A systematic approach to nonresonance conditions for periodically forced planar Hamiltonian systems

Christian Fabry and Alessandro Fonda

ABSTRACT. In the first part of the paper we consider periodic perturbations of some planar Hamiltonian systems. In a general setting, we detect conditions ensuring the existence and multiplicity of periodic solutions. In the second part, the same ideas are used to deal with some more general planar differential systems.

1 Introduction

The meaning of the word *resonance* is well understood for a linear equation of the type

$$x'' + \lambda\, x = q(t)\,,$$

where λ is a positive constant and $q(t)$ is a 2π-periodic forcing: resonance occurs when all the solutions are unbounded, both in the past and in the future. This may happen only when $\lambda = n^2$, for some integer n. On the contrary, if $\lambda \notin \{n^2 : n \in \mathbb{N}\}$, then all solutions of the differential equation are bounded, and among them there is a 2π-periodic solution, for any 2π-periodic forcing term $q(t)$.

For a more general nonlinear equation

$$x'' + g(x) = q(t)\,, \tag{1.1}$$

the meaning of *resonance* does not appear so clearly. However, it seems to be commonly accepted to consider as *nonresonance conditions* on the function $g(x)$ those ensuring that the differential equation admits at least one 2π-periodic solution, for any 2π-periodic forcing term $q(t)$.

Life becomes still more complicated if we consider equations of the type

$$x'' + g(x) = q(t, x)\,, \tag{1.2}$$

where $q(t, x)$, which is 2π-periodic in its first variable, is considered as some kind of perturbation of the autonomous equation. There is a huge literature on the existence of periodic solutions for this type of equations (see e.g. [13] and the references therein). In this case, "nonresonance conditions" necessarily involve both the functions $g(x)$ and $q(t, x)$, and they are supposed to guarantee the existence of at least one 2π-periodic solution of the differential equation.

In this paper, we are looking for "nonresonance conditions" for more general planar systems of the type

$$Jz' = \nabla H(z) + r(t, z)\,. \tag{1.3}$$

Here, and throughout the paper, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix, the *Hamiltonian function* $H : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, and $r : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is assumed to be continuous, and 2π-periodic in its first variable. We search for conditions on $H(z)$ and $r(t, z)$ guaranteeing that system (1.3) has at least one 2π-periodic solution.

Since we need uniqueness of solutions, we will typically assume $\nabla H(z)$ to be locally Lipschitz continuous, and $r(t, z)$ to have the same regularity property with respect to its second variable. However, we will also have to consider autonomous Hamiltonian systems without necessarily assuming the gradient of their Hamiltonian function to be locally Lipschitz continuous.

For the autonomous system

$$Jz' = \nabla H(z)\,, \tag{1.4}$$

associated with (1.3), it will be assumed that all large amplitude solutions are periodic. More precisely, we will assume that, for large energy levels E, the sets $H^{-1}(E)$ are closed curves corresponding to periodic solutions of (1.4) with some minimal period $T(E)$. In our approach for the study of system (1.3), we will consider it as some kind of perturbation of the autonomous system (1.4). Through a change of variables, we transform (1.3) into a system of differential equations having, as variables, the energy and a phase. By the systematic use of the energy as a parameter, our aim is to obtain sharp nonresonance results and to provide new insights into the nonresonance problem.

When the forcing term does not depend on z, i.e., when $r(t, z) = r(t)$, one expects that the system admits a 2π-periodic solution, unless the period 2π of the forcing term $r(t)$ interferes with the periods of the large amplitude free oscillations, meaning that $T(E)$ approaches $2\pi/n$, for some integer n, when E goes to infinity. Consequently, we expect that a nonresonance condition should be of the type

$$\lim_{E \to +\infty} T(E) \neq \frac{2\pi}{n}, \quad \textit{for all integer } n, \tag{1.5}$$

including the case where

$$\liminf_{E \to +\infty} T(E) \neq \limsup_{E \to +\infty} T(E). \tag{1.6}$$

It is the main objective of this paper to present conditions under which the inequality (1.5) guarantees the existence of a 2π-periodic solution for the system (1.3).

However, we believe that further restrictions on the Hamiltonian function H are required in order for (1.5) to become a valid nonresonance condition. We remark that, even for the particular case of the scalar equation (1.1), it has been shown in [7] that a nonresonance condition of type (1.5) is sufficient for the existence of a 2π-periodic solution provided that g is differentiable, *with a globally bounded derivative.* Some other set of restrictions on g can be found in [3].

The paper is organized as follows.

In Section 2 we present the general setting, showing how to cover some classical situations, like asymmetric oscillators, positively homogeneous Hamiltonians of degree 2, or Hamiltonians with separated variables, arising, e.g., from differential equations involving the scalar p-Laplacian operator.

Section 3 is devoted to the statement of our first existence theorem; it provides fairly general conditions which, combined with hypothesis (1.5), ensure that system (1.3) admits at least one 2π-periodic solution. Due to their generality, the assumptions of that theorem need to be analyzed in further detail. The proof of that existence theorem is carried out in Section 5.

In Section 4, we develop an approach to the nonresonance condition (1.5) by a comparison between the Hamiltonian function of equation (1.4), and two other Hamiltonians, which would typically be isochronous. Examples are provided by scalar second order equations and systems where the Hamiltonian function H has separated variables.

In Section 6 we extend our existence theorem in several directions. First, we consider the case when the limit of period function $T(E)$ is $+\infty$ as $E \to +\infty$. Then, we obtain an existence result in the critical case when, for some positive integer n_0,

$$\lim_{E\to+\infty} T(E) = \frac{2\pi}{n_0}\,,$$

assuming that the approach to resonance is "not too fast". As an example, we can deal with an equation of the type

$$x'' + k\,x^+ - a\,[x^-]^{p-1} = q(t,x)\,,$$

with $p > 2$ and $q(t,x)$ satisfying a growth condition at infinity. In the last part of the section, we also show how to formulate some Landesman–Lazer type conditions in our setting.

In Section 7, assuming that our system has a Hamiltonian structure, we show how the Poincaré–Birkhoff Theorem can be applied to provide multiplicity of periodic solutions when the time map has an oscillatory behaviour, i.e., when (1.6) holds. In order to be brief we limit our attention to this situation, where only the asymptotic behaviour of the time map is considered. Other situations could also be dealt with, following a similar approach; further developments along these lines are expected in the future.

Finally, in Section 8, we consider the more general system

$$Jz' = F(t,z),$$

and still obtain existence conditions for periodic solutions through a comparison with Hamiltonian systems.

In the following, we always denote by $\langle \cdot\,,\cdot\rangle$ the Euclidean scalar product in $\mathbb{R}^2$, with associated norm $|\cdot|$.

2 General setting and preliminaries

In the first part of this section, we list some conditions on the *Hamiltonian function* $H : \mathbb{R}^2 \to \mathbb{R}$ that will be assumed to hold throughout the paper. In the second part we define some notation, and we place emphasis on a result that will be useful in the sequel.

2.1 The structural assumptions

We recall that the function $H : \mathbb{R}^2 \to \mathbb{R}$ is supposed to be continuously differentiable. We start with two basic assumptions and their consequences for the autonomous equation

$$Jz' = \nabla H(z)\,. \tag{2.1}$$

A1. *The Hamiltonian function H is coercive:*

$$\lim_{|z|\to\infty} H(z) = +\infty\,. \tag{2.2}$$

A2. *There exists a number $\rho > 0$ such that*

$$\nabla H(z) \neq 0\,, \quad \textit{for } |z| \geq \rho.$$

With those hypotheses, there is uniqueness for the solutions of the associated initial value problems with a starting point of sufficiently large norm (cf. [28]), and it results from the Poincaré–Bendixson theory that all these solutions are periodic. Because of (2.2), we also see that the solutions of large amplitude circle the origin, and that the corresponding trajectories are oriented clockwise; there exists thus an annulus of closed orbits, extending to infinity. Notice that these orbits are not necessarily star-shaped. Among them, we select a particular one, denoted by Γ_1 and, by convention, we take

$$H(z) = 1\,, \quad \textit{for every } z \in \Gamma_1\,.$$

We assume that Γ_1 has been chosen in such a way that $|z| \geq \rho$, for all $z \in \Gamma_1$.

We want to parametrize the solutions of large amplitude of equation (2.1) by the energy. More precisely, we make the following structural assumption.

A3. *There exists a differentiable function $\varphi : \mathbb{R}\times\,]1,+\infty[\,\to \mathbb{R}$ such that*

$$J\,\frac{\partial\varphi}{\partial t}(t;E) = \nabla H(\varphi(t;E))\,, \quad \textit{for all } t \in \mathbb{R} \textit{ and } E > 1\,,$$

and

$$H(\varphi(0;E)) = E\,, \quad \textit{for all } E > 1\,.$$

As a consequence, the system (2.1) being conservative, we have that

$$H(\varphi(t;E)) = E\,, \quad \textit{for all } t \in \mathbb{R} \textit{ and } E > 1\,,$$

and, differentiating this relation,

$$\left\langle \nabla H(\varphi(t;E)), \frac{\partial\varphi}{\partial E}(t;E) \right\rangle = 1\,, \quad \textit{for all } t \in \mathbb{R} \textit{ and } E > 1\,. \tag{2.3}$$

2.2 About condition A3

Let us describe a standard way to construct the function φ. Fix a point z_0^* in Γ_1, and consider a solution $w(\tau)$ of the Cauchy problem

$$w'(\tau) = \frac{\nabla H(w(\tau))}{|\nabla H(w(\tau))|^2}\,, \quad w(1) = z_0^*\,. \tag{2.4}$$

Since

$$\frac{d}{d\tau}\,H(w(\tau)) = 1\,, \quad \textit{for every } \tau \geq 1\,,$$

integrating on $[1,E]$ we see that

$$H(w(E)) = E\,, \quad \textit{for every } E > 1\,.$$

In other words, E corresponds to the "energy" at the point $w(E)$, which motivates the notation E. Now, for any $E > 1$, let $\varphi(\cdot\,;E)$ be the solution of the Hamiltonian system (2.1) such that $\varphi(0;E) = w(E)$. It is clear that both equalities in A3 hold true. The regularity of φ is surely guaranteed if H is twice continuously differentiable, but we will see that it is satisfied also in some more general situations. As will appear below, advantage can be taken of the possibility of carefully choosing z_0^*, in order to get a function $\varphi(t;E)$ having some convenient properties.

From the definition of $\varphi(t;E)$, we also have that

$$\frac{\partial\varphi}{\partial E}(0;E) = \frac{\nabla H(\varphi(0;E))}{\big|\nabla H(\varphi(0;E))\big|^2}\,, \quad \textit{for every } E > 1\,; \tag{2.5}$$

this property of $\varphi(0;E)$ will play an important role in the sequel and we will assume it to hold throughout.

As a first example, consider a Hamiltonian function of the type

$$H(x,y) = \tfrac{1}{2}y^2 + G(x)\,, \tag{2.6}$$

with $G(0) = 0$. The autonomous system (2.1) is then equivalent to the scalar second order equation

$$x'' + g(x) = 0\,,$$

where $g(x) = G'(x)$. Notice that conditions A1 and A2 will be satisfied assuming

$$x\,g(x) > 0\,, \quad \textit{for } |x| \textit{ large}\,, \tag{2.7}$$

and

$$\lim_{|x|\to\infty} G(x) = +\infty\,. \tag{2.8}$$

In the above example, the regularity assumption in A3 is surely satisfied if g is continuously differentiable. However, it is also satisfied if, e.g.,

$$g(x) = a_+ x^+ - a_- x^-\,,$$

for some positive constants a_+, a_- (here, as usual, $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$). This last example leads us to study more carefully the case of *positively homogeneous* systems of degree 2.

Let the Hamiltonian function H be such that

$$0 < H(\lambda z) = \lambda^2 H(z)\,,\ \textit{for every } \lambda > 0 \textit{ and } z \in \mathbb{R}^2 \setminus \{0\}\,. \tag{2.9}$$

In this case, it is well known that the autonomous system (2.1) is isochronous, all the nonconstant orbits having the same minimal period $\widehat{T}$. (For the reader's convenience, this fact will also be proved below, as an easy consequence of Lemma 2.1.) Assumptions A1, A2 are readily verified. Concerning A3, let us show that in this case it is possible to choose z_0^* in (2.4) so that the resulting function $\varphi(t; E)$ satisfies

$$\varphi(t; E) = \sqrt{E}\,\varphi(t; 1)\,. \tag{2.10}$$

Indeed, if z is a solution of the autonomous equation (2.1) of energy equal to 1, because the function $t \mapsto |z(t)|^2$ reaches its extremal values in $[0, \widehat{T}]$, there exists a number $t^* \in [0, \widehat{T}]$ such that

$$\langle z(t^*), z'(t^*)\rangle = -\langle z(t^*), J\nabla H(z(t^*))\rangle = 0\,.$$

Consequently, since $\langle z, \nabla H(z)\rangle = 2H(z) > 0$, for $z \neq 0$, there exists $\nu > 0$ such that $z(t^*) = \nu\,\nabla H(z(t^*))$. Using the fact that $\nabla H(\lambda z) = \lambda \nabla H(z)$ for any $\lambda \geq 0$, if we then take $z_0^* = z(t^*)$ in (2.4), we note that a solution of this system is given by $w(\tau) = \sqrt{\tau}\,z(t^*)$, implying that $\varphi(0; E) = \sqrt{E}\,\varphi(0; 1)$, from which (2.10) follows. As a consequence, in this case we have

$$\frac{\partial \varphi}{\partial E}(t; E) = \frac{1}{2\sqrt{E}}\,\varphi(t; 1)\,. \tag{2.11}$$

Notice that, as a particular case, we could have $H(z) = \frac{1}{2}\langle \mathbb{A} z, z\rangle$, with a positive definite symmetric matrix $\mathbb{A}$.

