi_0 \geq 0\,.$$

We will assume that $\max_{\overline{\Omega}} \varphi_0 = 1$.

Remark 15. *In the applications to the elliptic case the function* φ_0 *can be taken constantly equal to* 1.

Here is the main result of this paper.

Theorem 16. *Let Assumptions 1–5 hold true. Let* (α, β) *be a pair of lower/upper solutions of (P) related to the partition* $(\mathcal{J}, \mathcal{K})$ *of* $\{1, \dots, M\}$ *which is strict with respect to the* k*-th component, for every* $k \in \mathcal{K}$, *except at most one.*

Assume that there exist two L^r-Carathéodory functions $f, g : \Omega \times \mathbb{R}^M \to \mathbb{R}^M$ with the following property: for every $k \in \mathcal{K}$,

$$F_k(x,u) = g_k(x,u)u_k + f_k(x,u)\,,$$

and there is an admissible pair (ψ_k, Ψ_k), and a function $h_k \in L^r(\Omega)$ such that

$$\psi_k(x) \le g_k(x,u) \le \Psi_k(x) \quad \text{and} \quad |f_k(x,u)| \le h_k(x)\,,$$

for almost every $x \in \Omega$ and every $u \in \mathbb{R}^M$. Then, problem (P) has a solution u such that

$(W_{\mathcal{J}})$ $\alpha_{\mathcal{J}} \le u_{\mathcal{J}} \le \beta_{\mathcal{J}}$ *;*

$(NW_{\mathcal{K}})$ $\alpha_k \not\leqslant\!\!\!\leqslant u_k$ *and* $u_k \not\leqslant\!\!\!\leqslant \beta_k$ *, for every* $k \in \mathcal{K}$ *.*

4 Examples and remarks

As an illustrative example, consider the Neumann problem

$$\begin{cases} -\Delta u_1 = |u_1|^\gamma \sin u_1 + w_1(x,u_1,u_2) & \text{in } \Omega\,, \\ -\Delta u_2 = \pm \arctan u_2 + w_2(x,u_1,u_2) & \text{in } \Omega\,, \\ \quad \partial_\nu u_1 = \partial_\nu u_2 = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Here γ is any positive exponent, and $w_1, w_2 : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and bounded functions, with

$$\|w_2\|_\infty := \sup\big\{|w_2(x,u_1,u_2)| : x \in \overline{\Omega}, u_1, u_2 \in \mathbb{R}\big\} < \frac{\pi}{2}\,.$$

Applying Theorem 16, we obtain the existence of infinitely many solutions $u = (u_1, u_2)$. Indeed, it is sufficient to choose the constant pairs of lower/upper solutions (α, β), with

$$\alpha = \Big(\frac{\pi}{2} + 2m\pi, \pm n\Big)\,, \quad \beta = \Big(\frac{3\pi}{2} + 2m\pi, \mp n\Big)\,,$$

for sufficiently large positive integers m, n. Notice that these will be well-ordered if the minus sign appears in the second differential equation, otherwise non-well-ordered in the second component.

As a second example, we consider the mixed Dirichlet–Neumann problem

$$\begin{cases} -\Delta u_1 = -u_1^3 + f_1(u_2) + w_1(x,u_1,u_2,u_3) & \text{in } \Omega\,, \\ -\Delta u_2 = -\arctan u_2 + w_2(x,u_1,u_2,u_3) & \text{in } \Omega\,, \\ -\Delta u_3 = \arctan u_3 + w_3(x,u_1,u_2,u_3) & \text{in } \Omega\,, \\ \quad u_1 = 0\,, \quad \partial_\nu u_2 = \partial_\nu u_3 = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Here $f_1 : \mathbb{R} \to \mathbb{R}$ is any continuous function, and $w_1, w_2, w_3 : \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ are continuous and bounded, with

$$\|w_2\|_\infty < \frac{\pi}{2}\,, \quad \|w_3\|_\infty < \frac{\pi}{2}\,.$$

Applying Theorem 16, we obtain the existence of at least one solution, taking the constant pairs of lower/upper solutions (α, β), with

$$\alpha = \Big(-m, -n, n \Big)\,, \quad \beta = \Big(m, n, -n \Big)\,,$$

for a sufficiently large positive integer n and $m = m(n)$. Indeed, it is sufficient to fix $n > \tan(\max\{\|w_2\|_\infty, \|w_3\|_\infty\})$ and

$$m > \Big(\max\big\{ |f_1(s)| : s \in [-n, n] \big\} + \|w_1\|_\infty \Big)^{1/3}.$$

Remark 17. *All the results of this paper hold if the nonlinearities depend also on the gradient ∇u, provided that a Nagumo-type condition is assumed. See [10] for the details.*

