

Multiple periodic solutions of infinite-dimensional pendulum-like equations

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Dedicated to Haïm Brezis, on the occasion of his 75th birthday

Abstract. We prove the multiplicity of periodic solutions for an equation in a separable Hilbert space H , with T -periodic dependence in time, of the type

$$\ddot{x} + \mathcal{A}x + \nabla_x V(t, x) = e(t).$$

Here, \mathcal{A} is a semi-negative definite bounded selfadjoint operator, with nontrivial null-space $\mathcal{N}(\mathcal{A})$, the function $V(t, x)$ is bounded above, periodic in x along a basis of $\mathcal{N}(\mathcal{A})$, with $\nabla_x V$ having its image in a compact set, and $e(t)$ has mean value in $\mathcal{N}(\mathcal{A})^\perp$. Our results generalize several well-known theorems in the finite-dimensional setting, as well as a recent existence result in [1].

1 Introduction

Motivated by the model of a periodically forced pendulum, the existence of *at least two* geometrically distinct T -periodic solutions for a scalar differential equation of the form

$$x'' + \partial_x V(t, x) = e(t)$$

was first proved in [16], using the direct method of the calculus of variations and the Mountain Pass Theorem, assuming $V(t, x)$ to be T -periodic with respect to t and τ -periodic with respect to x , and $e(t)$ to be T -periodic with zero mean, i.e.,

$$\int_0^T e(t) dt = 0. \tag{1}$$

This result extended an existence theorem first proved in [12], and later rediscovered independently in [5, 21].

Here, and in the sequel, for simplicity all functions will be assumed to be continuous. It can be seen that the multiplicity result in [16] is optimal if no further conditions are added. Different proofs have also been provided, e.g., in [8, 9, 11], by the use of some generalized versions of the Poincaré–Birkhoff theorem.

The result in [16] was later generalized in [17], through a similar approach, to the corresponding system in \mathbb{R}^N ,

$$x'' + \nabla_x V(t, x) = e(t), \quad (2)$$

when $V(t, x) = V(t, x_1, \dots, x_N)$ is T -periodic in t and τ_k -periodic in x_k , for every $k = 1, \dots, N$. The first aim of this paper is to extend such a result to an infinite-dimensional setting. So, let H be a separable Hilbert space, and let $(e_k)_{k \geq 1}$ be a Hilbert basis. We assume $V : \mathbb{R} \times H \rightarrow \mathbb{R}$ to be continuous, T -periodic with respect to its first variable t , and continuously differentiable with respect to its second variable x . Here is our result.

Theorem 1. *Assume that there exists a sequence of positive real numbers $(\tau_k)_{k \geq 1}$ such that*

$$V(t, x + \tau_k e_k) = V(t, x), \quad \text{for every } (t, x) \in [0, T] \times H \text{ and } k = 1, 2, \dots \quad (3)$$

If $\sum_{k=1}^{\infty} \tau_k^2 < +\infty$, then equation (2) has at least two geometrically distinct T -periodic solutions, for every $e(t)$ satisfying (1).

The above theorem generalizes [1, Theorem 6], where further regularity assumptions were made on V , in order to obtain the existence of *at least one* T -periodic solution. A Galerkin-type argument was used there to reduce the problem to a sequence of finite-dimensional differential systems, to which a generalized version of the Poincaré–Birkhoff theorem applies (cf. [9]), followed by a limit process.

The proof of Theorem 1 will follow the same ideas introduced in [16, 17], taking advantage of the compactness of the Hilbert cube $\prod_{k=1}^{\infty} [0, \tau_k]$. The first solution will be obtained by minimization of the action functional, while the second one will be of mountain pass type.

Using the Lusternik–Schnirelmann theory, it was proved in [15] that, under the same assumptions, system (2) in \mathbb{R}^N has indeed *at least* $N + 1$ geometrically distinct T -periodic solutions, thus generalizing the result in [17]. (Notice that, when $N \geq 2$, the multiplicity result is not optimal, as shown by the four equilibria of a double pendulum.) Even more, a system of the type

$$x'' + \mathcal{A}x + \nabla_x V(t, x) = e(t) \quad (4)$$

was considered there, involving a symmetric matrix \mathcal{A} . Other results in this direction, including the case of Hamiltonian systems leading to a strongly indefinite action functional, were studied, e.g., in [3, 4, 6, 7, 9, 10, 13, 14, 20].