In the sequel, we will also frequently refer to Hamiltonian functions of the form

$$H(x,y) = a\,|x|^p + b\,|y|^q,$$

with a and b positive constants, $p > 1$ and $q > 1$. It must be kept in mind that the gradient of such a function is not necessarily locally Lipschitz continuous. Nevertheless, its properties will prove useful when comparing the minimal period of solutions of various Hamiltonian systems. If $\varphi(t;1)$ is a solution of (2.1) of energy 1, we observe that

$$\varphi(t;E) = \mathrm{diag}(E^{1/p}, E^{1/q})\,\varphi(E^{\mu}\,t;1)\,, \tag{2.12}$$

where

$$\mu = 1 - \frac{1}{p} - \frac{1}{q}\,,$$

is also a solution of (2.1), implying that, for $\mu > 0$, the minimal period tends to 0 when E tends to $+\infty$, whereas, for $\mu < 0$, it tends to $+\infty$. On the other hand, if $\mu = 0$, i.e., if

$$\frac{1}{p} + \frac{1}{q} = 1\,, \tag{2.13}$$

the autonomous system is isochronous, and

$$\frac{\partial\varphi}{\partial E}(t;E) = \frac{1}{E}\,\mathrm{diag}\left(\frac{1}{p}\,,\frac{1}{q}\right)\varphi(t;E) = \mathrm{diag}\left(\frac{1}{p\,E^{1/q}}\,,\frac{1}{q\,E^{1/p}}\right)\varphi(t;1). \tag{2.14}$$

If we choose $\varphi(0;1) = ((1/a)^{1/p}, 0)$, it can be checked that (2.5) is satisfied.

2.3 A basic property

Let us denote by $\mathrm{int}(\Gamma_1)$ and $\mathrm{ext}(\Gamma_1)$ the bounded and the unbounded connected components of $\mathbb{R}^2 \setminus \Gamma_1$, respectively. For every $z_0 \in \mathrm{ext}(\Gamma_1)$, let $\mathcal{T}(z_0)$ be the minimal period of the solution issuing from it. We define the continuous function $T :]1,+\infty[\to \mathbb{R}$ as

$$T(E) = \mathcal{T}(\varphi(0;E))\,;$$

it expresses the period as a function of the energy. Moreover, for $E > 1$, we introduce the open bounded set

$$\Omega(E) = \{z \in \mathbb{R}^2 : H(z) < E\} \cup \mathrm{int}(\Gamma_1)\,.$$

Notice that, for E sufficiently large, $\Omega(E)$ is the bounded set delimited by the level curve $H^{-1}(E)$. The following lemma expresses a fundamental relation between the area $a(E)$ of $\Omega(E)$ and the minimal period $T(E)$.

Lemma 2.1. *Let the assumptions A1 to A3 hold. Then,*

$$a'(E) = T(E)\,, \quad \textit{for every } E > 1\,.$$

Proof. Given $E > 1$, let us consider the open sets

$$\mathcal{A} = \{(\tau, e) \in \mathbb{R}^2 : e \in \,]1, +\infty[\,,\ \tau \in \,]0, T(e)[\,\}\,,$$

$$\mathcal{B} = \mathbb{R}^2 \setminus \Big(\overline{\Omega(1)} \cup \varphi\big(\{0\}\times\,]1, +\infty[\,\big)\Big)\,.$$

Notice that $\mathcal{B}$ differs from $\mathbb{R}^2 \setminus \Omega(1)$ by a set of zero Lebesgue measure. Define the function $\Phi : \mathcal{A} \to \mathcal{B}$ as $\Phi(\tau, e) = \varphi(\tau; e)$. It is one-to-one and onto. Using (2.3), we have

$$\det \Phi'(\tau, e) = \left\langle J\varphi'(\tau; e), \frac{\partial\varphi}{\partial E}(\tau; e)\right\rangle = \left\langle \nabla H(\varphi(\tau; e)), \frac{\partial\varphi}{\partial E}(\tau; e)\right\rangle = 1\,,$$

for every $(\tau, e) \in \mathcal{A}$, so that Φ is a diffeomorphism. For $E > 1$, the area of $\Omega(E)$ is then given by

$$\begin{aligned} a(E) &= a(1) + \int_1^E \left(\int_0^{T(e)} |\det \Phi'(\tau, e)|\; d\tau\right) de \\ &= a(1) + \int_1^E T(e)\, de\,, \end{aligned}$$

and the conclusion directly follows. □

As a first example of application we can show that, when H satisfies (2.9), the system (2.1) is isochronous. Indeed, the homogeneity property implies that $\Omega(E) = \sqrt{E}\,\Omega(1)$, for any $E \geq 0$, so that $a(E) = E\,a(1)$. Consequently, the period is given by $T(E) = a'(E) = a(1)$.

Remark 2.2. Given a continuous function $T : [1, +\infty[\,\to\,]0, +\infty[\,$, it is always possible to construct a Hamiltonian function $H : \mathbb{R}^2 \to \mathbb{R}$ for which

$T(E)$ is the time map for the orbits of energy $E \geq 1$. Indeed, define $a : [1, +\infty[\to [\pi, +\infty[$ by

$$a(E) = \pi + \int_1^E T(e)\, de\,.$$

This function is strictly increasing and onto, hence invertible; consider a C^1-extension $a : [0, +\infty[\to [0, +\infty[$, for which we keep the same notation, with $a(0) = 0$ and $a'(E) > 0$ for every $E \geq 0$. Define $f : [0, +\infty[\to [0, +\infty[$ by $f(r) = a^{-1}(\pi\, r^2)$, and consider the associated Hamiltonian system (2.1), with $H(z) = f(|z|)$. It is clear that the orbit of energy $E > 0$ of this system is a circle with area $a(E)$. Consequently, by Lemma 2.1, for every $E \geq 1$ the minimal period of the corresponding solution is $T(E)$.

3 Existence of periodic solutions

In this section, we first state our existence theorem, its proof being postponed to Section 5. We then make some remarks on the assumptions of the theorem, and derive some useful corollaries.

We will make use of the following regularity conditions on H and r.

L1. *The function $H : \mathbb{R}^2 \to \mathbb{R}$ is differentiable with a locally Lipschitz continuous gradient.*

L2. *The function $r : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, 2π-periodic in its first variable, and locally Lipschitz continuous in its second variable.*

3.1 Statement of the existence result

Here is the main result of this section.

Theorem 3.1. *Let the assumptions A1 to A3 hold, as well as L1, L2, and the following nonresonance conditions:*

A4. *The function $T(E)$ is controlled as follows:*

$$\limsup_{E\to+\infty} T(E) > 0\,, \qquad \liminf_{E\to+\infty} T(E) < +\infty\,.$$

A5. *For any integer n,*

$$\lim_{E\to+\infty} T(E) \neq \frac{2\pi}{n}\,.$$

Assume also that:

A6. *There is a constant $C > 0$ such that*

$$\limsup_{|z|\to\infty} \frac{\big|\langle J\nabla H(z), r(t,z)\rangle\big|}{H(z)} \leq C\,, \;\; \textit{uniformly in } t \in [0, 2\pi]\,,$$

and

A7. *For any given compact interval $I \subseteq \mathbb{R}$,*

$$\lim_{E\to+\infty} \left\langle \frac{\partial\varphi}{\partial E}(s;E), r(t,\varphi(s;E)) \right\rangle = 0\,, \;\; \textit{uniformly for } (t,s) \in [0,2\pi]\times I\,.$$

Then, equation (1.3) *admits at least one 2π-periodic solution.*

Before going to the proof of Theorem 3.1, we make some comments and draw some consequences.

3.2 About conditions A4 and A5

It is to be understood that assumptions A4 and A5 hold if and only if

- either $\liminf_{E\to+\infty} T(E) \neq \limsup_{E\to+\infty} T(E)\,,$
- or $\liminf_{E\to+\infty} T(E) = \limsup_{E\to+\infty} T(E) \in \,]0,+\infty[\, \setminus \left\{ \frac{2\pi}{n} \;\Big|\; n = 1,2,\dots \right\}.$

Therefore, assuming A4 and A5 is equivalent to assuming the existence of a sequence $(E_k)_k$ such that

$$\lim_k E_k = +\infty \quad \textit{and} \quad \lim_k T(E_k) \in \,]0,+\infty[\, \setminus \left\{ \frac{2\pi}{n} \;\Big|\; n = 1,2,\dots \right\}.$$

Notice that, by Lemma 2.1 and the general l'Hôpital rule,

$$\liminf_{E\to+\infty} T(E) \leq \liminf_{E\to+\infty} \frac{a(E)}{E} \leq \limsup_{E\to+\infty} \frac{a(E)}{E} \leq \limsup_{E\to+\infty} T(E)\,,$$

so that conditions A4 and A5 will surely be verified if the following two hold.

A4'. *The function $a(E)/E$ is controlled as follows:*

$$\limsup_{E\to+\infty} \frac{a(E)}{E} > 0\,, \quad \liminf_{E\to+\infty} \frac{a(E)}{E} < +\infty\,.$$

A5'. *For any integer n,*

$$\lim_{E\to+\infty} \frac{a(E)}{E} \neq \frac{2\pi}{n}\,.$$

We can thus state the following immediate consequence of Theorem 3.1.

Corollary 3.2. *Let the assumptions A1 to A3 hold, as well as L1, L2, A6, and A7. Suppose that the nonresonance conditions A4', A5' are satisfied. Then, equation* (1.3) *admits at least one 2π-periodic solution.*

The interest of the above corollary lies in the observation that, under hypotheses A1 to A3, A6 and A7, a 2π-periodic solution will exist if, for some integer n_0,

$$\frac{2\pi}{n_0+1} < \liminf_{E\to+\infty} \frac{a(E)}{E} \leq \limsup_{E\to+\infty} \frac{a(E)}{E} < \frac{2\pi}{n_0}\,. \tag{3.1}$$

This condition is less stringent than the condition

$$\frac{2\pi}{n_0+1} < \liminf_{E\to+\infty} T(E) \leq \limsup_{E\to+\infty} T(E) < \frac{2\pi}{n_0}\,; \tag{3.2}$$

it is also likely to be easier to check, since estimates on $\liminf_{E\to+\infty} a(E)/E$ and $\limsup_{E\to+\infty} a(E)/E$ are deduced from estimates on $a(E)$, which, in turn, can be obtained by comparing the Hamiltonian H to other Hamiltonians, as will be shown below.

3.3 About conditions A6 and A7

In order to better understand conditions A6 and A7, let us first consider the particular case when the Hamiltonian function is positively homogeneous of degree 2, i.e., when (2.9) holds. In this case, taking into account (2.10), condition A6 holds if there exists a constant $c > 0$ such that

$$|r(t,z)| \leq c(1+|z|)\,, \quad \textit{for every } (t,z) \in [0,2\pi]\times\mathbb{R}^2.$$

On the other hand, taking into account (2.11), condition A7 holds if

$$\lim_{|z|\to\infty} \frac{r(t,z)}{|z|} = 0\,, \quad \textit{uniformly in } t \in [0,2\pi]\,. \tag{3.3}$$

We thus have the following.

Corollary 3.3. *Assume that L1, L2, and* (2.9) *hold. Let* $\widehat{T}$ *be the minimal period of the solutions of the isochronous system* (2.1)*. If* $\widehat{T} \neq 2\pi/n$*, for all integers* n*, and the forcing term satisfies* (3.3)*, then equation* (1.3) *admits at least one* 2π*-periodic solution.*

In order to deal with more general situations, we introduce an assumption which will ensure that A7 is satisfied for a function H which is twice continuously differentiable.

A8. *There exist a continuous function* $\mathcal{D} : [1, +\infty) \to \mathcal{GL}(\mathbb{R}^2)$ *(the group of invertible* 2×2 *real matrices) and a continuous function* $\kappa : \mathbb{R}^2 \to]0, +\infty[$ *such that, for* $E > 1$,

$$\big\langle \mathcal{D}(E) J H''(\varphi(t; E)) \mathcal{D}^{-1}(E)\, v, v \big\rangle \geq -\kappa(\varphi(t; E))\, |v|^2, \qquad \text{for all } t \in \mathbb{R} \text{ and } v \in \mathbb{R}^2, \tag{3.4}$$

and

$$\int_0^t \kappa(\varphi(s; E))\, ds \qquad \text{remains bounded for } E \to +\infty, \qquad \text{independently of } t \text{ in compact sets.} \tag{3.5}$$

Moreover, for any given compact interval $I \subseteq \mathbb{R}$,

$$\lim_{E \to +\infty} \frac{|\mathcal{D}(E)\, \nabla H(\varphi(0; E))|}{|\nabla H(\varphi(0; E))|^2} \left| (\mathcal{D}^T(E))^{-1} r(t, \varphi(s; E)) \right| = 0\,, \qquad \text{uniformly for } (t, s) \in [0, 2\pi] \times I\,. \tag{3.6}$$

Let us show that A8 implies A7. Indeed, from the variational equation

$$\frac{d}{dt} \frac{\partial \varphi}{\partial E}(t; E) = -J\, H''(\varphi(t; E))\, \frac{\partial \varphi}{\partial E}(t; E)\,,$$

we see that

$$\begin{aligned} \frac{d}{dt} \left| \mathcal{D}(E)\, \frac{\partial \varphi}{\partial E}(t; E) \right|^2 &= 2 \Big\langle \mathcal{D}(E)\, \frac{\partial \varphi}{\partial E}(t; E)\,,\, \mathcal{D}(E)\, \frac{d}{dt} \frac{\partial \varphi}{\partial E}(t; E) \Big\rangle \\ &= -2 \Big\langle \mathcal{D}(E)\, \frac{\partial \varphi}{\partial E}(t; E)\,,\, \mathcal{D}(E)\, J\, H''(\varphi(t; E))\, \frac{\partial \varphi}{\partial E}(t; E) \Big\rangle, \end{aligned}$$

and we deduce by (3.4) and Gronwall Lemma that

$$\left| \mathcal{D}(E)\, \frac{\partial \varphi}{\partial E}(t; E) \right| \leq \left| \mathcal{D}(E)\, \frac{\partial \varphi}{\partial E}(0; E) \right| \, \exp \left| \int_0^t \kappa(\varphi(s; E))\, ds \right|, \tag{3.7}$$

for every $t \in \mathbb{R}$. Using (2.5) and (3.5), we now see that A7 results from (3.6).