Remark 18. *Asymmetric nonlinearities can also be considered, as in [13, 6, 10]. We do not enter into details, for briefness.*

Remark 19. *Concerning a system with a p-Laplacian differential operator, some difficulties may arise. If we consider, e.g., the associated Dirichlet problem, then the inverse function $(L - \sigma I)^{-1}$ transforms any $h \in L^\infty(\Omega)$ into $(L - \sigma I)^{-1} h \in W_0^{1,p}(\Omega) \cap C^{1,\nu}(\overline{\Omega})$, for some $\nu > 0$, and this function might not have regular second order derivatives. In [6], this problem is overcome by defining lower and upper solutions in a weak form, and carrying out the same construction as for the linear case. A similar procedure can also be practiced in our situation, leading to an existence result analogous to Theorem 16.*

Remark 20. *The periodic problem for a system of ordinary differential equations has been treated in [9]. Infinite-dimensional systems were also considered there. It is an open problem whether it could be possible to extend the results of the present paper to an infinite-dimensional setting.*

5 Proof of Theorem 16

Notice that the case $\mathcal{K} = \varnothing$ reduces to Theorem 4. We thus assume $\mathcal{K} \neq \varnothing$ and, without loss of generality, we take either $\mathcal{J} = \varnothing$, or $\mathcal{J} = \{1, \ldots, J\}$ and $\mathcal{K} = \{J + 1, \ldots, M\}$ for a certain $J \in \{1, \ldots, N\}$. We moreover suppose that the component on which the lower/upper solution is possibly not strict is the last one, i.e., $k = M$. Indeed, mixing the coordinates we can always reduce to such a situation. We continue the proof in the case $\mathcal{J} \neq \varnothing$. (The case $\mathcal{J} = \varnothing$ can be treated essentially in the same way.)

We need to suitably modify problem (P). For $j = 1, \ldots, J$ we define

$$G_j(x,u) = F_j(x, \gamma_1(x,u_1), \ldots, \gamma_J(x,u_J), u_{J+1}, \ldots, u_M) + u_j - \sigma\gamma_j(x,u_j)\,,$$

where the functions γ_j are the ones introduced in the proof of Theorem 4.

Using Lemma 11 and the fact that α_k and β_k are bounded, we can find a constant $c > 0$ such that, for $k \in \mathcal{K}$,

$$-c\varphi_1 - c \le \alpha_k \le c\varphi_1\,, \quad -c\varphi_1 \le \beta_k \le c\varphi_1 + c\,.$$

For any $k \in \mathcal{K}$ and $\Lambda > 0$ large enough, to be fixed, we define

$$\tilde g_k(x,u) = \begin{cases} \lambda_1 & \text{if } \ u_k \le -(\Lambda\varphi_1(x) + \frac{2c}{\mu}\varphi_0(x))\,, \\ \ldots & \\ g_k(x,u) & \text{if } \ |u_k| \le \Lambda\varphi_1(x) + \frac{c}{\mu}\varphi_0(x)\,, \\ \ldots & \\ \lambda_1 & \text{if } \ u_k \ge \Lambda\varphi_1(x) + \frac{2c}{\mu}\varphi_0(x)\,, \end{cases}$$

$$\tilde f_k(x,u) = \begin{cases} \frac{3c\lambda_1+1}{\mu} & \text{if } \ u_k \le -(\Lambda\varphi_1(x) + \frac{2c}{\mu}\varphi_0(x))\,, \\ \ldots & \\ f_k(x,u) & \text{if } \ |u_k| \le \Lambda\varphi_1(x) + \frac{c}{\mu}\varphi_0(x)\,, \\ \ldots & \\ -\frac{3c\lambda_1+1}{\mu} & \text{if } \ u_k \ge \Lambda\varphi_1(x) + \frac{2c}{\mu}\varphi_0(x)\,, \end{cases}$$

(here, the dots mean "linear interpolation"), and

$$G_k(x,u) = \tilde g_k(x,u)u_k + \tilde f_k(x,u)\,.$$

We consider the problem

$$(\widetilde P_\Lambda) \qquad \begin{cases} \mathcal{L}u_n = G_n(x, u_1, \ldots, u_M) & \text{in } \Omega\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad n = 1, \ldots, M\,.$$

Proposition 21. *If u is a solution of $(\widetilde P_\Lambda)$, for any constant $\Lambda > 0$, then $\alpha_{\mathcal{J}} \le u_{\mathcal{J}} \le \beta_{\mathcal{J}}$.*