The second aim of this paper is to obtain multiplicity results for an infinite-dimensional system of the type (4), when $\mathcal{A} : H \rightarrow H$ is a semi-negative definite bounded selfadjoint operator, whose spectrum contains 0 as an isolated eigenvalue, $V(t, x)$ is bounded above and T -periodic in t , and the image of $\nabla_x V$ is contained in a compact set of H . Denoting by $\mathcal{N}(\mathcal{A})$ the null space of \mathcal{A} , we distinguish two cases.

If $\mathcal{N}(\mathcal{A})$ has finite dimension N and $V(t, x)$ satisfies a periodicity condition of the type (3), with the e_k replaced by the elements of an orthonormal basis of $\mathcal{N}(\mathcal{A})$, the existence of *at least* $N+1$ geometrically distinct T -periodic solutions is proved, when the mean value of $e(t)$ belongs to $\mathcal{N}(\mathcal{A})^\perp$. The precise statement will be given in Section 2. The proof, provided in Section 3, will be carried out by the use of an abstract theorem, given in [18] and inspired by [19], providing the multiplicity of critical points of some functionals in a Banach space X which are bounded below, invariant under the action of some discrete subgroups of X , and satisfy a suitable Palais–Smale condition.

If $\mathcal{N}(\mathcal{A})$ has infinite dimension, assuming in addition that $\sum_{k=1}^{\infty} \tau_k^2 < +\infty$, after finding the first solution by minimization of the action functional, a second one is provided by the Mountain Pass Theorem. We thus get, in this case, the existence of *at least two* geometrically distinct T -periodic solutions.

The paper ends with an example and a corresponding open problem.

2 The main result

Let H be a separable Hilbert space, with scalar product (\cdot, \cdot) and corresponding norm $|\cdot|$. In this space, we consider the equation

$$\ddot{x} + \mathcal{A}x + \nabla_x V(t, x) = e(t), \quad (5)$$

where $\mathcal{A} \in \mathcal{L}(H)$ is a bounded selfadjoint operator, and $e : \mathbb{R} \rightarrow H$ is continuous and T -periodic. Concerning the function $V : \mathbb{R} \times H \rightarrow \mathbb{R}$, it is continuous, T -periodic in its first variable t , and differentiable with respect to its second variable x , with corresponding continuous gradient $\nabla_x V : \mathbb{R} \times H \rightarrow H$.

Let us introduce our assumptions. We denote by $\mathcal{N}(\mathcal{A})$ the null-space of \mathcal{A} , and by $\sigma(\mathcal{A})$ its spectrum. We take a Hilbert basis $(a_k)_k$ of $\mathcal{N}(\mathcal{A})$, considered as a subspace of H . If $\mathcal{N}(\mathcal{A})$ has a finite dimension, its basis will

be given by (a_1, \dots, a_N) ; if it is infinite-dimensional, we will have a sequence of vectors (a_1, a_2, \dots) .

A1. *The selfadjoint operator \mathcal{A} is semi-negative definite, with $\mathcal{N}(\mathcal{A}) \neq \{0\}$, and*

$$\sup(\sigma(\mathcal{A}) \setminus \{0\}) < 0.$$

So, 0 is an isolated point of $\sigma(\mathcal{A})$.

A2. *The mean value of $e(t)$ is orthogonal to $\mathcal{N}(\mathcal{A})$, i.e.,*

$$\int_0^T e(t) dt \in \mathcal{N}(\mathcal{A})^\perp.$$

Then, we have that

$$\int_0^T (e(t), a_k) dt = 0, \quad \text{for every } k = 1, 2, \dots$$

A3. *There exists a sequence of positive real numbers $(\tau_k)_{k \geq 1}$ such that*

$$V(t, x + \tau_k a_k) = V(t, x), \quad \text{for every } (t, x) \in [0, T] \times H \text{ and } k = 1, 2, \dots$$

A4. *There is a nonnegative constant C such that*

$$V(t, x) \leq C, \quad \text{for every } (t, x) \in [0, T] \times H.$$

A5. The set $\nabla_x V([0, T] \times H)$ is precompact in H .