Based on the above considerations, the following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.4. *Let the assumptions A1 to A3, and L2 hold, as well as A6 and A8, H being twice continuously differentiable. Then, if the nonresonance conditions A4, A5 are satisfied, equation* (1.3) *admits at least one* 2π*-periodic solution.*

A noteworthy situation is the case where H'' is globally bounded. Taking as $\mathcal{D}(E)$ the identity matrix, we see that (3.4) is plainly satisfied with κ being a constant function, whereas (3.6) holds if, for any given compact interval $I \subseteq \mathbb{R}$,

$$\lim_{E\to+\infty} \frac{r(t,\varphi(s;E))}{|\nabla H(\varphi(0;E))|} = 0\,, \quad \textit{uniformly for } (t,s) \in [0,2\pi]\times I\,. \tag{3.8}$$

Under this condition, we will show that hypothesis A6 also holds, obtaining the following corollary.

Corollary 3.5. *Let the assumptions A1 to A3 hold, as well as L2. Assuming H to be twice continuously differentiable, let H'' be globally bounded and suppose that* (3.8) *holds. Then, if the nonresonance conditions A4, A5 are satisfied, equation* (1.3) *admits at least one* 2π*-periodic solution.*

Proof. We first observe that, with H'' globally bounded, (3.4) implies

$$\frac{d}{dt}|\nabla H(\varphi(t;E))|^2 \le 2\kappa\, |\nabla H(\varphi(t;E))|^2\,;$$

hence, by Gronwall Lemma,

$$|\nabla H(\varphi(s;E))| \le |\nabla H(\varphi(0;E))|\, e^{\kappa|s|}\,, \tag{3.9}$$

for every $s\in\mathbb{R}$. Taking (3.8) into account, it then follows that condition A6 will hold if

$$\limsup_{E\to\infty} \frac{|\nabla H(\varphi(0;E))|^2}{E} < +\infty\,. \tag{3.10}$$

But, this inequality is a consequence of the boundedness property of H''. Indeed, differentiating $|\nabla H(\varphi(0;E))|^2$ with respect to E, we get, using (2.5),

$$\begin{aligned}\frac{\partial}{\partial E}\,|\nabla H(\varphi(0;E))|^2 &= 2\left\langle H''(\varphi(0;E))\,\frac{\partial\varphi}{\partial E}(0;E), \nabla H(\varphi(0;E))\right\rangle\\ &= \frac{1}{|\nabla H(\varphi(0;E))|^2}\,\langle H''(\varphi(0;E))\,\nabla H(\varphi(0;E)), \nabla H(\varphi(0;E))\rangle\,.\end{aligned}$$

Since H'' is assumed to be globally bounded, it follows that there exist constants C_1, C_2 such that

$$|\nabla H(\varphi(0;E))|^2 \leq C_1 E + C_2 \,,$$

implying (3.10). The proof is thus completed. $\square$

Remark 3.6. Because of (3.9), it is clear that, when H'' is globally bounded, condition (3.8) will hold if

$$\lim_{|z|\to\infty} \frac{r(t,z)}{|\nabla H(z)|} = 0 \,, \quad \textit{uniformly in } t \in [0, 2\pi] \,. \tag{3.11}$$

Remark 3.7. Consider the scalar second order equation

$$x'' + g(x) = r(t) \,, \tag{3.12}$$

with g continuously differentiable, g' being globally bounded, and r continuous and 2π-periodic. Denoting by G a primitive of g, we can associate with this equation the Hamiltonian function defined by (2.6). If we assume that, for some $d > 0$,

$$xg(x) > 0 \,, \quad \textit{for } |x| > d \,, \tag{3.13}$$

it is clear that A1, A2 hold. Condition A3 also holds since g is continuously differentiable. Moreover, if we take $\varphi(0;E) = (0, \sqrt{E})$, it is immediate that (3.8) is satisfied. Consequently, we are in a situation where Corollary 3.5 applies, and we may conclude that (3.12) admits at least one 2π-periodic solution, provided that the nonresonance conditions A4, A5 are satisfied. This result had been obtained in [7], under a more restrictive condition than (3.13), namely, that there exists a positive constant $c > 0$ such that $g(x)/x \geq c$, for $|x| > d$.

The above remark can be adapted to the equation

$$x'' + g(x) = q(t,x) \,, \tag{3.14}$$

provided that

$$\lim_{|x|\to\infty} \frac{q(t,x)}{x} = 0 \,, \quad \textit{uniformly in } t \in [0, 2\pi] \,. \tag{3.15}$$

Assuming the existence of $\eta > 0$ such that

$$G(x) \geq \eta\,|x| \,, \quad \textit{for } |x| > d \,, \tag{3.16}$$

we see that (3.8) is satisfied if we take $\varphi(0;E) = (0, \sqrt{E})$. The following corollary can thus be deduced from Corollary 3.5.

Corollary 3.8. *Let the function $g(x)$ be continuously differentiable, with a globally bounded derivative, and such that* (3.13) *and* (3.16) *hold. Assume moreover that $q(t,x)$ is continuous, 2π-periodic in t, locally Lipschitz continuous in x, and satisfies* (3.15)*. Then, if the nonresonance conditions A4, A5 are satisfied, equation* (3.14) *admits at least one 2π-periodic solution.*

4 Comparison between Hamiltonians

In this section we provide some corollaries of the results of the previous sections. The main idea is to compare the Hamiltonian function H with other Hamiltonians for which the assumptions are easier to check.

4.1 Comparison with isochronous Hamiltonians

Suppose that, for some number $\rho > 0$,

$$H_1(z) \leq H(z) \leq H_2(z)\,, \quad \textit{for } |z| \geq \rho\,, \tag{4.1}$$

the three functions H, H_1, H_2 being continuously differentiable and satisfying the hypotheses A1 to A3 of Section 2. We do not require however the gradients of H_1, H_2 to be locally Lipschitz continuous. For E large enough, we denote by $\Omega_1(E)$, $\Omega(E)$, $\Omega_2(E)$ the bounded sets delimited by the curves $H_1^{-1}(E)$, $H^{-1}(E)$, $H_2^{-1}(E)$, and by $a_1(E)$, $a(E)$, $a_2(E)$ their areas, respectively. Then,

$$\Omega_2(E) \subseteq \Omega(E) \subseteq \Omega_1(E)\,,$$

and hence

$$a_2(E) \leq a(E) \leq a_1(E)\,.$$

So, if for some integer n_0, one has that

$$\frac{2\pi}{n_0+1} < \liminf_{E\to+\infty} \frac{a_2(E)}{E} \leq \limsup_{E\to+\infty} \frac{a_1(E)}{E} < \frac{2\pi}{n_0}\,,$$

it is clear that (3.1) is satisfied. Moreover, if H_1, H_2 are *isochronous* Hamiltonians with respective minimal periods T_1, T_2, by Lemma 2.1 we will have that $a_1(E) = T_1E + C_1$ and $a_2(E) = T_2E + C_2$, for some constants C_1, C_2, and necessarily $T_2 \leq T_1$. This leads to the following consequence of Theorem 3.1.

Corollary 4.1. *Let H, H_1, H_2 satisfy* (4.1) *and the hypotheses A1 to A3, the Hamiltonians H_1, H_2 being isochronous with respective minimal periods T_1, T_2. Assume moreover that H and r satisfy L1, L2, A6, A7. Then, equation* (1.3) *admits at least one 2π-periodic solution, provided that, either $T_2 > 2\pi$, or, for some integer n_0,*

$$\frac{2\pi}{n_0+1} < T_2 \leq T_1 < \frac{2\pi}{n_0}\,. \tag{4.2}$$

If we now recall the situation considered in Corollary 3.5, we immediately get the following.

Corollary 4.2. *Let H, H_1, H_2 satisfy* (4.1) *and the hypotheses A1 to A3, the Hamiltonians H_1, H_2 being isochronous with respective minimal periods T_1, T_2. With H twice continuously differentiable and r satisfying L2, let H'' be globally bounded, and assume that* (3.8) *holds. Then, equation* (1.3) *admits at least one 2π-periodic solution, provided that, either $T_2 > 2\pi$, or, for some integer n_0, condition* (4.2) *holds.*

Notice that, for condition (3.8), advantage can be taken of the freedom of choice in the construction of $\varphi(0;E)$.

4.2 Scalar second order equations

We illustrate the above results with an application to the second order equation (3.14). As before, we assume the function $g(x)$ to be continuously differentiable, and $q(t,x)$ to be continuous, 2π-periodic in t, and locally Lipschitz continuous in x. We denote by G a primitive of g, and associate with this equation the Hamiltonian function defined by (2.6). Assume that G has a quadratic growth; more precisely, suppose that there exist numbers G_-, G_+, G^-, G^+ such that

$$0 < G_\pm = \liminf_{x\to\pm\infty} \frac{2\,G(x)}{x^2} \leq \limsup_{x\to\pm\infty} \frac{2\,G(x)}{x^2} = G^\pm < +\infty\,. \tag{4.3}$$

We fix a small $\varepsilon > 0$, and define

$$\begin{aligned} H_1(x,y) &= \tfrac{1}{2}\big(G_+[x^+]^2 + G_-[x^-]^2 + y^2 - \varepsilon(x^2+y^2)\big)\,,\\ H_2(x,y) &= \tfrac{1}{2}\big(G^+[x^+]^2 + G^-[x^-]^2 + y^2 + \varepsilon(x^2+y^2)\big)\,. \end{aligned}$$

The Hamiltonians H_1, H_2 being positively homogeneous, conditions A1 to A3 are satisfied, and we see that (4.1) holds for ρ sufficiently large. On the other hand, the minimal periods $T_{1,\varepsilon}$, $T_{2,\varepsilon}$ associated with H_1, H_2 are such that

$$\widehat{T}_1 := \lim_{\varepsilon\to 0_+} T_{1,\varepsilon} = \frac{\pi}{\sqrt{G_+}} + \frac{\pi}{\sqrt{G_-}}\,, \quad \widehat{T}_2 := \lim_{\varepsilon\to 0_+} T_{2,\varepsilon} = \frac{\pi}{\sqrt{G^+}} + \frac{\pi}{\sqrt{G^-}}\,.$$

In order to apply Corollary 3.8, we assume that $g'(x)$ is globally bounded. Hypothesis (3.16) of that corollary is satisfied because there exists a constant $\eta > 0$ such that $G(x) \geq \eta\, x^2$ for $|x|$ sufficiently large. Using the fact that ε can be chosen arbitrarily small, we then deduce the following result.

Corollary 4.3. *Let the function $g(x)$ be continuously differentiable, with a globally bounded derivative, and such that* (3.13) *holds. Denoting by $G(x)$ a primitive of $g(x)$, assume that positive numbers $G_\pm, G^\pm$ exist for which* (4.3) *holds, these numbers being such that, either*

$$\frac{\pi}{\sqrt{G^+}} + \frac{\pi}{\sqrt{G^-}} > 2\pi\,,$$

or, for some integer n_0,

$$\frac{2\pi}{n_0+1} < \frac{\pi}{\sqrt{G^+}} + \frac{\pi}{\sqrt{G^-}} \leq \frac{\pi}{\sqrt{G_+}} + \frac{\pi}{\sqrt{G_-}} < \frac{2\pi}{n_0}\,. \tag{4.4}$$

Assume moreover that $q(t,x)$ is continuous, 2π-periodic in t, locally Lipschitz continuous in x, and satisfies (3.15)*. Then, equation* (3.14) *admits at least one 2π-periodic solution.*

The conditions (4.4) can be interpreted in terms of the Fučík spectrum for the 2π-periodic boundary value problem. They amount to requiring that the rectangle $[G_+, G^+] \times [G_-, G^-]$ lies between two successive Fučík curves (or below the first one). This is an improvement with respect to the "classical" conditions of Drábek and Invernizzi [8] for the equation considered here, which is a perturbation of a Hamiltonian equation. Indeed, our hypotheses are based on the limits, for $x \to \pm\infty$, of the ratio $2\,G(x)/x^2$, rather than on the limits of $g(x)/x$.

Notice that the assumption (4.4) does not necessarily imply that condition (3.2) holds for the periods $T(E)$. We illustrate this with the equation

$$x'' + \frac{5}{2}\,x + \frac{5}{3}\,x\,\sin(\ln(1 + 2x^+ + 3x^-)) = q(t,x)\,, \tag{4.5}$$

to which we can associate a Hamiltonian function like the one in (2.6), with

$$G(x) = \frac{5\,x^2}{4} + \frac{\sqrt{5}\,x^2}{3}\sin\Big(\ln\big(1 + 2x^+ + 3x^-\big) + \frac{\pi}{4}\Big) + O(|x|)\,, \quad \textit{for } |x| \to \infty\,.$$

We deduce from the above expression that

$$G^+ = G^- = \frac{5}{2} + \frac{2\sqrt{5}}{3}\,, \quad G_+ = G_- = \frac{5}{2} - \frac{2\sqrt{5}}{3}\,,$$

so that (4.4) is satisfied for $n_0 = 1$. However, numerical computations show that

$$\liminf_{E\to+\infty} T(E) \simeq 3.09\ldots < \pi \qquad \textit{and} \qquad \limsup_{E\to+\infty} T(E) \simeq 6.51\ldots > 2\pi\,.$$

4.3 Hamiltonians with separated variables

We now consider systems with Hamiltonian functions of the form

$$H(x,y) = G(x) + K(y)\,;$$

we will suppose that $G, K : \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable, their first derivatives being denoted respectively by $g(x)$, $k(y)$. We assume that, for some numbers $p > 1$, $q > 1$ related by (2.13),

$$g'(x) = O(|x|^{p-2})\,, \quad \textit{for } x \to \pm\infty\,, \qquad k'(y) = O(|y|^{q-2})\,, \quad \textit{for } y \to \pm\infty\,.$$

More precisely, we introduce numbers C_1, C_2 such that

$$|g'(x)| \le C_1\,|x|^{p-2}, \ \ \textit{for } |x| \ge 1\,, \qquad |k'(y)| \le C_2\,|y|^{q-2}, \ \textit{for } |y| \ge 1\,. \tag{4.6}$$

From there, it is easy to deduce constants C_1', C_2' such that

$$|g(x)| \le C_1'\,|x|^{p-1}, \ \textit{for } |x| \ge 1\,, \qquad |k(y)| \le C_2'\,|y|^{q-1}, \ \textit{for } |y| \ge 1\,. \tag{4.7}$$

We will also assume that for some positive constants c_1, c_2,

$$x\,g(x) \ge c_1\,|x|^p, \ \textit{for } |x| \ge 1\,, \qquad y\,k(y) \ge c_2\,|y|^q, \ \textit{for } |y| \ge 1\,. \tag{4.8}$$

The above hypotheses imply that there exist constants $L \ge \ell > 0$ and $C > 0$ such that

$$\ell(|x|^p + |y|^q) - C \le H(x,y) \le L(|x|^p + |y|^q) + C\,, \ \ \textit{for all } (x,y) \in \mathbb{R}^2, \tag{4.9}$$

and it can be checked that the function H satisfies the assumptions A1 to A3. We can also observe that the curves $H^{-1}(E)$ are star-shaped for E large.