Proof It is easily adapted from Step 2 of the proof of Theorem 4. □

We define $\tilde\alpha_{\mathcal{K}}$ and $\tilde\beta_{\mathcal{K}}$ by setting $\tilde\alpha_k = -(\Lambda\varphi_1 + \frac{3c}{\mu}\varphi_0)$ and $\tilde\beta_k = \Lambda\varphi_1 + \frac{3c}{\mu}\varphi_0$, for every $k \in \mathcal{K}$. Notice that, taking $\Lambda > c$,

$$\tilde\alpha_{\mathcal{K}} \ll \alpha_{\mathcal{K}} \ll \tilde\beta_{\mathcal{K}}\,, \qquad \tilde\alpha_{\mathcal{K}} \ll \beta_{\mathcal{K}} \ll \tilde\beta_{\mathcal{K}}\,.$$

Finally, we choose $\tilde\alpha = (\alpha_{\mathcal{J}}, \tilde\alpha_{\mathcal{K}})$ and $\tilde\beta = (\beta_{\mathcal{J}}, \tilde\beta_{\mathcal{K}})$.

Let us prove that $(\tilde{\alpha}, \tilde{\beta})$ is pair of lower/upper solutions of $(\widetilde{P}_\Lambda)$. We have not modified the components of $\tilde{\alpha}_\mathcal{J}$ and $\tilde{\beta}_\mathcal{J}$, so we just need to check what happens for $\tilde{\alpha}_\mathcal{K}$ and $\tilde{\beta}_\mathcal{K}$. For every $k \in \mathcal{K}$ we have

$$\mathcal{L}\tilde{\alpha}_k(x) \leq -\Lambda\lambda_1\varphi_1(x) = \lambda_1\tilde{\alpha}_k(x) + \frac{3c\lambda_1}{\mu}\varphi_0(x) < \tilde{g}_k(x, \tilde{\alpha}(x))\tilde{\alpha}_k(x) + \tilde{f}_k(x, \tilde{\alpha}(x)),$$

and $\mathcal{B}\tilde{\alpha}_k = -\frac{3c}{\mu}\mathcal{B}\varphi_0 \leq 0$. Similar computations can be done for $\tilde{\beta}_k$. So, $(\tilde{\alpha}, \tilde{\beta})$ is pair of lower/upper solutions for $(\widetilde{P}_\Lambda)$.

Let us prove that $(\tilde{\alpha}, \tilde{\beta})$ is strict with respect to the k-th component, for every $k \in \mathcal{K}$. Let u be a solution such that $u_k \geq \tilde{\alpha}_k$. We want to show that $u_k > \tilde{\alpha}_k$. By contradiction, let $v_k = u_k - \tilde{\alpha}_k$ be such that $\min v_k = 0$. Let $z_k = v_k - \frac{c}{\mu}\varphi_0$, so that $\min z_k < 0$ and

$$\mathcal{B}z_k = \mathcal{B}u_k - \mathcal{B}\tilde{\alpha}_k - \frac{c}{\mu}\mathcal{B}\varphi_0 = \frac{2c}{\mu}\mathcal{B}\varphi_0 \geq 0.$$

By Assumption 1 there is a $x_0 \in \Omega$ such that $z_k(x_0) < 0$ and there is no neighborhood U of x_0 on which $(\mathcal{L}z_k)(x) > 0$, for almost every $x \in U \cap \Omega$. By continuity, there is a neighborhood V of x_0 on which $z_k < 0$. So, on V, we have that $u_k < \tilde{\alpha}_k + \frac{c}{\mu}\varphi_0$, and so

$$\begin{aligned}
\mathcal{L}z_k &= \mathcal{L}u_k - \mathcal{L}\tilde{\alpha}_k - \frac{c}{\mu}\mathcal{L}\varphi_0 \\
&= \Big(\lambda_1 u_k + \frac{3c\lambda_1 + 1}{\mu}\Big) + \Lambda\lambda_1\varphi_1 + \frac{2c}{\mu}\mathcal{L}\varphi_0 \\
&\geq \Big(\lambda_1\tilde{\alpha}_k + \frac{3c\lambda_1 + 1}{\mu}\Big) + \Lambda\lambda_1\varphi_1 \\
&= -\frac{3c\lambda_1}{\mu}\varphi_0 + \frac{3c\lambda_1 + 1}{\mu} > 0,
\end{aligned}$$

a contradiction. Similar estimates can be written for $\tilde{\beta}_k$, so we conclude that the pair of lower/upper solutions is strict with respect to the k-th component.

Let $\mathcal{X}$ be the subset of $[W(\Omega)]^M$ made of those solutions u of $(\widetilde{P}_\Lambda)$, for any $\Lambda > c$, satisfying $\alpha_\mathcal{K} \not\ll u_\mathcal{K}$ and $u_\mathcal{K} \not\ll \beta_\mathcal{K}$.