In the above setting, we can now state the main result of this paper.

Theorem 2. *Assume that conditions A1 to A5 hold. If $\mathcal{N}(\mathcal{A})$ is finite-dimensional, then equation (5) has at least $\dim \mathcal{N}(\mathcal{A}) + 1$ geometrically distinct T -periodic solutions. On the other hand, if $\mathcal{N}(\mathcal{A})$ is infinite-dimensional and $\sum_{k=1}^{\infty} \tau_k^2 < +\infty$, then there are at least two of them.*

Notice that, once a T -periodic solution $x(t)$ has been found, any function obtained by adding to it some integer multiples of $\tau_k a_k$ is still a T -periodic solution. We say that two T -periodic solutions are *geometrically distinct* if they cannot be obtained one from the other in this way.

Concerning assumption A5, we remark that it will surely be satisfied if the following holds.

A5'. *There exists a Hilbert basis $(e_k)_{k \geq 1}$ and a nonnegative sequence $(M_k)_k$, with $\sum_{k=1}^{\infty} M_k^2 < +\infty$, such that*

$$\left| \frac{\partial V}{\partial e_k}(t, x) \right| \leq M_k, \quad \text{for every } (t, x) \in [0, T] \times H \text{ and } k = 1, 2, \dots$$

Indeed, A5' implies that $\nabla_x V([0, T] \times H)$ is contained in a Hilbert cube, which is a compact set in H .

Notice that Theorem 1 is a direct consequence of Theorem 2, taking $\mathcal{A} = 0$ and $(a_k)_k = (e_k)_k$, a Hilbert basis of H . Indeed, the periodicity assumption in Theorem 1 and the compactness of the set $[0, T] \times \prod_{k=1}^{\infty} [0, \tau_k]$ show that A4 and A5 are surely satisfied.

In the proof of Theorem 2, we will need a result from [18], which we now recall, for the reader's convenience.

Let G be a discrete subgroup of a Banach space X and $\pi : X \rightarrow X/G$ be the canonical surjection. A subset S of X is G -invariant if $\pi^{-1}(\pi(S)) = S$, and a function f defined on X is G -invariant if $f(u + g) = f(u)$, for every $u \in X$ and every $g \in G$. If $\varphi \in C^1(X, \mathbb{R})$ is G -invariant, then φ' is also G -invariant, and if u is a critical point of φ , the same is true for $u + g$ for all $g \in G$. The corresponding set $\{u + g : g \in G\}$ is called a critical orbit of φ .

A G -invariant differentiable function $\varphi : X \rightarrow \mathbb{R}$ satisfies the $(PS)_G$ condition if, for every sequence $(u_n)_n$ in X such that $\varphi(u_n)$ is bounded and $\varphi'(u_n) \rightarrow 0$, the sequence $(\pi(u_n))_n$ contains a convergent subsequence.

The following multiplicity result for the critical points of G -invariant functionals is stated as Theorem 4.12 in [18].

Theorem 3. *Let $\varphi \in C^1(X, \mathbb{R})$ be a G -invariant functional satisfying the $(PS)_G$ condition. If φ is bounded from below and if the dimension N of the space generated by G is finite, then φ has at least $N + 1$ critical orbits.*

3 Proof of Theorem 2

Let $L^2([0, T], H)$ be the space of measurable functions $x : [0, T] \rightarrow H$ such that $|x|$ is square integrable. It is a Hilbert space equipped with the scalar product

$$\langle x, y \rangle_2 = \int_0^T (x(t), y(t)) dt,$$

and corresponding norm

$$\|x\|_2 = \left(\int_0^T |x(t)|^2 dt \right)^{\frac{1}{2}}.$$

We consider the space $H^1([0, T], H)$, made of those functions x belonging to $L^2([0, T], H)$ with weak derivative \dot{x} also in $L^2([0, T], H)$. It is a Hilbert space, as well, with the scalar product

$$\langle x, y \rangle = \langle x, y \rangle_2 + \langle \dot{x}, \dot{y} \rangle_2 = \int_0^T [(x(t), y(t)) + (\dot{x}(t), \dot{y}(t))] dt,$$

and corresponding norm

$$\|x\| = (\|x\|_2^2 + \|\dot{x}\|_2^2)^{\frac{1}{2}} = \left(\int_0^T [|x(t)|^2 + |\dot{x}(t)|^2] dt \right)^{\frac{1}{2}}.$$