Writing $r(t,x,y) = (r_x(t,x,y), r_y(t,x,y))$, we want to apply Corollary 4.1 to the system

$$x' = k(y) + r_y(t,x,y)\,, \quad -y' = g(x) + r_x(t,x,y)\,, \tag{4.10}$$

assuming r_x, r_y to be continuous, 2π-periodic in t, and locally Lipschitz continuous in (x,y). For simplicity, we only deal here with symmetric conditions on G, K, i.e. with bounds for the limits independent of the signs of x and y. We therefore take positive numbers α_1, α_2, β_1, β_2 such that

$$\alpha_1 < \liminf_{x\to\pm\infty} \frac{p\,G(x)}{|x|^p} \le \limsup_{x\to\pm\infty} \frac{p\,G(x)}{|x|^p} < \alpha_2\,,$$
$$\beta_1 < \liminf_{y\to\pm\infty} \frac{q\,K(y)}{|y|^q} \le \limsup_{y\to\pm\infty} \frac{q\,K(y)}{|y|^q} < \beta_2\,,$$

and consider the Hamiltonian functions H_1, H_2 defined by

$$H_1(x,y) = \alpha_1\,\frac{|x|^p}{p} + \beta_1\,\frac{|y|^q}{q}\,, \quad H_2(x,y) = \alpha_2\,\frac{|x|^p}{p} + \beta_2\,\frac{|y|^q}{q}\,.$$

When the exponents p, q satisfy condition (2.13), both H_1 and H_2 are isochronous (cf. (2.12), with $\mu = 0$); their respective minimal periods are given by

$$T_1 = \frac{2\,\pi_p}{\alpha_1^{1/p}\,\beta_1^{1/q}}\,, \quad T_2 = \frac{2\,\pi_p}{\alpha_2^{1/p}\,\beta_2^{1/q}}\,,$$

where

$$\pi_p = 2\,(p-1)^{1/p}\,\frac{\pi/p}{\sin(\pi/p)} \tag{4.11}$$

(see, e.g., [6, 24]). It is clear that

$$H_1(x,y) \le H(x,y) \le H_2(x,y)\,, \quad \textit{for } |(x,y)| \textit{ sufficiently large}\,.$$

In order to apply Corollary 4.1, we need to show that

$$\liminf_{E\to+\infty} T(E) > 0\,, \tag{4.12}$$

$T(E)$ denoting, as before, the minimal period of solutions of (2.1), for $E > 1$. Actually, using (4.7) and (4.8), it can be shown that $T(E)$ is bounded away from 0, as well as bounded above. More precisely,

$$\widetilde{T}_2 \le \liminf_{E\to+\infty} T(E) \le \limsup_{E\to+\infty} T(E) \le \widetilde{T}_1\,,$$

$\widetilde{T}_1$, $\widetilde{T}_2$ being the minimal period of the nontrivial solutions of the equations associated with the Hamiltonians

$$\widetilde{H}_1(x,y) = c_1\frac{|x|^p}{p} + c_2\frac{|y|^q}{q}\,, \quad \widetilde{H}_2(x,y) = C_1'\frac{|x|^p}{p} + C_2'\frac{|y|^q}{q}\,,$$

where c_1, c_2, C_1', C_2' are the constants appearing in (4.7) and (4.8). We omit the proof, for briefness, the arguments being similar to those of Theorem 8.2 below. Notice that

$$\widetilde{T}_2 \le T_2 \le T_1 \le \widetilde{T}_1\,,$$

strict inequalities being possible.

Considering that (4.12) holds, we deduce the following result from Corollary 4.1.

Corollary 4.4. *Let $H(x,y) = G(x) + K(y)$, with G, K twice continuously differentiable functions, their first derivatives being denoted by $g(x)$, $k(y)$, respectively. Assume that the conditions* (4.6), (4.8) *hold, and that the exponents p, q satisfy condition* (2.13). *Assume moreover that*

$$r_x(t,x,y) = o(|x|^{p-1})\,, \quad \text{for } x \to \pm\infty, \ \text{uniformly in } t, y\,, \tag{4.13}$$
$$r_y(t,x,y) = o(|y|^{q-1})\,, \quad \text{for } y \to \pm\infty, \ \text{uniformly in } t, x\,. \tag{4.14}$$

Then, system (4.10) *admits at least one 2π-periodic solution, provided that, either $T_2 > 2\pi$, or there exists an integer n_0 such that* (4.2) *holds.*

Proof. We have seen that assumptions A1 to A3 are satisfied. Moreover, it results from (4.7), (4.9), (4.13), (4.14), using Young's inequality again, that condition A6 is satisfied.

Let us show that, if we take $\mathcal{D}(E) = \operatorname{diag}(E^{\frac{1}{q}}, E^{\frac{1}{p}})$, condition A8 will be satisfied, implying that the same holds true for A7.

Let $\varphi(t;E)$ be a solution of energy E for the Hamiltonian system

$$x' = k(y)\,, \quad -y' = g(x)\,.$$

As explained below, we will need later a solution built in a particular way; but, at this stage, this particular construction plays no role. Denoting by $\varphi_x(t;E)$, $\varphi_y(t;E)$, the components of the solution $\varphi(t;E)$ we compute, with $v = (v_1, v_2)$,

$$\big\langle \mathcal{D}(E) J H''(\varphi(t;E)) \mathcal{D}^{-1}(E)\, v, v \big\rangle = \Big(E^{1-\frac{2}{q}}\, g'(\varphi_x(t;E)) - E^{1-\frac{2}{p}}\, k'(\varphi_y(t;E)) \Big)\, v_1 v_2\,.$$

The inequality (3.4) then holds if we define

$$\kappa(x,y) = E^{1-\frac{2}{q}}\,|g'(x)| + E^{1-\frac{2}{p}}\,|k'(y)|\,.$$

We now have to to show that, with this definition, κ satisfies (3.5). Let us consider, for instance, the case where $p \geq 2$ (the case $q \geq 2$ being analogous). Using (4.6), (4.9) and the relation (2.13) between p and q, we observe that $E^{1-\frac{2}{q}}\,g'(\varphi_x(t;E))$ is bounded, independently of t and E. The same property holds for $E^{1-\frac{2}{p}}\,k'(\varphi_y(t;E))$, as long as $|\varphi_y(t;E)| \geq 1$. To prove (3.5), it remains to show that, given any compact interval I,

$$\int_{\Sigma(E)} E^{1-\frac{2}{p}}\,|k'(\varphi_y(s;E))|\,ds \quad \textit{remains bounded for } E \to +\infty\,,$$

where $\Sigma(E) = \{t \in I : |\varphi_y(t;E)| \leq 1\}$. The curve $H^{-1}(E)$ being star-shaped for E large, and the motion being clockwise, the set $\Sigma(E)$ is contained in the union of intervals, corresponding to transitions between the values -1 and $+1$ for the function $\varphi_y(t;E)$. Because of (4.12), the number of those intervals can be assumed to be finite. Considering for instance one of those intervals, let $t_1, t_2 \in [0, T(E)]$, with $t_1 < t_2$, be such

$$\varphi_y(t_1;E) = +1\,, \quad \varphi_y(t_2;E) = -1\,, \quad \varphi_y(t;E) \in [-1,1]\,, \quad \textit{for } t \in [t_1,t_2]\,.$$

Assuming E sufficiently large, we have $\varphi_x(t;E) > 0$, for $t \in [t_1,t_2]$. Moreover, by (4.8) and (4.9), we see that if $H(x,y) = E$ and if $|y| \leq 1$, then $|g(x)| \geq c_0\,E^{1-\frac{2}{p}}$ for some constant $c_0 > 0$, so that the equation $-y' = g(x)$ leads to

$$2 = \int_{t_1}^{t_2} g(\varphi_x(s;E)\,ds \geq (t_2 - t_1)\,c_0\,E^{1-\frac{2}{p}}.$$

It then follows that $t_2 - t_1 = O\big(E^{\frac{2}{p}-1}\big)$ for $E \to +\infty$. Consequently, since $|\varphi_y(t;E)| \leq 1$ for $t \in [t_1,t_2]$, we have that

$$\int_{t_1}^{t_2} E^{1-\frac{2}{p}}\,|k'(\varphi_y(s;E))|\,ds \quad \textit{remains bounded for } E \to +\infty\,.$$

This, combined with the observations made above, finally proves that the function κ indeed satisfies (3.5).

To establish A8, we still have to prove that (3.6) holds. From (4.13) and (4.14) we deduce that

$$\lim_{E\to+\infty} (\mathcal{D}^T(E))^{-1} r(t,z) = 0\,, \text{ uniformly for } t \in [0,2\pi], z \in H^{-1}(E)\,,$$

so that (3.6) will be satisfied if

$$\frac{|\mathcal{D}(E)\,\nabla H(\varphi(0;E))|}{|\nabla H(\varphi(0;E))|^2} \;\; \textit{remains bounded, for } E \to +\infty\,. \tag{4.15}$$

We will show that this is the case if the function $\varphi(0;E)$ has been built in an appropriate way. Notice first that, by (4.8), there exists y_0 such that $k(y_0) = 0$. Moreover, the function G is strictly increasing for $x \geq 1$, so that, for $E \geq G(1) + K(y_0)$, we may define $\varphi(0;E)$ by

$$\varphi(0;E) = (x_E, y_0)\,,$$

with $x_E \geq 1$ such that $G(x_E) = E - K(y_0)$. Since $k(y_0) = 0$, we have $\nabla H(\varphi(0;E)) = (g(x_E), 0)$ and it is easily checked that $\varphi(0;E)$ satisfies (2.5), at least for E "large". Condition (4.15) then amounts to finding a constant $C > 0$ such that

$$E^{1/q} \leq C\, g(x_E), \text{ independently of } E \geq G(1) + K(y_0)\,.$$

Taking into account the relation (2.13) between p and q, the fact that such a constant can be found is then deduced from (4.7), (4.8). Condition A8 is thus satisfied, and the conclusion then follows from Corollary 4.1. □

The case where $g(x) = \alpha_0|x|^p$, $k(y) = |y|^q$ is covered by results of Jiang [22], who also deals with asymmetric functions $g(x)$. (See also [2] for a more general system.) It must be emphasized again that our hypotheses are based on the limits of the ratios $G(x)/|x|^p$, $K(y)/|y|^q$, yielding less restrictive conditions with respect to more classical assumptions based on the limits of the ratios $x\,g(x)/|x|^p$, $y\,k(y)/|y|^q$. By Theorem 6.1 below, it will also be possible to deal with the case where $(1/p) + (1/q) > 1$; in that situation, besides conditions (4.6), (4.8) and (4.13) – (4.14), no further hypotheses will be needed.

5 Proof of Theorem 3.1

The proof is based upon degree arguments. We will use the homotopy

$$Jz' = \nabla H(z) + \lambda\, r(t,z)\,, \tag{5.1}$$

with $\lambda \in [0,1]$, and denote by $P_{2\pi}^{(\lambda)}$ the Poincaré map for the period 2π, associated with the above equation. We look for fixed points of $P_{2\pi}^{(1)}$, which correspond to 2π-periodic solutions of (1.3).

We first need to prove that $P_{2\pi}^{(\lambda)}$ is well-defined for $\lambda \in [0,1]$, and continuous. Since we will assume $\nabla H(z)$ and $r(t,z)$ to be locally Lipschitz continuous in z, uniqueness of the solutions of (5.1) and continuity with respect to initial conditions are guaranteed. It remains to show that the solutions of (5.1) do not escape to infinity. This is a consequence of condition A6. Indeed, if $z^{(\lambda)}(t)$ denotes a solution and if we define $e^{(\lambda)}(t) = H(z^{(\lambda)}(t))$, we have

$$(e^{(\lambda)})'(t) = \lambda \langle J\nabla H(z^{(\lambda)}(t)), r(t, z^{(\lambda)}(t))\rangle\,. \tag{5.2}$$

We then deduce from A6 that, for some positive constants C', C'',

$$|(e^{(\lambda)})'(t)| \le C' e^{(\lambda)}(t) + C'', \tag{5.3}$$

showing that $e^{(\lambda)}(t)$ remains bounded on any compact interval. It then follows, by the coercivity condition (2.2), that $z^{(\lambda)}(t)$ can be extended to the whole real line.