Claim. There exists a constant $C_1 > 0$ (independent of Λ) such that, for any $u \in \mathcal{X}$, one has $|u_k| \leq C_1\varphi_1$, for every $k \in \mathcal{K}$.

Proof of the Claim. We first prove that there is a constant $K > 0$ (independent of Λ) such that, for any $u \in \mathcal{X}$, one has $\|u_k\|_\infty \leq K$, for every $k \in \mathcal{K}$. By contradiction, let $(u^n)^n$ be a sequence in $\mathcal{X}$, such that $\|u_k^n\|_\infty \to \infty$, for some $k \in \mathcal{K}$. Let us now fix such $k \in \mathcal{K}$.

We know that u^n is a solution of $(\widetilde{P}_{\Lambda_n})$, for some $\Lambda_n > c$. Let us denote by $\tilde{g}_k^n$ and $\tilde{f}_k^n$ the corresponding modified functions. Then $w_k^n = u_k^n/\|u_k^n\|_\infty$ satisfies

$$\mathcal{L}w_k^n(x) = \tilde{g}_k^n(x, u^n(x))w_k^n(x) + \frac{1}{\|u_k^n\|_\infty}\tilde{f}_k^n(x, u^n(x)), \quad \mathcal{B}w_k^n = 0\,.$$

Let us consider the set of functions

$$\mathcal{D}_k = \{p \in L^r(\Omega) : \psi_k(x) \le p(x) \le \Psi_k(x), \quad \text{for a.e. } x \in \Omega\},$$

which is bounded, closed and convex, hence weakly compact. Since the sequence $(\tilde{g}_k^n(\cdot, u^n(\cdot)))_n$ belongs to $\mathcal{D}_k$, up to a subsequence it weakly converges in $L^r(\Omega)$ to some $q(\cdot) \in \mathcal{D}_k$, while

$$\frac{1}{\|u_k^n\|_\infty}\tilde{f}_k^n(x, u^n(x)) \to 0 \quad \text{in } L^r(\Omega)\,.$$

Let $\widetilde{N}_k^n : C^1_{\mathcal{B}}(\overline{\Omega}) \to L^r(\Omega)$ be defined as

$$(\widetilde{N}_k^n w)(x) = \tilde{g}_k^n(x, u^n(x))w(x) + \frac{1}{\|u_k^n\|_\infty}\tilde{f}_k^n(x, u^n(x))\,.$$

Let $\sigma \in \mathbb{R}$ be the number given by Assumption 2, and let $\widetilde{\mathcal{S}}_k^n : C^1_{\mathcal{B}}(\overline{\Omega}) \to C^1_{\mathcal{B}}(\overline{\Omega})$ be defined as

$$\widetilde{\mathcal{S}}_k^n v = (L - \sigma I)^{-1}(\widetilde{N}_k^n v - \sigma v)\,.$$

Notice that

$$w_k^n = \widetilde{\mathcal{S}}_k^n w_k^n\,.$$

Since $(\widetilde{N}_k^n w_k^n - \sigma w_k^n)_n$ is bounded in $L^r(\Omega)$ and $(L - \sigma I)^{-1} : L^r(\Omega) \to C^1_{\mathcal{B}}(\overline{\Omega})$ is compact, there is a $w \in C^1_{\mathcal{B}}(\overline{\Omega})$ such that, up to a subsequence,

$$\widetilde{\mathcal{S}}_k^n w_k^n = (L - \sigma I)^{-1}(\tilde{N}_k^n w_k^n - \sigma w_k^n) \to w \quad \text{in } C^1_{\mathcal{B}}(\overline{\Omega})\,.$$

Hence, $w_k^n \to w$ in $C^1_{\mathcal{B}}(\overline{\Omega})$. Since $\tilde{N}_k^n w_k^n - \sigma w_k^n$ weakly converges to $q(\cdot)w - \sigma w$, we conclude that

$$w = (L - \sigma I)^{-1}(q(\cdot)w - \sigma w)\,,$$

so that $w \in W_{\mathcal{B}}(\Omega)$ and

$$Lw = q(\cdot)w\,,$$

i.e., w satisfies (P_{lin}). Then, either $w = 0$, or $w \ll 0$, or $w \gg 0$. Since $\|w_k^n\|_\infty = 1$, for every n, we know that $w \neq 0$. Assume for instance $w \gg 0$ (the case $w \ll 0$ is similar). By Lemma 11, there is a constant $\hat{c}_k > 0$ such that $\alpha_k \le \hat{c}_k\varphi_1$. By Lemma 14, $w \ge c_{\psi_k,\Psi_k}\varphi_1 \gg (c_{\psi_k,\Psi_k}/2)\varphi_1$, since $\varphi_1 \gg 0$, by Assumption 2. So,

$$w \gg \frac{c_{\psi_k,\Psi_k}}{2}\varphi_1 \ge \frac{c_{\psi_k,\Psi_k}}{2\hat{c}_k}\alpha_k := b_k\alpha_k\,.$$

By Assumption 3, for n large enough, $w_k^n \gg b_k\alpha_k$, and increasing n still more, $u_k^n = \|u_k^n\|_\infty w_k^n \gg \|u_k^n\|_\infty b_k\alpha_k \ge \alpha_k$, a contradiction.