Moreover, $H^1([0, T], H)$ is continuously embedded in $C([0, T], H)$, the space of continuous functions, with the usual norm

$$\|x\|_\infty = \max\{|x(t)| : t \in [0, T]\}.$$

(For further information on the space $H^1([0, T], H)$ we refer, e.g., to [2].)

Let

$$H_T^1 = \{x \in H^1([0, T], H) : x(0) = x(T)\},$$

and define the functional $\varphi : H_T^1 \rightarrow \mathbb{R}$ as

$$\varphi(x) = \int_0^T \left[\frac{1}{2} |\dot{x}(t)|^2 - \frac{1}{2} (\mathcal{A}x(t), x(t)) - V(t, x(t)) + (e(t), x(t)) \right] dt.$$

It is continuously differentiable, and its critical points correspond to the T -periodic solutions of (5). Moreover, by A2 and A3,

$$\varphi(x + \tau_k a_k) = \varphi(x), \quad \text{for every } x \in H_T^1 \text{ and } k \geq 1. \quad (6)$$

As usual, we identify the constant functions with their constant value. So, having identified H with the space of constant functions, it will be a subspace of H_T^1 . Hence, we can write

$$H_T^1 = H \oplus W = \mathcal{N}(\mathcal{A}) \oplus \mathcal{N}(\mathcal{A})^\perp \oplus W = \mathcal{N}(\mathcal{A}) \oplus \widetilde{W}.$$

Here, W is the orthogonal space to H in H_T^1 , $\mathcal{N}(\mathcal{A})^\perp$ is the orthogonal to $\mathcal{N}(\mathcal{A})$ in H , and $\widetilde{W} = \mathcal{N}(\mathcal{A})^\perp \oplus W$. Correspondingly, we will write each $x \in H_T^1$ as $x(t) = \bar{x} + \tilde{x}(t)$, with $\bar{x} \in \mathcal{N}(\mathcal{A})$ and $\tilde{x} \in \widetilde{W}$. Moreover, we will write $\tilde{x}(t) = \hat{x} + \check{x}(t)$, with $\hat{x} \in \mathcal{N}(\mathcal{A})^\perp$ and $\check{x} \in W$. Notice that, for any $x \in H_T^1$,

$$[x] := \frac{1}{T} \int_0^T x(t) dt = \bar{x} + \hat{x}, \quad \frac{1}{T} \int_0^T \check{x}(t) dt = 0. \quad (7)$$

Proposition 4. For every $x \in H_T^1$, one has

$$\|\tilde{x}\|_\infty \leq \sqrt{T} \|\dot{x}\|_2. \quad (8)$$

Proof. Let $(e_k)_{k \geq 1}$ be a Hilbert basis of H . Then, for any function $x \in H_T^1$, we may write

$$\tilde{x}(t) = \sum_{k=1}^{\infty} (\tilde{x}(t), e_k) e_k = \sum_{k=1}^{\infty} \tilde{x}_k(t) e_k.$$