We are now in a position to formulate a lemma which describes the guiding idea of the proof of Theorem 3.1. We recall that, for $E_0 > 1$,

$$\Omega(E_0) = \{z \in \mathbb{R}^2 : H(z) < E_0\} \cup \mathrm{int}(\Gamma_1)\,.$$

Lemma 5.1. *Let the assumptions A1 to A3 hold, as well as L1, L2, and let $E_0 > 1$ be such that $T(E_0) \neq 2\pi/n$, for any integer n. Then, we have* $\deg(I - P_{2\pi}^{(0)}, \Omega(E_0), 0) = 1$. *Moreover, if*

$$P_{2\pi}^{(\lambda)}(z_0) \neq z_0\,,\ \textit{for any } z_0 \in H^{-1}(E_0) \textit{ and any } \lambda \in [0,1]\,, \tag{5.4}$$

then $\deg(I - P_{2\pi}^{(1)}, \Omega(E_0), 0) = 1$ *and, consequently, equation* (1.3) *admits at least one 2π-periodic solution.*

Proof. The degree $\deg(I - P_{2\pi}^{(0)}, \Omega(E_0), 0)$ is clearly well-defined if $T(E_0) \neq 2\pi/n$, for all integer n. Indeed, we then have $P_{2\pi}^{(0)}(z_0) \neq z_0$, for all z_0 belonging to the boundary of $\Omega(E_0)$. Moreover, the closed set $\overline{\Omega(E_0)}$, which is homeomorphic to a closed ball, is mapped into itself by $P_{2\pi}^{(0)}$, so that the result concerning $I - P_{2\pi}^{(0)}$ follows from Brouwer's theorem. Finally, using hypothesis (5.4), the property of invariance of the degree with respect to a homotopy implies that

$$\deg(I - P_{2\pi}^{(1)}, \Omega(E_0), 0) = \deg(I - P_{2\pi}^{(0)}, \Omega(E_0), 0) = 1\,,$$

so that $P_{2\pi}^{(1)}$ has a fixed point in $\Omega(E_0)$. $\square$

To apply the above lemma for proving Theorem 3.1, we need to estimate $P_{2\pi}^{(\lambda)}(z_0)$. This will be done by considering the large amplitude solutions of equation (5.1) as perturbations of the solutions of the autonomous equation (1.4). We therefore write the solutions of (5.1) under the form

$$z(t) = \varphi\big(t + \tau^{(\lambda)}(t); e^{(\lambda)}(t)\big)\,,$$

so that

$$\begin{aligned}\nabla H\big(\varphi\big(t+\tau^{(\lambda)}(t);e^{(\lambda)}(t)\big)\big)(\tau^{(\lambda)})'(t) + J\frac{\partial\varphi}{\partial E}\big(t+\tau^{(\lambda)}(t);e^{(\lambda)}(t)\big)(e^{(\lambda)})'(t) =\\ = \lambda\, r\big(t, \varphi\big(t+\tau^{(\lambda)}(t);e^{(\lambda)}(t)\big)\big)\,.\end{aligned}$$

Simple calculations making use of (2.3) then lead to the system

$$(\tau^{(\lambda)})' = \lambda\,\Big\langle\, \frac{\partial\varphi}{\partial E}\big(t+\tau^{(\lambda)};e^{(\lambda)}\big), r\big(t,\varphi\big(t+\tau^{(\lambda)};e^{(\lambda)}\big)\big)\Big\rangle\,, \tag{5.5}$$

$$(e^{(\lambda)})' = \lambda\,\big\langle J\nabla H\big(\varphi\big(t+\tau^{(\lambda)};e^{(\lambda)}\big)\big), r\big(t,\varphi\big(t+\tau^{(\lambda)};e^{(\lambda)}\big)\big)\big\rangle\,. \tag{5.6}$$

Notice by the way that this last equation is just a rewrite of (5.2). Let us denote the solution of the above system for the initial conditions $\tau^{(\lambda)}(0) = \tau_0$, $e^{(\lambda)}(0) = E_0$ by $(\tau^{(\lambda)}(t;\tau_0,E_0), e^{(\lambda)}(t;\tau_0,E_0))$, or briefly by $(\tau^{(\lambda)}(t), e^{(\lambda)}(t))$, when there is no risk of ambiguity in omitting the initial conditions. The basic point for the proof of our existence results is the observation that (5.4) will be satisfied unless, for some integer n, some $\tau_0 \in [0, T(E_0)]$, and some $\lambda \in [0,1]$, we have

$$e^{(\lambda)}(2\pi;\tau_0,E_0) = E_0\,, \quad 2\pi + \tau^{(\lambda)}(2\pi;\tau_0,E_0) = \tau_0 + n\,T(E_0)\,. \tag{5.7}$$

By assumptions A4 and A5, it is always possible to find a sequence $(E_k)_k$, with $\lim_k E_k = +\infty$, such that $T(E_k)$ converges to some strictly positive finite value T^*, with $T^* \neq 2\pi/n$ for any integer n. Hence, a number $\eta > 0$ exists such that, for k sufficiently large, $|2\pi - n\,T(E_k)| \geq \eta$, for any integer n. On the other hand, it follows from (5.3) that

$$\lim_k e^{(\lambda)}(t;\tau_0,E_k) = +\infty\,,\ \textit{uniformly in } (t,\tau_0,\lambda) \in [0,2\pi]\times[0,2T^*]\times[0,1]\,,$$

and, considering (5.5), we deduce from A7 that

$$\lim_k \tau^{(\lambda)}(t;\tau_0,E_k) = \tau_0\,,\ \textit{uniformly in } (t,\tau_0,\lambda) \in [0,2\pi]\times[0,2T^*]\times[0,1]$$

(the above choice of the interval $[0,2T^*]$ is somehow arbitrary; what is needed, is just an interval going beyond T^*). Consequently, the second equality in (5.7) is impossible for any integer n, when E_0 is replaced by a sufficiently large element E_k. □

6 Some extensions of Theorem 3.1

In this section, we extend Theorem 3.1 in several directions. First, in Section 6.1, we consider the case when the limit of the period function $T(E)$ is $+\infty$ as $E \to +\infty$. Then, the critical case when the limit of $T(E)$ is equal to some $2\pi/n_0$ is considered, with two different approaches: in Section 6.2 we approach resonance, but "not too fast", while in Section 6.3 we add some conditions of Landesman–Lazer type.

6.1 The case when $\lim_{E\to+\infty} T(E) = +\infty$

The nonresonance conditions A4 - A5 are satisfied when

$$\limsup_{E\to+\infty} T(E) > 2\pi \quad \textit{and} \quad \liminf_{E\to+\infty} T(E) < +\infty\,,$$

so that Theorem 3.1 can be invoked to deal with such situations. On the other hand, Theorem 3.1 does not apply when

$$\lim_{E\to+\infty} T(E) = +\infty\,. \tag{6.1}$$

However, it is still possible to obtain existence conditions for this last case, as shown by the next theorem, where an auxiliary Hamiltonian function H_0 is introduced. We associate with it a function $\varphi_0(0;E)$ defined by an equation of the type (2.4).

Theorem 6.1. *Let the assumptions A1 to A3 hold, as well as L1, L2, A6, A7, and*

$$\limsup_{E\to+\infty} T(E) > 2\pi \,. \tag{6.2}$$

Assume also the following condition:

A9. *There exist a differentiable function $H_0 : \mathbb{R}^2 \to \mathbb{R}$, satisfying the assumptions A1 to A3, and a constant $E^* > 0$ such that*

$$\langle \nabla H_0(z), \nabla H(z) + \lambda\, r(t,z)\rangle > 0 \,, \text{ with } z = \varphi_0(0;E)\,,$$
$$\text{for every } (t,\lambda) \in [0,2\pi]\times[0,1] \text{ and } E \geq E^*.$$

Then, equation (1.3) *admits at least one 2π-periodic solution.*

Proof. As already observed, we only need to consider the case when (6.1) holds. Referring to Lemma 5.1, we aim to show that, for sufficiently large values of E_0,

$$P_{2\pi}^{(\lambda)}(z_0) \neq z_0 \,, \text{ for any } z_0 \in H^{-1}(E_0) \text{ and any } \lambda \in [0,1]\,,$$

where $P_{2\pi}^{(\lambda)}$ denotes, as before, the Poincaré map for the period 2π, associated with equation (5.1). Assume by contradiction that this is not true. Let $z^{(\lambda)}(t;0,z_0)$ denote the solution of (5.1) corresponding to the initial condition $z(0) = z_0$; that solution is assumed to be 2π-periodic. Arguing as in the proof of Theorem 3.1, and using hypothesis A6, we can show that $H(z^{(\lambda)}(t;0,z_0))$ can be made arbitrarily large by choosing E_0 large enough, uniformly for $t \in [0,2\pi]$. Provided that E_0 is taken sufficiently large, it then results from A9 that, when the trajectory of the solution $z^{(\lambda)}(t;0,z_0)$ crosses the gradient curve $E \mapsto \varphi_0(0;E)$, associated with the Hamiltonian H_0, the crossing occurs in the clockwise direction. That solution being, by assumption, 2π-periodic, we then conclude that, on the interval $[0,2\pi]$, the trajectory must make at least one turn around the origin, in the clockwise direction. We will show that the other hypotheses prevent this possibility.

For this aim, we want to use the same arguments as in the proof of Theorem 2. Some modification is needed however, because condition A7 holds only for s in a compact set, whereas the natural domain of this variable is $[0,T(E)]$, with the period $T(E)$ of the free oscillations going to $+\infty$, for $E \to +\infty$. We will therefore manage to consider only values of the argument s of $\partial\varphi(s;E)/\partial E$ in an interval slightly larger than $[0,2\pi]$.

The curve $[0, 2\pi] \to \mathbb{R}^2 : t \mapsto z^{(\lambda)}(t; 0, z_0)$, making at least one turn in the phase plane, must also cross all the gradient curves associated with the Hamiltonian H, provided that $E_0 = H(z_0)$ is large enough. It will cross in particular the curve $E \mapsto \varphi(0; E)$. Therefore, we can find a value $t^* \in [0, 2\pi]$ such that

$$z^{(\lambda)}(t^*; 0, z_0) = \varphi(0; H(z^{(\lambda)}(t^*; 0, z_0)))\,.$$

Define then

$$\widetilde{z}(t) = z^{(\lambda)}(t + t^*; 0, z_0)\,,$$

a solution of

$$J\widetilde{z}\,' = \nabla H(\widetilde{z}) + \lambda\, r(t + t^*, \widetilde{z})\,,$$

with the initial condition $\widetilde{z}(0) = z^{(\lambda)}(t^*; 0, z_0) = \varphi(0; H(\widetilde{z}(0)))$. We have already observed that $H(z^{(\lambda)}(t^*; 0, z_0))$ can be made arbitrarily large by choosing E_0 large enough. We will now compare $\widetilde{z}(t)$ with $\varphi(t; H(\widetilde{z}(0))$ for $t \in [0, 2\pi]$, and show that, because of hypothesis A7, the difference in "phase" remains "small". More precisely, letting

$$\widetilde{z}(t) = \varphi\big(t + \tau^{(\lambda)}(t); e^{(\lambda)}(t)\big)\,,$$

we see that the function $\tau^{(\lambda)}$ satisfies an equation similar to (5.5), i.e.,

$$(\tau^{(\lambda)})' = \lambda \left\langle \frac{\partial \varphi}{\partial E}\big(t + \tau^{(\lambda)}; e^{(\lambda)}\big), r\big(t + t^*, \varphi\big(t + \tau^{(\lambda)}; e^{(\lambda)}\big)\big) \right\rangle,$$

with the initial conditions $\tau^{(\lambda)}(0) = 0$, $e^{(\lambda)}(0) = H(\widetilde{z}(0))$. The hypothesis of the contradiction argument would then imply that

$$2\pi + \tau^{(\lambda)}(2\pi; 0, H(\widetilde{z}(0))) = n\, T(H(\widetilde{z}(0)))\,, \tag{6.3}$$

for some integer $n \neq 0$. Using A7 (with s in an interval slightly larger than $[0, 2\pi]$) and working as in the proof of Theorem 3.1, it can be proved that

$$\lim_{E_0 \to +\infty} \tau^{(\lambda)}(2\pi; 0, H(\widetilde{z}(0))) = 0\,, \ \textit{uniformly in } z_0 \in H^{-1}(E_0)\,,$$

showing that the equality (6.3) is impossible if E_0 is taken sufficiently large, since $\lim_{E \to +\infty} T(E) = +\infty$. □

It is, of course, admissible to choose $H_0 = H$ in assumption A9. Therefore, that condition is fulfilled if there exists a $E^* > 1$ such that

$$|r(t, \varphi_0(0; E))| < |\nabla H(\varphi_0(0; E))|\,, \ \ \textit{for every } t \in [0, 2\pi] \textit{ and } E \geq E^*\,.$$

This observation, together with those made at the end of Section 3.3, lead to the following corollary.

Corollary 6.2. *Let the assumptions A1 to A3 hold, as well as* (6.2)*. With H twice continuously differentiable and r satisfying L2, let H'' be globally bounded, and assume that* (3.8) *holds. Then, equation* (1.3) *admits at least one 2π-periodic solution.*

We remark that a similar situation has been considered by Fernandes and Zanolin in [12] for a second order scalar equation of the type (1.1). See also [19, 20].

6.2 Approaching resonance

When

$$\lim_{E\to+\infty} T(E) = \frac{2\pi}{n_0}\,, \quad \textit{for some positive integer } n_0\,, \tag{6.4}$$

Theorem 3.1 cannot be invoked to prove the existence of 2π-periodic solutions for equation (1.3). But, adapting the arguments of the proof of Theorem 3.1, it is still possible to provide some existence conditions. This is the object of the next theorem.