We have thus seen that $\mathcal{X}_\mathcal{K}$, the projection of the set $\mathcal{X}$ on the $\mathcal{K}$-th component, is uniformly bounded. Now recall that problem $(\widetilde{P}_\Lambda)$ is equivalent to a fixed point problem

$$u = (L - \sigma I)^{-1}(\widetilde{N}u - \sigma u) .$$

By Assumption 2, we deduce that $\mathcal{X}_\mathcal{K}$ is indeed bounded in $[W(\Omega)]^{\sharp\mathcal{K}}$. Then, by Lemma 11, we find a constant $C_1 > 0$ such that $|u_k| \le C_1\varphi_1$, for every $k \in \mathcal{K}$. The proof of the Claim is thus completed. □

From now on, we fix $\Lambda \ge C_1$. We are going to compute the Leray-Schauder degree of $I - \mathcal{F}$ on a family of open sets, where

$$\mathcal{F} : [C^1_\mathcal{B}(\overline{\Omega})]^M \to [C^1_\mathcal{B}(\overline{\Omega})]^M , \qquad \mathcal{F}(u) = (L - \sigma I)^{-1}(\widetilde{N}u - \sigma u) .$$

Let us define the functions

$$\breve{\alpha}_j = \alpha_j - \varphi_0 , \quad \text{and} \quad \breve{\beta}_j = \beta_j + \varphi_0 ,$$

for every $j \in \mathcal{J}$.

We need to introduce a multi-index $\vec{\eta} = (\eta_{J+1}, \dots, \eta_M) \in \{1, 2, 3\}^{M-J}$, in order to define the open sets

$$\Omega_{\vec{\eta}} := \left\{u \in [C^1_\mathcal{B}(\overline{\Omega})]^M \;:\; \breve{\alpha}_\mathcal{J} \ll u_\mathcal{J} \ll \breve{\beta}_\mathcal{J} \text{ and } (\mathcal{O}^{\eta_k}_k) \text{ holds for every } k \in \mathcal{K}\right\} ,$$

where

$(\mathcal{O}^1_k)$ $\tilde{\alpha}_k \ll u_k \ll \tilde{\beta}_k$,

$(\mathcal{O}^2_k)$ $\tilde{\alpha}_k \ll u_k \ll \beta_k$,

$(\mathcal{O}^3_k)$ $\alpha_k \ll u_k \ll \tilde{\beta}_k$.

We now end the proof of Thoerem 16 assuming first that the lower/upper solutions are *strict with respect to all the components* $k \in \mathcal{K}$.

Proposition 22. *For every multi-index $\vec{\eta}$, the degree $d(I - \mathcal{F}, \Omega_{\vec{\eta}})$ is well-defined, and*

$$d(I - \mathcal{F}, \Omega_{\vec{\eta}}) = 1 .$$

Proof Assume by contradiction that there is $u \in \partial\Omega_{\vec{\eta}}$ such that $(I - \mathcal{F})u = 0$, i.e., u is a solution of $(\widetilde{P}_\Lambda)$. All the several different situations which may arise lead back to the following four cases.

Case A. For some index $j \in \mathcal{J}$, $\breve{\alpha}_j \le u_j \le \breve{\beta}_j$, and either $\breve{\alpha}_j \not\ll u_j$, or $u_j \not\ll \breve{\beta}_j$. We have seen in the proof of Theorem 4 that $\alpha_j \le u_j \le \beta_j$. Since $\varphi_0 > 0$, by Assumption 5, we then have $\breve{\alpha}_j < u_j < \breve{\beta}_j$, hence $\breve{\alpha}_j \ll u_j \ll \breve{\beta}_j$, a contradiction.

Case B. For some index $k \in \mathcal{K}$, $\tilde{\alpha}_k \leq u_k \leq \tilde{\beta}_k$, and either $\tilde{\alpha}_k \not\ll u_k$, or $u_k \not\ll \tilde{\beta}_k$. This is impossible, since $(\tilde{\alpha}, \tilde{\beta})$ is strict with respect to the k-th component, for every $k \in \mathcal{K}$.

Case C. For some index $k \in \mathcal{K}$, $\tilde{\alpha}_k \ll u_k \leq \beta_k$, and $u_k \not\ll \beta_k$. Such a situation cannot arise, by assumption.