Being \tilde{x}_k continuous, T -periodic with zero mean, there is a $t_k \in [0, T]$ for which $\tilde{x}_k(t_k) = 0$, hence

$$|\tilde{x}_k(t)| = \left| \tilde{x}_k(t_k) + \int_{t_k}^t \dot{x}_k(s) ds \right| \leq \int_0^T |\dot{x}_k(s)| ds \leq \sqrt{T} \left(\int_0^T |\dot{x}_k(s)|^2 ds \right)^{\frac{1}{2}},$$

for every $t \in [0, T]$. As a consequence,

$$|\tilde{x}(t)|^2 = \sum_{k=1}^{\infty} |\tilde{x}_k(t)|^2 \leq T \int_0^T \sum_{k=1}^{\infty} |\dot{x}_k(s)|^2 ds = T \int_0^T |\dot{x}(s)|^2 ds,$$

for every $t \in [0, T]$, whence the conclusion. \square

By A1, A2, A4, (7) and (8), setting $\delta := -\sup(\sigma(\mathcal{A}) \setminus \{0\})$,

$$\begin{aligned} \varphi(x) &= \int_0^T \left[\frac{1}{2} |\dot{x}(t)|^2 - \frac{1}{2} (\mathcal{A}\tilde{x}(t), \tilde{x}(t)) - V(t, x(t)) + (e(t), \tilde{x}(t)) \right] dt \\ &\geq \int_0^T \left[\frac{1}{2} |\dot{x}(t)|^2 - \frac{1}{2} (\mathcal{A}\hat{x}, \hat{x}) - \frac{1}{2} (\mathcal{A}\tilde{x}(t), \tilde{x}(t)) \right] dt - CT - T\|e\|_\infty \|\tilde{x}\|_\infty \\ &\geq \int_0^T \frac{1}{2} |\dot{x}(t)|^2 dt - \frac{1}{2} T (\mathcal{A}\hat{x}, \hat{x}) - CT - T\|e\|_\infty (\|\hat{x}\| + \|\tilde{x}\|_\infty) \\ &\geq \frac{1}{2} \|\dot{x}\|_2^2 + \frac{1}{2} T \delta |\hat{x}|^2 - CT - T^{\frac{3}{2}} \|e\|_\infty \|\dot{x}\|_2 - T\|e\|_\infty |\hat{x}|. \end{aligned}$$

Hence, since $\delta > 0$, there are two positive constants $c > 0$ and $c' > 0$ for which

$$\varphi(x) \geq c (\|\dot{x}\|_2^2 + |\hat{x}|^2) - c', \quad (9)$$

and the functional φ is bounded below.

For $u \in C([0, T], H)$, we denote by Pu the indefinite integral defined on $[0, T]$ by

$$Pu(t) = \int_0^t u(s) ds.$$

Lemma 5. *Let $E \subseteq C([0, T], H)$ be such that $A := \{u([0, T]) : u \in E\}$ is precompact in H . Then:*

(a) *the set $B := \{\int_0^T u(t) dt : u \in E\}$ is precompact in H ;*

(b) *the set $S := \{Pu : u \in E\}$ is precompact in $C([0, T], H)$.*

Proof. (a) Let $\varepsilon > 0$. There exists a finite sequence (v_1, \dots, v_m) in H such that, denoting by $B(u, \rho)$ any open ball of center u and radius ρ ,

$$A \subseteq \bigcup_{k=1}^m B(v_k, \varepsilon).$$

We denote by Q_0 the orthogonal projection from H to the space V generated by (v_1, \dots, v_m) . The set

$$C = \left\{ \int_0^T Q_0 u(t) dt : u \in E \right\}$$

is bounded in V , hence precompact in V . This implies the existence of a finite sequence (w_1, \dots, w_m) in V such that

$$C \subseteq \bigcup_{k=1}^m B(w_k, \varepsilon).$$

For every $u \in E$, we have

$$\left| \int_0^T u(t) dt - \int_0^T Q_0 u(t) dt \right| \leq \int_0^T |u(t) - Q_0 u(t)| dt \leq \varepsilon T.$$

It follows that

$$B \subseteq \bigcup_{k=1}^m B(w_k, (1 + T)\varepsilon).$$

Since ε is arbitrary, B is precompact in H .