Theorem 6.3. *Let the Hamiltonian H satisfy assumptions A1 to A3, and be such that* (6.4) *holds. Assume that L1, L2 hold, that*

A6'.

$$\lim_{|z|\to\infty} \frac{\langle J\nabla H(z), r(t,z)\rangle}{H(z)} = 0\,, \quad \textit{uniformly in } t\in[0,2\pi]\,,$$

and that

A7'. *there exists a number $\gamma > 0$, and a constant $C \geq 0$ such that, for any given compact interval $I \subseteq \mathbb{R}$,*

$$\limsup_{E\to+\infty} E^\gamma \left| \left\langle \frac{\partial\varphi}{\partial E}(s;E), r(t,\varphi(s;E)) \right\rangle \right| \leq C\,,$$
$$\textit{uniformly for } (t,s)\in[0,2\pi]\times I\,.$$

Then, equation (1.3) *admits at least one 2π-periodic solution, provided that, either*

$$\limsup_{E\to+\infty} E^\gamma \left(n_0\, T(E) - 2\pi\right) > 2\pi\, C\,, \tag{6.5}$$

or

$$\liminf_{E\to+\infty} E^\gamma \left(n_0\, T(E) - 2\pi\right) < -2\pi\, C\,. \tag{6.6}$$

Proof. By hypotheses (6.5), (6.6), it is possible to find a sequence $(E_k)_k$, with $\lim_k E_k = +\infty$, such that

$$T(E_k) \neq \frac{2\pi}{n_0}\,, \quad \textit{for all } k\,,$$

so that, by (6.4) and Lemma 5.1, the degree $\deg(I - P_{2\pi}^{(0)}, \Omega(E_k), 0)$ is equal to 1. Working as in the proofs of Theorem 3.1 and Theorem 6.1, we want to show that, for k sufficiently large,

$$P_{2\pi}^{(\lambda)}(z_0) \neq z_0\,, \textit{ for any } z_0 \in H^{-1}(E_k), \textit{ and any } \lambda \in [0,1]\,.$$

where $P_{2\pi}^{(\lambda)}$ denotes, as usual, the Poincaré map for the period 2π associated with equation (5.1). Writing, as in the proof of Theorem 3.1, the solutions of equation (5.1) under the form

$$z(t) = \varphi\big(t + \tau^{(\lambda)}(t); e^{(\lambda)}(t)\big)\,,$$

we have to find values E_k such that, for any integer n,

$$2\pi + \tau^{(\lambda)}(2\pi; \tau_0, E_k) \neq \tau_0 + n\,T(E_k), \textit{ for all } \tau_0 \in \left[0, \frac{4\pi}{n_0}\right] \textit{ and } \lambda \in [0,1]. \quad (6.7)$$

As already explained, the choice of the interval $[0, 4\pi/n_0]$ is somehow arbitrary, as long as it contains $2\pi/n_0$ in its interior. Notice that, A6', A7' being stronger than A6, A7, it is clear that

$$\lim_k \tau^{(\lambda)}(t; \tau_0, E_k) = \tau_0\,, \quad \textit{uniformly in } t \in [0, 2\pi], \tau_0 \in \left[0, \frac{4\pi}{n_0}\right], \lambda \in [0,1]\,,$$

so that, by (6.4), the above inequality is certainly verified if $n \neq n_0$. Referring to the system (5.5)–(5.6), and using the same notation $\tau^{(\lambda)}(t; \tau_0, E_0)$, $e^{(\lambda)}(t; \tau_0, E_0)$ as in the proof of Theorem 3.1, we deduce from A6' that

$$\lim_{E_0 \to +\infty} \frac{e^{(\lambda)}(t; \tau_0, E_0))}{E_0} = 1\,, \textit{ uniformly in } t \in [0, 2\pi], \tau_0 \in \left[0, \frac{4\pi}{n_0}\right], \lambda \in [0,1]\,.$$

Using that result, it follows from A7' that

$$\limsup_{E_0 \to +\infty} E_0^{\gamma} \left|\tau^{(\lambda)}(t; \tau_0, E_0)) - \tau_0\right| \leq 2\pi\, C\,,$$

$$\textit{uniformly in } t \in [0, 2\pi], \tau_0 \in \left[0, \frac{4\pi}{n_0}\right], \lambda \in [0,1]\,. \quad (6.8)$$

The combination of (6.8) with (6.5) or (6.6) then implies that large values E_k can be found for which (6.7) holds also with $n = n_0$. □

We notice that similar nonresonance results have been proposed by Hao and Ma in [21] for the second order equation (1.1), with $\gamma = 1/2$. A somewhat related approach to resonance was also proposed by Omari and Zanolin in [26].

We now provide an example of application of Theorem 6.3 to the second order equation

$$x'' + \frac{m^2}{4}\, x^+ - a\,[x^-]^{p-1} = q(t,x)\,, \tag{6.9}$$

where m is an integer, $p > 2$, $a > 0$, and where $q(t,x)$ is assumed to be continuous and 2π-periodic in t, with

$$q(t,x) = o(|x|^{2/p})\,,\ \textit{for } |x| \to \infty\,,\ \textit{uniformly in } t \in [0,2\pi]\,. \tag{6.10}$$

This condition is surely satisfied if $q(t,x)$ is globally bounded. The Hamiltonian function associated to the unperturbed equation is

$$H(x,y) = \frac{y^2}{2} + \frac{m^2}{8}\,[x^+]^2 + \frac{a}{p}\,[x^-]^p, \tag{6.11}$$

and the minimal periods of the free oscillations are given by

$$T(E) = \frac{2\pi}{m} + T^-(E)\,,$$

where $T^-(E)$ is the transit time in the negative phase plane. It can be computed that

$$T^-(E) = \frac{2^{\frac{1}{2}+\frac{1}{p}}\sqrt{\pi}\ \Gamma\left(1+\frac{1}{p}\right)}{a^{\frac{1}{p}}\,E^{\frac{1}{2}-\frac{1}{p}}\,\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}\,,$$

where $\Gamma(\cdot)$ is the Euler gamma function. Consequently,

$$\begin{aligned} &\lim_{E\to+\infty} T(E) = \frac{2\pi}{m}\,,\\ &\lim_{E\to+\infty} E^{\frac{1}{2}-\frac{1}{p}}\left(T(E)-\frac{2\pi}{m}\right) = \frac{2^{\frac{1}{2}+\frac{1}{p}}\sqrt{\pi}\ \Gamma\left(1+\frac{1}{p}\right)}{a^{\frac{1}{p}}\,\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}\,. \end{aligned}$$

The condition (6.5) is thus fulfilled with $\gamma = (1/2) - (1/p)$, for some positive constant C. Consider now the solution $\varphi(t;E)$ associated with the autonomous equation for the Hamiltonian (6.11), built from the initial value

$\varphi(0;1) = (0, \sqrt{2})$. It is fairly immediate that

$$\varphi(t;E) = \left(\sqrt{2E}\,\sin\left(\frac{m\,t}{2}\right), \sqrt{2E}\,\cos\left(\frac{m\,t}{2}\right)\right), \ \textit{for } t \in \left[0, \frac{2\pi}{m}\right].$$

On the other hand, denoting by φ_x, φ_y the components of φ, and adapting (2.12) with $\mu = 1 - (1/2) - (1/p)$, we see that

$$\varphi(t;E) = \left(E^{\frac{1}{p}}\,\varphi_x(E^{\mu}\,t;1), E^{\frac{1}{2}}\,\varphi_y(E^{\mu}\,t;1)\right), \ \textit{for } t \in \left[\frac{2\pi}{m}, \frac{2\pi}{m} + T^-(E)\right].$$

From there, it can be checked that $\sqrt{E}\,\partial\varphi_x(t;E)/\partial E$ remains bounded for $E \to +\infty$, uniformly for t in compact sets. Hence, using (6.10), it follows that A7' holds with $\gamma = (1/2) - (1/p)$, the limit being equal to 0. Moreover, under (6.10), hypothesis A6' is also satisfied. We conclude by Theorem 6.3 that equation (6.9) admits at least one 2π-periodic solution.

Remark 6.4. Actually, the conclusion still holds for the equation

$$x'' + k\,x^+ - a\,[x^-]^{p-1} = q(t,x)\,,$$

no matter what value the coefficient $k > 0$ takes. Indeed, if $k \neq m^2/4$ for any integer m, Theorem 3.1 applies.

Remark 6.5. Similar situations have been considered in [4, 5, 15, 31] for more general nonlinearities. Since problems “near resonance” are concerned, some restrictions must be imposed on the nonlinearity (in our approach, condition (6.10)). They may take the form of conditions of Landesman–Lazer type.

6.3 Landesman–Lazer conditions

A huge literature exists concerning existence conditions for periodic solutions based on the so-called Landesman–Lazer conditions (see, for instance, [13] and [25] for references). We want to discuss briefly the relation between those conditions and the results presented above.

Since the forcing term $r(t,z)$ in equation (1.3) is 2π-periodic in t, Landesman–Lazer conditions would typically concern situations where the Hamiltonian function H is isochronous (at least for solutions of large amplitude), with solutions having a minimal period of the form $2\pi/n_0$, for some positive integer n_0. In that case, it is no longer possible to resort directly to Lemma 5.1, since $\deg(I - P_{2\pi}^{(0)}, \Omega(E_0), 0)$ is no longer defined. But the arguments still work with an adapted homotopy, i.e.,

$$Jz' = [1 + (1-\lambda)\,\sigma]\,\nabla H(z) + \lambda\, r(t,z)\,, \tag{6.12}$$

the choice of $\sigma \neq 0$, its sign in particular, being explained below. It is immediate that, for $\lambda = 0$, the minimal period of the large solutions now becomes $2\pi/(n_0(1+\sigma))$. Hence, if $P_{2\pi}^{(\lambda)}$ now denotes the Poincaré map for the period 2π associated with equation (6.12), the arguments used in Lemma 5.1 tell us that, provided that $\sigma \neq 0$ is taken small enough, the degree $\deg(I - P_{2\pi}^{(0)}, \Omega(E_0), 0)$ is well-defined for E_0 large, and equal to 1. In order to obtain an existence result, it remains once more to find conditions ensuring that, for well-chosen values E_0,

$$P_{2\pi}^{(\lambda)}(z_0) \neq z_0\,, \quad \textit{for any } z_0 \in H^{-1}(E_0) \textit{ and any } \lambda \in [0,1]\,. \tag{6.13}$$

Here is our result.

Theorem 6.6. *Let the assumptions A1 to A3 hold, as well as L1, L2, A6' and A7'. Let H be isochronous with period $2\pi/n_0$. With γ the constant appearing in assumption A7', assume that there exists $\eta > 0$ such that either, for every $\tau_0 \in [0, 4\pi/n_0]$,*

$$\int_0^{2\pi} \liminf_{E\to+\infty} E^\gamma \min_{|s-t|\le\eta} \left\langle \frac{\partial\varphi}{\partial E}(\tau_0+s;E), r(t,\varphi(\tau_0+s;E)) \right\rangle dt > 0\,, \tag{6.14}$$

or, for every $\tau_0 \in [0, 4\pi/n_0]$,

$$\int_0^{2\pi} \limsup_{E\to+\infty} E^\gamma \min_{|s-t|\le\eta} \left\langle \frac{\partial\varphi}{\partial E}(\tau_0+s;E), r(t,\varphi(\tau_0+s;E)) \right\rangle dt < 0\,. \tag{6.15}$$

Then, equation (1.3) *admits at least one 2π-periodic solution.*

Proof. As in the proof of Theorem 3.1, we write the solutions of equation (6.12) under the form

$$z(t) = \varphi\big(t + \tau^{(\lambda)}(t); e^{(\lambda)}(t)\big)\,.$$

Adapting (5.5), (5.6) leads to

$$(\tau^{(\lambda)})' = (1-\lambda)\,\sigma + \lambda \left\langle \frac{\partial \varphi}{\partial E}\big(t+\tau^{(\lambda)}; e^{(\lambda)}\big), r\big(t, \varphi\big(t+\tau^{(\lambda)}; e^{(\lambda)}\big)\big) \right\rangle, \tag{6.16}$$

$$(e^{(\lambda)})' = [1+(1-\lambda)\,\sigma]\,\big\langle J\nabla H\big(\varphi\big(t+\tau^{(\lambda)}; e^{(\lambda)}\big)\big), r\big(t, \varphi\big(t+\tau^{(\lambda)}; e^{(\lambda)}\big)\big)\big\rangle. \tag{6.17}$$

Assume that (6.14) holds. In this case, since $T(E_0) = 2\pi/n_0$, we will take $\sigma > 0$ small enough, in order to show that, for any integer n,

$$2\pi + \tau^{(\lambda)}(2\pi; \tau_0, E_0) \neq \tau_0 + n\,\frac{2\pi}{n_0}\,, \ \textit{for all } \tau_0 \in \left[0, \frac{4\pi}{n_0}\right] \textit{ and } \lambda \in [0,1]\,. \tag{6.18}$$

Using A6' in (6.17), we have that

$$\lim_{E_0 \to +\infty} \frac{e^{(\lambda)}(t; \tau_0, E_0))}{E_0} = 1\,, \ \textit{uniformly in } t \in [0, 2\pi], \tau_0 \in \left[0, \frac{4\pi}{n_0}\right], \lambda \in [0,1]\,,$$

while using A7' in (6.16), we get, for any compact interval $I \subseteq \mathbb{R}$,

$$\lim_{E \to +\infty} \left\langle \frac{\partial \varphi}{\partial E}(s; E), r(t, \varphi(s; E)) \right\rangle = 0\,,$$
$$\textit{uniformly for}\,(t,s) \in [0, 2\pi] \times I\,,$$

and hence, by Lebesgue's Theorem,

$$\limsup_{E_0 \to +\infty} |\tau^{(\lambda)}(t; \tau_0, E_0) - \tau_0| \leq 2\pi\,\sigma\,,$$
$$\textit{uniformly in } t \in [0, 2\pi], \tau_0 \in \left[0, \frac{4\pi}{n_0}\right], \lambda \in [0,1]\,.$$

Then, taking $\sigma > 0$ small enough, we see that (6.18) could hold only if $n = n_0$; moreover, by (6.14),

$$\int_0^{2\pi} \liminf_{E \to +\infty} E^{\gamma} \min_{|s-t| \leq 3\pi\sigma} \left\langle \frac{\partial \varphi}{\partial E}(\tau_0 + s; E), r(t, \varphi(\tau_0 + s; E)) \right\rangle dt > 0\,,$$

for every $\tau_0 \in [0, 4\pi/n_0]$. It then follows from Fatou's Lemma and A7' that there exists a number $\eta > 0$ such that

$$\liminf_{E_0 \to +\infty} E_0^{\gamma} \left(\tau^{(\lambda)}(2\pi; \tau_0, E_0) - \tau_0\right) \geq \eta\,,$$
$$\textit{uniformly in } \tau_0 \in \left[0, \frac{4\pi}{n_0}\right], \lambda \in [0,1]\,. \tag{6.19}$$

Hence, if E_0 is taken large enough,

$$\tau^{(\lambda)}(2\pi;\tau_0,E_0)\neq\tau_0\,,\quad \textit{for any } \tau_0\in\left[0,\frac{4\pi}{n_0}\right]\ \textit{ and } \lambda\in[0,1]\,.$$

showing that (6.18) cannot hold even if $n=n_0$.

The case when (6.15) holds can be treated similarly. □

Remark 6.7. We notice that it is not really necessary that H be isochronous of period $2\pi/n_0$; under the above conditions, it suffices that

$$\lim_{E_0\to+\infty}E_0^\gamma\left(T(E_0)-\frac{2\pi}{n_0}\right)=0\,.$$

Remark 6.8. Conditions like (6.14) and (6.15) appear in [1, 14, 17] for Hamiltonians which are positively homogeneous of degree 2. Also the "double resonance" situation has been considered there (see also [9, 10, 11]).