Case D. For some index $k \in \mathcal{K}$, $\alpha_k \leq u_k \ll \tilde{\beta}_k$, and $\alpha_k \not\ll u_k$. Such a situation cannot arise, by assumption.

Since the sets $\Omega_{\vec{\eta}}$ provide us a well-ordered pair of strict lower/upper solutions of problem $(\widetilde{P}_\Lambda)$, the conclusion is a consequence of Theorem 7. $\square$

We now start an iterative process, defining a series of open sets and computing the corresponding degrees. This process will eventually lead us to the conclusion.

For every $\ell \in \{1,2,3\}$ and any $\vec{\eta} = (\eta_{J+2}, \dots, \eta_M) \in \{1,2,3\}^{M-(J+1)}$, we now define the open sets

$$\Omega^0_{(\ell,\vec{\eta})} = \Omega_{(\ell,\eta_{J+2},\dots,\eta_M)} \,.$$

Notice that $\Omega^0_{(2,\vec{\eta})}$ and $\Omega^0_{(3,\vec{\eta})}$ are disjoint subsets of $\Omega^0_{(1,\vec{\eta})}$. We also define the open set

$$\Omega^0_{(4,\vec{\eta})} = \Omega^0_{(1,\vec{\eta})} \setminus \overline{\Omega^0_{(2,\vec{\eta})} \cup \Omega^0_{(3,\vec{\eta})}} \,.$$

Proposition 23. *For every multi-index $\vec{\eta}$, the degree $d(I - \mathcal{F}, \Omega^0_{(4,\vec{\eta})})$ is well-defined, and*

$$d(I - \mathcal{F}, \Omega^0_{(4,\vec{\eta})}) = -1 \,.$$

Proof Using the fact that the sets $\Omega^0_{(\ell,\vec{\eta})}$ are open, for $\ell = 1,2,3$, we can see that

$$\partial\Omega^0_{(4,\vec{\eta})} \subseteq \partial\Omega^0_{(1,\vec{\eta})} \cup \partial\Omega^0_{(2,\vec{\eta})} \cup \partial\Omega^0_{(3,\vec{\eta})} \,.$$

Since we already know that there are no solutions of $(\widetilde{P}_\Lambda)$ on $\partial\Omega^0_{(\ell,\vec{\eta})}$, for $\ell = 1,2,3$, we consequently have that there are no solutions of $(\widetilde{P}_\Lambda)$ on $\partial\Omega^0_{(4,\vec{\eta})}$, hence the degree is well-defined. By the additivity property of the degree and Proposition 22,

$$d(I - \mathcal{F}, \Omega^0_{(4,\vec{\eta})}) = d(I - \mathcal{F}, \Omega^0_{(1,\vec{\eta})}) - d(I - \mathcal{F}, \Omega^0_{(2,\vec{\eta})}) - d(I - \mathcal{F}, \Omega^0_{(3,\vec{\eta})}) = -1 \,,$$

so that the proof is completed. $\square$

Now, for every $\ell \in \{1,2,3\}$ and any $\vec{\eta} = (\eta_{J+3}, \dots, \eta_M) \in \{1,2,3\}^{M-(J+2)}$, we define the open sets

$$\Omega^1_{(\ell,\vec{\eta})} = \Omega_{(4,\ell,\eta_{J+3},\dots,\eta_M)} \,.$$

Notice that $\Omega^1_{(2,\vec{\eta})}$ and $\Omega^1_{(3,\vec{\eta})}$ are disjoint subsets of $\Omega^1_{(1,\vec{\eta})}$. We also define the open set

$$\Omega^1_{(4,\vec{\eta})} = \Omega^1_{(1,\vec{\eta})} \setminus \overline{\Omega^1_{(2,\vec{\eta})} \cup \Omega^1_{(3,\vec{\eta})}}\,.$$

Proceeding by induction, for $K \in \{0,1,\dots,M-(J+1)\}$, any $\ell \in \{1,2,3\}$ and any $\vec{\eta} = (\eta_{J+K+2},\dots,\eta_M) \in \{1,2,3\}^{M-(J+K+1)}$ we can define the open sets

$$\Omega^K_{(\ell,\vec{\eta})} = \Omega_{(\underbrace{4,\dots,4}_{K\,times},\ell,\eta_{J+K+2},\dots,\eta_M)}\,.$$

Notice that $\Omega^K_{(2,\vec{\eta})}$ and $\Omega^K_{(3,\vec{\eta})}$ are disjoint subsets of $\Omega^K_{(1,\vec{\eta})}$. We also define the open set

$$\Omega^K_{(4,\vec{\eta})} = \Omega^K_{(1,\vec{\eta})} \setminus \overline{\Omega^K_{(2,\vec{\eta})} \cup \Omega^K_{(3,\vec{\eta})}}\,.$$