(b) Let us define

$$R := \{P(Q_0 u) : u \in E\}.$$

The set $\{P(Q_0 u)(t) : t \in [0, T], u \in E\}$ is bounded in V , hence precompact in V . For $0 \leq t_1 \leq t_2 \leq T$, we have

$$|P(Q_0 u)(t_2) - P(Q_0 u)(t_1)| = \left| \int_{t_1}^{t_2} Q_0(u)(s) ds \right| \leq c(t_2 - t_1)$$

for some $c > 0$. By the Ascoli–Arzelà theorem, the set R is precompact in $C([0, T], V)$. This implies the existence of a finite sequence (f_1, \dots, f_N) in $C([0, T], V)$ such that

$$R \subseteq \bigcup_{k=1}^N B(f_k, \varepsilon).$$

Since, for every $u \in E$ and $t \in [0, T]$, we have

$$|Pu(t) - P(Q_0u)(t)| \leq \int_0^t |u(s) - Q_0u(s)| ds \leq T\varepsilon,$$

we conclude that

$$S \subseteq \bigcup_{k=1}^N B(f_k, (1+T)\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, S is precompact in $C([0, T], H)$. \square

We now prove the following.

Proposition 6. *If $(x^n)_n$ is a sequence in H_T^1 such that $(\varphi(x^n))_n$ is bounded and $\nabla\varphi(x^n) \rightarrow 0$, then $(\tilde{x}^n)_n$ has a convergent subsequence.*

Proof. Since $(\varphi(x^n))_n$ is bounded, by (8) and (9) we have that $(\tilde{x}^n)_n$ is bounded in H_T^1 . On the other hand, we can modify \tilde{x}^n into some \bar{z}^n such that the scalar product (\bar{z}^n, a_k) belongs to $[0, \tau_k]$, for every k , and $(\bar{z}^n, a_k) \equiv (\tilde{x}^n, a_k) \pmod{\tau_k}$. Defining $z^n = \bar{z}^n + \tilde{x}^n$, we have a new sequence for which $\varphi(z^n) = \varphi(x^n)$ and $\nabla\varphi(z^n) = \nabla\varphi(x^n)$, by (6). Moreover, $(z^n)_n$ is bounded, hence there is a subsequence, still denoted by $(z^n)_n$, which weakly converges to some $z^* \in H_T^1$. We want to show that $(z^n)_n$ strongly converges to z^* in H_T^1 .

Since $\nabla\varphi(z^n) \rightarrow 0$ and $(z^n)_n$ weakly converges to z^* , we have that

$$\langle \nabla\varphi(z^n) - \nabla\varphi(z^*), z^n - z^* \rangle \rightarrow 0,$$

i.e.,

$$\lim_n \int_0^T [|\dot{z}^n(t) - \dot{z}^*(t)|^2 - (\mathcal{A}(z^n(t) - z^*(t)), z^n(t) - z^*(t)) - (\nabla_x V(t, z^n(t)) - \nabla_x V(t, z^*(t)), z^n(t) - z^*(t))] dt = 0. \quad (10)$$

Since $(z^n)_n$ weakly converges to z^* in $L^2([0, T], H)$,

$$\lim_n \int_0^T (\nabla_x V(t, z^*(t)), z^n(t) - z^*(t)) dt = 0. \quad (11)$$

We want to prove that also, up to a subsequence,

$$\lim_n \int_0^T (\nabla_x V(t, z^n(t)), z^n(t) - z^*(t)) dt = 0. \quad (12)$$

Define on $[0, T]$ the continuous functions

$$w^n(t) = \nabla_x V(t, z^n(t)), \quad y^n(t) = z^n(t) - z^*(t),$$

having values in H . Using the notation in (7), we have

$$\begin{aligned} \int_0^T (w^n(t), y^n(t)) dt &= \int_0^T ([w^n] + \check{w}^n(t), [y^n] + \check{y}^n(t)) dt \\ &= T([w^n], [y^n]) + \int_0^T (\check{w}^n(t), \check{y}^n(t)) dt. \end{aligned}$$

Since $(y^n)_n$ weakly converges to 0 in $L^2([0, T], H)$, we see that $([y^n])_n$ weakly converges to 0 in H . Indeed, for every $\eta \in H$, considering it as a constant function in $L^2([0, T], H)$, we have that

$$([y^n], \eta) = \left(\frac{1}{T} \int_0^T y^n(t) dt, \eta \right) = \frac{1}{T} \int_0^T (y^n(t), \eta) dt \rightarrow 0.$$