As an illustration, consider the case of the Hamiltonian function

$$H(x,y)=\tfrac{1}{2}\Big(a_+[x^+]^2+a_-[x^-]^2+y^2\Big)\,,$$

with

$$\frac{1}{\sqrt{a_+}}+\frac{1}{\sqrt{a_-}}=\frac{2}{n_0},$$

n_0 being a positive integer, and let $r(t,x,y)=(r_x(t,x),0)$. We are thus dealing with the scalar second order differential equation

$$x''+a_+\,x^+-a_-\,x^-+r_x(t,x)=0\,.$$

The solutions of the autonomous equation all have the same minimal period $2\pi/n_0$. Denoting by φ_x,φ_y the components of φ, using (2.11), condition (6.14) with $\gamma=1/2$ reduces to

$$\int_0^{2\pi}\liminf_{\lambda\to+\infty}\min_{|s-t|\le\eta}[\varphi_x(\tau_0+s;1)\,r_x(t,\lambda\,\varphi_x(\tau_0+s;1))]\,dt\ >0\,,$$

for every $\tau_0\in[0,2\pi/n_0]$. Since η can be taken arbitrarily small, if we assume $r_x(t,x)$ to be globally bounded, this condition is fulfilled when, for every

$\tau_0 \in [0, 2\pi/n_0]$,

$$\int_{\varphi_x(\tau_0+\,\cdot\,;1)>0} \varphi_x(\tau_0 + t\,;1) \liminf_{x\to+\infty} r_x(t,x)\,dt + \\ + \int_{\varphi_x(\tau_0+\,\cdot\,;1)<0} \varphi_x(\tau_0 + t\,;1) \limsup_{x\to-\infty} r_x(t,x)\,dt > 0\,.$$

This is the classical Landesman–Lazer condition (first introduced in [23] for the Dirichlet problem). For the example considered here, the problem has been treated by Dancer [4, 5] (in the case where r does not depend on x). See also [13] and the references therein.

7 Multiplicity of periodic solutions

When the equation (1.3) has a Hamiltonian structure, i.e., when the perturbation $r(t,z)$ is a gradient, with respect to z, of a function $R(t,z)$, it is possible to obtain multiplicity results for the periodic solutions by the use of a generalized version of the Poincaré–Birkhoff Theorem, cf. [13, 18, 28].

Let us then consider the system

$$Jz' = \nabla H(z) + \nabla_z R(t,z)\,. \tag{7.1}$$

We will prove the existence of an infinite number of periodic solutions under the assumptions of Theorem 3.1, only replacing A5 by the following.

A5″. *The function $T(E)$ is such that*

$$0 \le \liminf_{E\to+\infty} T(E) < \limsup_{E\to+\infty} T(E) < +\infty\,.$$

The idea of a using a condition on the period of the free oscillations in order to apply the Poincaré–Birkhoff Theorem can also be found, for instance, in [16].

We need a preliminary result concerning the forced system (1.3); it does not require $r(t,z)$ to be a gradient. In the following, we will denote by $z(t;z_0)$ the solution of (7.1) with initial condition $z(0) = z_0$.

Lemma 7.1. *Let the function $H : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable. Assume that the hypotheses of Theorem 3.1 hold, except for A5, not needed here. Then, for any given compact interval $I \subseteq \mathbb{R}$,*

$$\lim_{E_0 \to +\infty} \big[T(H(z(t; z_0)) - T(H(z_0)) \big] = 0 \,, \qquad \textit{uniformly in } (t, z_0) \in I \times H^{-1}(E_0) \,.$$

Proof. Let us write

$$\Delta(t; z_0) = T(H(z(t; z_0)) - T(H(z_0)) \,.$$

Since the function H is twice continuously differentiable, it is well-known that the function $E \mapsto T'(E)$ is continuously differentiable (see, for instance, [27, Theorem 3.4.1]). Since $\Delta(0; z_0) = 0$, the lemma will be proven if we can show that

$$\lim_{E_0 \to +\infty} \frac{\partial \Delta}{\partial t}(t; z_0) = 0 \,, \quad \textit{uniformly in } (t, z_0) \in I \times H^{-1}(E_0) \,.$$

Writing $z(t)$ for $z(t; z_0)$, since $z(t)$ is a solution of (1.3), we have

$$\begin{aligned} \frac{\partial \Delta}{\partial t}(t; z_0) &= T'(H(z(t))) \, \langle \nabla H(z(t)), z'(t) \rangle \\ &= T'(H(z(t))) \, \langle J \nabla H(z(t)), r(t, z(t)) \rangle \,. \end{aligned} \tag{7.2}$$

Assuming E_0 to be sufficiently large, working as in Theorem 3.1, we can write $z(t)$ under the form

$$z(t) = \varphi(t + \tau(t); e(t)) \,,$$

where $e(t) = H(z(t))$, and $\varphi(t; E)$ being, as before, a solution of the autonomous system (2.1), as introduced in Section 2.1. Since $\varphi(t; E)$ is, by definition, of period $T(E)$ in t, we have obviously

$$\varphi(t + T(E); E) = \varphi(t; E) \,, \quad \textit{for every } t \in \mathbb{R} \,,$$

from which follows, differentiating with respect to E,

$$\frac{\partial \varphi}{\partial t} \, (t + T(E); E) \, T'(E) = \frac{\partial \varphi}{\partial E} \, (t; E) - \frac{\partial \varphi}{\partial E} \, (t + T(E); E) \,,$$

or

$$-J \nabla H(\varphi(t; E)) T'(E) = \frac{\partial \varphi}{\partial E} \, (t; E) - \frac{\partial \varphi}{\partial E} \, (t + T(E); E) \,. \tag{7.3}$$

Since the above relation holds for all $t \in \mathbb{R}$, and all E sufficiently large, we can also write

$$-J\nabla H(z(t))\, T'(e(t)) = \frac{\partial \varphi}{\partial E}\,(t+\tau(t);e(t)) - \frac{\partial \varphi}{\partial E}\,(t+\tau(t)+T(e(t));e(t))\,.$$

Taking the scalar product with $r(t,z(t))$ and remembering that

$$\lim_{E_0\to+\infty} e(t) = \lim_{E_0\to+\infty} H(z(t;z_0)) = +\infty\,, \quad \textit{uniformly in } (t,z_0) \in I\times H^{-1}(E_0)\,,$$

and

$$\varphi(t+\tau(t)+T(e(t));e(t)) = \varphi(t+\tau(t);e(t))\,,$$

the conclusion follows from (7.2) and assumption A7. □

We are now in a position to state a multiplicity result based on a generalized version of the Poincaré–Birkhoff Theorem.

Theorem 7.2. *Let the function $H:\mathbb{R}^2\to\mathbb{R}$ be twice continuously differentiable and satisfy*

$$\langle \nabla H(z), z\rangle > 0\,, \quad \textit{for } H(z) \textit{ sufficiently large}\,. \tag{7.4}$$

Assume that the hypotheses of Theorem 3.1 hold, with $r(t,z) = \nabla_z R(t,z)$, except for the nonresonance condition A5, replaced by A5″. Moreover, let

$$\lim_{E\to+\infty} T'(E)\, R(t,z) = 0\,, \quad \textit{uniformly for } (t,z) \in [0,2\pi]\times H^{-1}(E)\,. \tag{7.5}$$

Then, system (7.1) *admits an infinite number of periodic solutions.*

Proof. By assumption A5″, it is possible to find positive integers m,n, and a positive number ε, such that

$$\liminf_{E\to+\infty} T(E) < \frac{2\pi m}{n} - 2\,\varepsilon < \frac{2\pi m}{n} + 2\,\varepsilon < \limsup_{E\to+\infty} T(E)\,.$$

We then build increasing sequences $(E_k)_k$, $(E_k^*)_k$, with $E_k\to+\infty$, such that $E_{2k} < E_{2k}^* < E_{2k+1}^* < E_{2k+1}$ and

$$\begin{aligned}
&T(E_{2k}) = \frac{2\pi m}{n} - 2\,\varepsilon\,, \qquad T(E_{2k}^*) = \frac{2\pi m}{n} - \varepsilon\,,\\
&\frac{2\pi m}{n} - 2\,\varepsilon < T(E) < \frac{2\pi m}{n} - \varepsilon\,, \quad \textit{for } E \in \,]E_{2k}, E_{2k}^*[\,,\\
&T(E_{2k+1}^*) = \frac{2\pi m}{n} + \varepsilon\,, \qquad T(E_{2k+1}) = \frac{2\pi m}{n} + 2\,\varepsilon\,,\\
&\frac{2\pi m}{n} + \varepsilon < T(E) < \frac{2\pi m}{n} + 2\,\varepsilon, \quad \textit{for } E \in \,]E_{2k+1}^*, E_{2k+1}[\,.
\end{aligned}$$

We may assume that $\varepsilon > 0$ has been chosen small enough so that

$$3\,\varepsilon \leq \frac{2\pi m}{n(n+1)}\,,$$

which implies that

$$|n^* T(E) - 2\pi m| \geq \varepsilon\, n^*, \qquad \text{for any integer } n^* \text{ and any } E \in [E_{2k}, E_{2k}^*] \cup [E_{2k+1}^*, E_{2k+1}]\,. \tag{7.6}$$

To the above defined sequences we associate, for large values of k, the annuli

$$\mathcal{A}_k^* = \Omega(E_{2k+1}^*) \setminus \Omega(E_{2k}^*)\,, \qquad \mathcal{A}_k = \Omega(E_{2k+1}) \setminus \Omega(E_{2k})\,.$$

Notice that $\mathcal{A}_k^*$ is contained in $\mathcal{A}_k$.

For some sufficiently large k, to be fixed below, we now modify our system. The idea is to leave it unchanged in $\mathcal{A}_k^*$, and to cancel the forcing term on $\mathbb{R}^2 \setminus \mathcal{A}_k$, the transition in the zone $\mathcal{A}_k \setminus \mathcal{A}_k^*$ being built as described below. We thus replace $R(t,z)$ by

$$\widetilde{R}(t,z) = \begin{cases} R(t,z)\,, & \text{for } z \in \mathcal{A}_k^*\,, \\ \chi(T(H(z)))\, R(t,z)\,, & \text{for } z \in \mathcal{A}_k \setminus \mathcal{A}_k^*\,, \\ 0\,, & \text{for } z \in \mathbb{R}^2 \setminus \mathcal{A}_k\,, \end{cases}$$

with $\chi : \mathbb{R} \to \mathbb{R}$ a C^∞-function, built in such a way that

$$\chi(s) = \begin{cases} 1\,, & \text{if } s \in \left[\frac{2\pi m}{n} - \frac{5}{4}\varepsilon, \frac{2\pi m}{n} + \frac{5}{4}\varepsilon\right]\,, \\ 0\,, & \text{if } s \in \left[\frac{2\pi m}{n} - \frac{7}{4}\varepsilon, \frac{2\pi m}{n} + \frac{7}{4}\varepsilon\right]\,. \end{cases}$$

We may assume that $|\chi'(s)| \leq 4/\varepsilon$, for every $s \in \mathbb{R}$.

We now consider the modified system

$$Jz' = \nabla H(z) + \nabla_z \widetilde{R}(t,z)\,. \tag{7.7}$$

Since

$$\nabla_z \widetilde{R}(t,z) = \begin{cases} \nabla_z R(t,z)\,, \quad \text{for } z \in \mathcal{A}_k^*\,, \\ \chi(T(H(z)))\, \nabla_z R(t,z) + \\ \qquad + \chi'(T(H(z)))\, T'(H(z))\, R(t,z) \nabla H(z)\,, \quad \text{for } z \in \mathcal{A}_k \setminus \mathcal{A}_k^*\,, \\ 0, \ \text{for } z \in \mathbb{R}^2 \setminus \mathcal{A}_k\,, \end{cases}$$

it is readily seen that $\widetilde{r}(t,z) := \nabla_z \widetilde{R}(t,z)$ inherits from $r(t,z)$ the regularity properties required for the application of Theorem 3.1. Moreover, using hypothesis (7.5), we observe that $\nabla_z \widetilde{R}(t,z)$ satisfies the same conditions A6 and A7 as $r(t,z)$. Consequently, arguing as in the proof of Theorem 3.1, if we write a solution $\widetilde{z}(t;z_0)$ of the modified system (7.7) under the form

$$\widetilde{z}(t;z_0) = \varphi(t + \widetilde{\tau}(t;\widetilde{\tau}_0,\widetilde{e}_0); \widetilde{e}(t;\widetilde{\tau}_0,\widetilde{e}_0))\,, \tag{7.8}$$

we can show that, if k is large enough, we will have

$$|\widetilde{\tau}(t;\widetilde{\tau}_0,\widetilde{e}_0) - \widetilde{\tau}_0| \le \frac{\varepsilon}{4}\,, \quad \textit{for every } (\widetilde{\tau}_0,\widetilde{e}_0) \in \mathbb{R} \times [E_{2k}, E_{2k+1}]\,. \tag{7.9}$$

Let us first show that, if k is sufficiently large, any solution $\widetilde{z}(t;z_0)$ of system (7.7) issued from a point z_0 in $\mathcal{A}_k^*$ does not enter the zone where the equation has been modified, when $t \in [0, 2\pi m]$. Indeed, with $z_0 \in \mathcal{A}_k^*$, assume that for some $t_0 \in [0, 2\pi m]$ either $H(\widetilde{z}(t_0;z_0)) = E_{2k}^*$, or $H(\widetilde{z}(t_0;z_0)) = E_{2k+1}^*$, and

$$E_{2k}^* \le H(\widetilde{z}(s;z_0)) \le E_{2k+1}^*\,, \quad \text{for every } s \in [0, t_0]\,.$$

Applying Lemma 7.1 with $\widetilde{z}(t_0;z_0)$ instead of z_0, if k is taken large enough, then

$$|T(H(\widetilde{z}(t;z_0))) - T(H(\widetilde{z}(t_0;z_0)))| \le \frac{\varepsilon}{4}\,, \quad \textit{for any } t \in [0, 2\pi m]\,,$$

from which follows that

$$2\pi\frac{m}{n} - \frac{5\,\varepsilon}{4} \le T(H(\widetilde{z}(t;z_0))) \le 2\pi\frac{m}{n} + \frac{5\,\varepsilon}{4}\,, \quad \textit{for any } t \in [0, 2\pi m]\,.$$

This means that, for any $t \in [0, 2\pi m]$, either $\widetilde{z}(t;z_0)$ belongs to $\mathcal{A}_k^*$, or $\chi(T(H(\widetilde{z}(t;z_0)))) = 1$. In any case, we have that $\widetilde{r}(t, \widetilde{z}(t;z_0)) = r(t, \widetilde{z}(t;z_0))$, for $t \in [0, 2\pi m]$, meaning that $\widetilde{z}(t;z_0)$ is actually a solution of the original system (7.1).