Proposition 24. *For every $K \in \{0,1,\dots,M-(J+2)\}$ and every multi-index $\vec{\eta}$, the degree $d(I-\mathcal{F},\Omega^K_{(4,\vec{\eta})})$ is well-defined, and*

$$d(I-\mathcal{F},\Omega^K_{(4,\vec{\eta})}) = (-1)^{K+1}\,.$$

Proof We proceed by induction. The validity of the statement for $K=0$ follows by Proposition 23. Assume that it holds for some $K \in \{0,1,\dots,M-(J+3)\}$. The same argument in the proof of Proposition 23 shows us that the degree is well-defined. Then, for every $\vec{\eta} = (\eta_{J+K+3},\dots,\eta_M) \in \{1,2,3\}^{M-(J+K+2)}$,

$$\begin{aligned}
&d(I-\mathcal{F},\Omega^{K+1}_{(4,\vec{\eta})}) = d(I-\mathcal{F},\Omega^{K+1}_{(1,\vec{\eta})}) - d(I-\mathcal{F},\Omega^{K+1}_{(2,\vec{\eta})}) - d(I-\mathcal{F},\Omega^{K+1}_{(3,\vec{\eta})})\\
&= d(I-\mathcal{F},\Omega_{(\underbrace{4,\dots,4}_{K+1\,times},1,\eta_{J+K+3},\dots,\eta_M)})-\\
&\quad - d(I-\mathcal{F},\Omega_{(\underbrace{4,\dots,4}_{K+1\,times},2,\eta_{J+K+3},\dots,\eta_M)}) - d(I-\mathcal{F},\Omega_{(\underbrace{4,\dots,4}_{K+1\,times},3,\eta_{J+K+3},\dots,\eta_M)})\\
&= d(I-\mathcal{F},\Omega_{(\underbrace{4,\dots,4}_{K\,times},4,1,\eta_{J+K+3},\dots,\eta_M)})-\\
&\quad - d(I-\mathcal{F},\Omega_{(\underbrace{4,\dots,4}_{K\,times},4,2,\eta_{J+K+3},\dots,\eta_M)}) - d(I-\mathcal{F},\Omega_{(\underbrace{4,\dots,4}_{K\,times},4,3,\eta_{J+K+3},\dots,\eta_M)})\\
&= d(I-\mathcal{F},\Omega^K_{(4,1,\vec{\eta})}) - d(I-\mathcal{F},\Omega^K_{(4,2,\vec{\eta})}) - d(I-\mathcal{F},\Omega^K_{(4,3,\vec{\eta})})\\
&= (-1)^{K+1} - (-1)^{K+1} - (-1)^{K+1} = (-1)^{K+2}\,,
\end{aligned}$$

yielding the conclusion. □

By the previous proposition, in the special case $K = M-(J+2)$ we have that for every $\ell \in \{1,2,3\}$,

$$d_\ell := d(I-\mathcal{F},\Omega_{(\underbrace{4,\dots\dots,4}_{M-J-2\,times},4,\ell)}) = (-1)^{M-(J+1)}\,.$$

We now consider the set

$$\Omega_{(4,\dots,4,4)} = \Omega_{(4,\dots,4,1)} \setminus \overline{\Omega_{(4,\dots,4,2)} \cup \Omega_{(4,\dots,4,3)}}\,.$$

By the same argument as above,

$$d(I-\mathcal{F},\Omega_{(4,\dots,4)}) = d_1 - d_2 - d_3 = (-1)^{M-J}\,.$$

As a consequence, there exists a solution u of problem $(\widetilde{P}_\Lambda)$ in the set $\Omega_{(4,\dots,4)}$. Recalling the above a priori bounds, we see that the solution u is indeed a solution of problem (P) and satisfies $(W_{\mathcal{J}})$ and $(NW_{\mathcal{K}})$. The proof is thus completed, in the case when the lower/upper solutions are strict with respect to all the components $k \in \mathcal{K}$.

If the lower/upper solutions are not strict with respect to the M-th component, the previous propositions all continue to hold provided that $\eta_M = 1$, but we cannot ensure that the degree is well-defined if $\eta_M = 2$ or $\eta_M = 3$. We thus have that

$$d(I-\mathcal{F},\Omega_{(4,\dots,4,1)}) = (-1)^{M-(J+1)}\,,$$

and there are two possibilities: either, there is a solution of problem $(\widetilde{P}_\Lambda)$ on $\partial\Omega_{(4,\dots,4,2)} \cup \partial\Omega_{(4,\dots,4,3)}$, or the degrees $d(I-\mathcal{F},\Omega_{(4,\dots,4,2)})$ and $d(I-\mathcal{F},\Omega_{(4,\dots,4,3)})$ are well-defined, and we conclude as above. □

6 The parabolic case

In this section we briefly describe how our results can be adapted to the study of systems of parabolic type. For the details, we refer to [10, Section 7].