Moreover, by A5, the set $\{w^n(t) : t \in [0, T], n \in \mathbb{N}\}$ is precompact in H . Hence, by Lemma 5(a), the sequence $([w^n])_n$ is contained in a compact subset of H . Then, up to a subsequence,

$$\lim_n ([w^n], [y^n]) = 0.$$

On the other hand, defining

$$\xi^n(t) = \int_0^t \check{w}^n(s) ds = (P\check{w}^n)(t),$$

we have that $\xi^n(T) = \xi^n(0)$, and recalling that $\check{y}^n(t)$ and $y^n(t)$ differ by a constant, integrating by parts we have

$$\int_0^T (\check{w}^n(t), \check{y}^n(t)) dt = - \int_0^T (\xi^n(t), \dot{y}^n(t)) dt.$$

We know that $(\dot{y}^n)_n$ weakly converges to 0 in $L^2([0, T], H)$. Moreover, since $\{w^n(t) : t \in [0, T], n \in \mathbb{N}\}$ is precompact in H , by Lemma 5(b), the sequence $(\xi^n)_n$ is contained in a compact subset of $C([0, T], H)$ and hence, up to a subsequence,

$$\lim_n \int_0^T (\xi^n(t), \dot{y}^n(t)) dt = 0,$$

thus proving (12).

Going back to (10), by (11) and (12), we get

$$\lim_n \int_0^T [|\dot{z}^n(t) - \dot{z}^*(t)|^2 - (\mathcal{A}(z^n(t) - z^*(t)), z^n(t) - z^*(t))] dt = 0.$$

By A1, being \mathcal{A} semi-negative definite, we deduce that

$$\lim_n \int_0^T |\dot{z}^n(t) - \dot{z}^*(t)|^2 dt = 0,$$

and

$$\lim_n \int_0^T (\mathcal{A}(z^n(t) - z^*(t)), z^n(t) - z^*(t)) dt = 0,$$

i.e.,

$$\lim_n \int_0^T [(\mathcal{A}(\hat{z}^n - \hat{z}^*), \hat{z}^n - \hat{z}^*) + (\mathcal{A}(\tilde{z}^n(t) - \tilde{z}^*(t)), \tilde{z}^n(t) - \tilde{z}^*(t))] dt = 0,$$

Hence, $\dot{z}^n \rightarrow \dot{z}^*$ in $L^2([0, T], H)$, and, by A1, also $\hat{z}^n \rightarrow \hat{z}^*$. Being $z^n = \hat{z}^n + \tilde{z}^n$, we have proved that $(z^n)_n$ converges in H_T^1 . This fact leads to the conclusion of the proof. \square

We now distinguish the two cases. If $\mathcal{N}(\mathcal{A})$ has finite dimension N , then Theorem 3 applies, because Proposition 6 provides the $(PS)_G$ condition for

$$G = \left\{ \sum_{k=1}^N m_k \tau_k a_k : m_k \in \mathbb{Z} \right\},$$

which is a subgroup of H_T^1 , and φ is G -invariant. We thus get $N + 1$ critical orbits of φ .

Assume now that $\mathcal{N}(\mathcal{A})$ is infinite-dimensional. We first prove that φ has a minimum. To this aim, let $(x^n)_n$ be a sequence in H_T^1 such that $\varphi(x^n) \rightarrow \iota := \inf \varphi(H_T^1)$. By the Ekeland Principle, there is a sequence $(y^n)_n$ such that

$$\|x^n - y^n\| \rightarrow 0, \quad \varphi(y^n) \rightarrow \iota, \quad \nabla \varphi(y^n) \rightarrow 0.$$

Moreover, by (6), we can argue as in beginning of the proof of Proposition 6 and assume without loss of generality that

$$\bar{y}^n \in K := \left\{ y = \sum_{k=1}^{\infty} y_k a_k : y_k \in [0, \tau_k] \text{ for } k = 1, 2, \dots \right\},$$

for every n . The set K is compact, being isometric to the Hilbert cube $\prod_{k=1}^{\infty} [0, \tau_k]$ in ℓ^2 , since $\sum_{k=1}^{\infty} \tau_k^2 < +\infty$. Using this and Proposition 6, there is a subsequence of $(y^n)_n$ converging to some $y^* \in H_T^1$. Then, $\varphi(y^*) = \iota$, and $\nabla \varphi(y^*) = 0$. We have thus found a minimum point for the functional φ .