Let Ω be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. Given $T>0$, set $Q = \Omega\times\,]0,T[\,$. Taking $r > N+2$, we define the operator $\mathcal{L} : W^{2,1}_r(Q) \to L^r(Q)$ as follows:

$$\mathcal{L}w = \partial_t w - \sum_{l,m=1}^{N} a_{lm}(x,t)\partial^2_{x_l x_m} w + \sum_{i=1}^{N} a_i(x,t)\partial_{x_i} w + a_0(x,t)w\,.$$

Here $a_{lm} \in C(\overline{Q})$, $a_{lm} = a_{ml}$, $a_{lm}(x,0) = a_{lm}(x,T)$ in $\overline{\Omega}$, for $l,m = 1,\dots,N$, there exists $\bar a > 0$ such that

$$\sum_{l,m=1}^{N} a_{lm}(x,t)\xi_i\xi_j \geq \bar a\|\xi\|^2\,, \quad \text{for every } (x,t,\xi) \in \overline{Q}\times\mathbb{R}^N\,,$$

and $a_i \in L^\infty(Q)$, for $i = 0,\dots,N$.

Assume that $\partial\Omega$ is the disjoint union of two closed sets Γ_1 and Γ_2 (the cases $\Gamma_1 = \varnothing$ or $\Gamma_2 = \varnothing$ are admitted). Let τ_s be the operator defined by

$$(\tau_s w)(x,t) = w(x,t+s)\,,$$

and consider the boundary operator $\mathcal{B} : C^{1,0}(\overline{Q}) \to C(\partial Q)$ defined as

$$\mathcal{B}w := \begin{cases} w & \text{on } \Gamma_1 \times [0,T]\,, \\ \displaystyle\sum_{i=1}^{N} b_i(x,t)\partial_{x_i} w + b_0(x,t)w & \text{on } \Gamma_2 \times [0,T]\,, \\ w - \tau_T w & \text{in } \Omega \times \{0\}\,, \\ \tau_{(-T)} w - w & \text{in } \Omega \times \{T\}\,. \end{cases}$$

Here $b_i \in C^1(\partial\Omega \times [0,T])$, $b_i(x,0) = b_i(x,T)$ in $\partial\Omega$, for $i = 0, \dots, N$, and there exists $\bar{b} > 0$ such that

$$b_0(x,t) \geq 0 \quad \text{and} \quad \sum_{i=1}^{N} b_i(x,t)\nu_i(x) \geq \bar{b}\,, \quad \text{for every } (x,t) \in \partial\Omega \times]0,T[\,.$$

We thus have Dirichlet-periodic conditions on Γ_1, and Robin-periodic on Γ_2.

We can deal with the problem

$$\begin{cases} \mathcal{L}u_n = F_n(x,t,u_1,\dots,u_M) & \text{in } Q\,, \\ \mathcal{B}u_n = 0 & \text{on } \partial Q\,, \end{cases} \qquad n = 1,\dots,M\,.$$

Also in this setting our choice of taking the same differential operator and boundary conditions for all components has only the aim of simplifying the exposition. A *solution* of problem (P) is a function $u \in W_r^{2,1}(Q)$ which satisfies the differential equation almost everywhere in Q and the boundary conditions pointwise. A function with these properties is usually called “strong solution” in the literature. All the existence results of this paper can be adapted to this situation. See [10] for the verification of the corresponding Assumptions 1–5.

As a final example, we can consider the system of the mixed Dirichlet-periodic and Neumann-periodic problem

$$\begin{cases} \partial_t u_1 - \Delta u_1 = -u_1^3 + w_1(x,t,u_1,u_2) & \text{in } Q\,, \\ \partial_t u_2 - \Delta u_2 = \arctan u_2 + w_2(x,t,u_1,u_2) & \text{in } Q\,, \\ u_1 = 0\,, \quad \partial_\nu u_2 = 0 & \text{on } \partial\Omega \times [0,T]\,, \\ u_1(x,0) = u_1(x,T)\,, \quad u_2(x,0) = u_2(x,T) & \text{on } \Omega\,. \end{cases}$$

If $w_1, w_2 : \overline{Q} \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and bounded, with $\|w_2\|_\infty < \pi/2$, we obtain the existence of at least one solution, taking the constant pairs of lower/upper solutions (α, β), with $\alpha = (-m, n)$ and $\beta = (m, -n)$, for sufficiently large positive integers m and n.

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