If y^* is not an isolated minimum point, then there are infinitely many minimum points near y^* . In this case, then, there are infinitely many geometrically distinct critical points of φ .

Otherwise, if y^* is an isolated minimum point, there is a constant $r > 0$ such that

$$\varphi(u) > \min \varphi, \quad \text{for every } u \in \overline{B}(y^*, r) \setminus \{y^*\}.$$

(We denote by $B(y^*, r)$ the open ball centered at y^* , with radius $r > 0$, and by $\overline{B}(y^*, r)$ its closure.) Let us prove that

$$\inf_{\partial B(y^*, r)} \varphi > \min \varphi.$$

By contradiction, assume that there is a sequence $(\xi^n)_n$ in $\partial B(y^*, r)$ such that $\varphi(\xi^n) \rightarrow \min \varphi$. Using the Ekeland Principle, it is possible to find a sequence $(\eta^n)_n$ in H_T^1 such that $\varphi(\eta^n) \rightarrow \min \varphi$, $\|\eta^n - \xi^n\| \rightarrow 0$ and $\nabla \varphi(\eta^n) \rightarrow 0$. By (6), we can assume without loss of generality that $\bar{\eta}^n \in K$, for every n . Then, by Proposition 6, there is a subsequence of $(\eta^n)_n$ which converges to some y in H_T^1 . Being $\partial B(x, r)$ a closed set, we have that $y \in \partial B(y^*, r)$, and by continuity $\varphi(y) = \min \varphi$, a contradiction.

Choosing, e.g., $y^{**} = y^* + \tau_1 a_1$, if $r > 0$ small enough we have that $y^{**} \notin B(y^*, r)$, and

$$\varphi(y^{**}) = \varphi(y^*) < \inf_{\partial B(y^*, r)} \varphi.$$

So, the Mountain Pass Theorem applies: setting

$$\Gamma = \{\gamma \in C([0, 1], H_T^1) : \gamma(0) = y^*, \gamma(1) = y^{**}\},$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \varphi(\gamma(t)),$$

there is a sequence $(u^n)_n$ in H_T^1 such that

$$\lim_n \varphi(u^n) = c, \quad \lim_n \nabla \varphi(u^n) = 0.$$

Moreover, $\varphi(y^*) < c$. Proceeding as in the first part of the proof, we can assume without loss of generality that $\bar{u}^n \in K$, and we can find a subsequence of $(u^n)_n$ which converges to some $u^* \in H_T^1$, so that $\varphi(u^*) = c$ and $\nabla \varphi(u^*) = 0$. Since $\varphi(y^*) < \varphi(u^*)$, we have thus found two critical points, y^* and u^* , which are geometrically distinct. \square

4 An example and an open problem

Let $(e_k)_k$ be an orthonormal basis in H , and assume that the periodicity condition (3) holds. Assume moreover (1), i.e., that $e(t)$ has a zero mean. Defining, for every $N \geq 1$, the projection

$$P_N : H \rightarrow H, \quad x = \sum_{k=1}^{\infty} x_k e_k \mapsto \sum_{k=N+1}^{\infty} x_k e_k,$$

we have that

$$\mathcal{N}(P_N) = \text{span}\{e_1, \dots, e_N\}.$$

Then, taking $\mathcal{A} = -P_N$, Theorem 2 applies to the system

$$\ddot{x} - P_N x + \nabla_x V(t, x) = e(t),$$

and provides us with at least $N+1$ geometrically distinct T -periodic solutions.

Notice that the number of T -periodic solutions increases indefinitely together with N . However, passing to the limit on N , the system becomes

$$\ddot{x} + \nabla_x V(t, x) = e(t),$$

to which Theorem 2 still applies, but guarantees *only two* T -periodic solutions. It is an open problem to know if, in this last case, the existence of more than two T -periodic solutions can be proved.

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