On the other hand, we claim that there are no $2\pi m$-periodic solutions $\widetilde{z}(t;z_0)$ of system (7.7) starting from a point z_0 in $\mathcal{A}_k \setminus \mathcal{A}_k^*$, with k large enough. Indeed, if $\widetilde{z}(t;z_0)$ is such a solution, writing it under the form (7.8), we have

$$2\pi m + \widetilde{\tau}(t;\widetilde{\tau}_0,\widetilde{e}_0) = \widetilde{\tau}_0 + n^* T(H(z_0)))\,,$$

for some integer n^*. But, using (7.6) and (7.9), we see that, if k is large enough, this equality is impossible, for any integer n^*.

We will now apply the version of Poincaré–Birkhoff Theorem presented in [18, Theorem 1.2]. To this aim, first notice that, by assumption (7.4), for k large enough the two curves delimiting the set $\mathcal{A}_k$ are strictly star-shaped with respect to the origin; moreover, if $z_0 \in \mathcal{A}_k$, then $z(t; z_0) \neq (0,0)$, for every $t \in [0, 2\pi m]$. This allows us to consider continuous determinations $\arg z(t; z_0)$ of the argument function along these trajectories, and to define their rotation numbers

$$\mathrm{Rot}(z(t; z_0); [0, 2\pi m]) = \frac{\arg z(2\pi m; z_0) - \arg z(0; z_0)}{2\pi}.$$

Let us now estimate these rotation numbers when z_0 belongs to the inner and to the outer boundary of the annulus $\mathcal{A}_k$. For k sufficiently large, by (7.4) the function $\arg z(\cdot\,; z_0)$ is strictly decreasing when $z_0 \in H^{-1}(E_{2k})$, and by the definition of E_{2k} we get

$$\mathrm{Rot}(z(t; z_0), [0, 2\pi m]) < -n\,, \quad \textit{if } z_0 \in H^{-1}(E_{2k})\,.$$

By a similar argument, using the definition of E_{2k+1} we see that, for k sufficiently large,

$$\mathrm{Rot}(z(t; z_0), [0, 2\pi m]) > -n\,, \quad \textit{if } z_0 \in H^{-1}(E_{2k+1})\,.$$

Then, by [18, Theorem 1.2], system (7.7) has at least two $2\pi m$-periodic solutions $z^{(1)}(t)$, $z^{(2)}(t)$, starting from the interior of $\mathcal{A}_k$, such that

$$\mathrm{Rot}(z^{(1)}(t), [0, 2\pi m]) = \mathrm{Rot}(z^{(2)}(t), [0, 2\pi m]) = -n\,.$$

By the above arguments, these are $2\pi m$-periodic solutions of the original system (7.1), and the proof is thus completed. □

We end this section with two remarks.

Remark 7.3. When the system satisfies assumption A8, it is possible to deduce condition (7.5) in Theorem 7.2 from conditions based more directly on $H(z)$ and $R(t, z)$. In particular, if H is twice differentiable and H'' globally bounded, using (3.7) (with $\mathcal{D}(E)$ the identity matrix) and (3.9), it results from (7.3) that there exists a constant such that

$$|T'(E)|\,|\nabla H(\varphi(0; E))|^2 \leq C\,,$$

so that condition (7.5) reduces to

$$\lim_{|z|\to\infty} \frac{R(t,z)}{|\nabla H(z)|^2} = 0\,.$$

For the second order equation

$$x'' + g(x) = p(t)\,,$$

with g differentiable and g' globally bounded, this allows to recover multiplicity results obtained in [7] (see also [32] and the references therein).

Remark 7.4. In a situation of resonance, i.e. when

$$\lim_{E\to+\infty} T(E) = \frac{2\pi}{n_0}\,,$$

for some positive integer n_0, conditions of existence of infinitely many periodic solutions can still be obtained, the idea being then to use the hypotheses of Theorem 6.3 rather than those of Theorem 3.1. We will not give the details here, but simply indicate that a "twist condition" would be met by asking the conditions (6.5) and (6.6) to hold simultaneously, whereas (7.5) should be replaced by the stronger condition

$$\lim_{E\to+\infty} E^\gamma\, T'(E)\, R(t,z) = 0\,, \quad \textit{uniformly for } (t,z) \in [0,2\pi] \times H^{-1}(E)\,,$$

with $\gamma > 0$ the same constant as in (6.5) and (6.6). The resulting periodic solutions would then be of period 2π, making n_0 turns around the origin on a time interval of length 2π.

8 More general differential equations

In this section, we consider the planar system

$$Jz' = F(t,z)\,. \tag{8.1}$$

We will assume throughout that $F(t,z)$ is continuous, 2π-periodic in t, and locally Lipschitz continuous in z. We will provide existence conditions for 2π-periodic solutions, through a comparison with Hamiltonian systems.

We first state a preliminary result concerning the number of turns, around the origin, of a closed curve; il will be applied below to trajectories of (possible) periodic solutions of equations like (8.1).

Lemma 8.1. *Let $H^* : \mathbb{R}^2 \to \mathbb{R}$ satisfy conditions A1 to A3, and let $\varphi^*(t; E)$ be the function associated with H^* by hypothesis A3. Assume that $\varphi^*(t; E)$ is of minimal period T^* in t, the period being independent of E, for $E > 1$, and that $\varphi^*(0; E)$ satisfies (2.5). Consider a parametric curve $t \mapsto z(t) = \varphi^*(\sigma(t); e(t))$, with $z(T) = z(0)$, the functions $\sigma(t)$, $e(t)$ being differentiable on $\mathbb{R}$, and such that $e(t) > 1$, for all $t \in [0, T]$. Assume that*

$$\sigma(t) = 0 \bmod T^* \Longrightarrow \langle \nabla H^*(z(t)), Jz'(t) \rangle > 0\,. \tag{8.2}$$

If the curve $t \mapsto z(t)$ makes n turns around the origin on the interval $[0, T]$, then

$$\sigma(T) = \sigma(0) + n\,T^*. \tag{8.3}$$

Conversely, if the above equality holds for some nonnegative integer n, the curve $t \mapsto z(t)$ makes n turns around the origin on the interval $[0, T]$, in the clockwise sense.

Proof. The curve $z : t \mapsto z(t)$ crosses the gradient curve $E \mapsto \varphi^*(0; E)$, when and only when $\sigma(t) = 0 \bmod T^*$. The condition (8.2) means that the curve z is transversal to the gradient curve $\varphi^*(0; \cdot)$, the crossing occurring in the clockwise direction. This gradient curve extends from a point on the closed curve $H(z) = 1$ to infinity. Since the curve z remains in the unbounded set $\{z \in \mathbb{R}^2 \mid H(z) > 1\}$ for $t \in [0, T]$, the number of turns of z around the origin on the interval $[0, T)$ is equal to the number of crossings with the curve $\varphi^*(0; \cdot)$. Because of (8.2) and (2.5), we have $\sigma'(t) > 0$ when $\sigma(t) = 0 \bmod T^*$. Hence, each revolution of the curve z corresponds to an increase T^* of the parameter t. Therefore, the number n in (8.3) is necessarily the number of turns of the curve z, around the origin, on the interval $[0, T]$. $\square$

Notice that the above result also makes sense with $n = 0$.

8.1 An existence result

We now state an existence result for equation (8.1) based on a comparison between $F(t, z)$ and the gradients of two Hamiltonian functions H_1, H_2. The proof is based again on degree arguments, a homotopy

$$Jz' = (1 - \lambda)\nabla H(z) + \lambda\, F(t, z)\,. \tag{8.4}$$

being used. The hypotheses on the third Hamiltonian function H, which appears in that equation, aim essentially at controlling the energy of the possible periodic solutions, whereas the hypotheses on H_1, H_2 concern their number of revolutions in the plane.

Many existence results have been obtained in the past by considering this number of turns, namely by means of a so-called "rotation number" (see, for instance, [1, 33]). Our aim, here, is to present a result in the line of our approach of Section 3, transforming again the system (8.4) into a system whose variables are the energy and the phase. An objective will then be to compare, for an equation like (1.3), the results presented below to the results of Section 3. As will be seen, the main difference is that, in the present section, the hypotheses used for the comparison are based essentially on ∇H_1, ∇H_2, whereas, in Section 3, the hypotheses concern more directly the functions H_1, H_2 themselves, through the relations (4.1).

Let the three Hamiltonian functions H, H_1, H_2 satisfy conditions A1 to A3. Assume that the functions $\varphi_1(t;E)$, $\varphi_2(t;E)$, associated respectively with H_1, H_2, satisfy equation (2.5). The respective minimal periods of H, H_1, H_2, will be denoted by $T(E)$, T_1, T_2, the Hamiltonian functions H_1 and H_2 being assumed to be isochronous. Notice that $\nabla H_1(z)$, $\nabla H_2(z)$ are not required to be locally Lipschitz continuous.

Theorem 8.2. *Assume that the functions H, H_1, H_2 satisfy the assumptions A1 to A3, that ∇H is locally Lipschitz continuous, and that the Hamiltonian functions H_1 and H_2 are isochronous. Assume that*

$$\frac{\langle J\nabla H(z), F(t,z)\rangle}{H(z)} \text{ remains bounded for } |z| \to \infty, \text{ uniformly in } t\,. \tag{8.5}$$

With

$$F^{(\lambda)}(t,z) = (1-\lambda)\nabla H(z) + \lambda\, F(t,z)\,, \tag{8.6}$$

assume that

$$\langle \nabla H_1(\varphi_1(0;E)), F^{(\lambda)}(t,\varphi_1(0;E))\rangle > 0\,, \tag{8.7}$$

$$\langle \nabla H_2(\varphi_2(0;E)), F^{(\lambda)}(t,\varphi_2(0;E))\rangle > 0\,, \tag{8.8}$$

for $\lambda \in \{0,1\}$, $t \in [0,2\pi]$, and E sufficiently large. Assume moreover that, *for any given compact interval* $I \subseteq \mathbb{R}$,

$$1 \le \liminf_{E \to +\infty} \left\langle \frac{\partial \varphi_1}{\partial E}(s;E), F^{(\lambda)}(t, \varphi_1(s;E)) \right\rangle , \tag{8.9}$$

$$\limsup_{E \to +\infty} \left\langle \frac{\partial \varphi_2}{\partial E}(s;E), F^{(\lambda)}(t, \varphi_2(s;E)) \right\rangle \le 1 , \tag{8.10}$$

for $\lambda \in \{0,1\}$, *uniformly for* $(t,s) \in [0,2\pi] \times I$. *If for some integer* n_0,

$$\frac{2\pi}{n_0+1} < T_2 \quad \textit{and} \quad T_1 < \frac{2\pi}{n_0} , \tag{8.11}$$

then equation (8.1) *admits at least one* 2π*-periodic solution.*

The hypotheses in the above theorem may seem awkward, but have been stated at that level of generality to allow applications in a large variety of situations. We will provide below various sets of conditions ensuring that these hypotheses are satisfied.

Proof. We denote by $P_{2\pi}^{(\lambda)}$ the Poincaré map for the period 2π, associated with equation (8.4); reasoning as in the proof of Theorem 3.1, this map can be shown to be well defined. Indeed, letting $z^{(\lambda)}(t;z_0)$ denote the solution of (8.4) for the initial condition $z(0) = z_0$, and defining $e^{(\lambda)}(t;z_0) = H(z^{(\lambda)}(t;z_0))$, we have

$$(e^{(\lambda)})'(t;z_0) = \lambda \langle J\nabla H(z^{(\lambda)}(t;z_0)), F(t, z^{(\lambda)}(t;z_0)) \rangle . \tag{8.12}$$

We then deduce from (8.5) that $e^{(\lambda)}(t;z_0)$ remains bounded on any compact interval, from which follows, by the coercivity condition A1, that $z^{(\lambda)}(t;z_0)$ can be extended to the whole real line and $|z^{(\lambda)}(t;z_0)|$ tends to $+\infty$, for $E_0 \to +\infty$, uniformly for $z_0 \in H^{-1}(E_0)$. Notice that, by hypothesis A1 for H_1 and H_2, both $H_1(z^{(\lambda)}(t;z_0))$ and $H_2(z^{(\lambda)}(t;z_0))$ also tend to $+\infty$ for $E_0 \to +\infty$, uniformly for $z_0 \in H^{-1}(E_0)$, $t \in [0,2\pi]$, $\lambda \in [0,1]$.

The theorem will be proved if we can show that, for E_0 sufficiently large,

$$P_{2\pi}^{(\lambda)}(z_0) \ne z_0 , \quad \textit{for any } z_0 \in H^{-1}(E_0) \textit{ and } \lambda \in [0,1] . \tag{8.13}$$

In order to prove that (8.13) holds for E_0 sufficiently large, we will look at the number of turns of possible 2π-periodic solutions of (8.4) around the origin. For that purpose, we will, in a first stage, consider those solutions as perturbations of the solutions of equation

$$Jz' = \nabla H_1(z)\,. \tag{8.14}$$

We therefore write them under the form

$$z^{(\lambda)}(t; z_0) = \varphi_1\big(t + \tau_1^{(\lambda)}(t); e_1^{(\lambda)}(t)\big)\,.$$

Taking (2.3) into account, simple calculations lead to the system

$$(\tau_1^{(\lambda)})' = \left\langle \frac{\partial\varphi_1}{\partial E}\big(t + \tau_1^{(\lambda)}; e_1^{(\lambda)}\big), F^{(\lambda)}\Big(t, \varphi_1\big(t + \tau_1^{(\lambda)}; e_1^{(\lambda)}\big)\Big)\right\rangle - 1\,, \tag{8.15}$$

$$(e_1^{(\lambda)})' = \lambda\,\Big\langle J\nabla H_1\big(\varphi_1\big(t + \tau_1^{(\lambda)}; e_1^{(\lambda)}\big)\big), F^{(\lambda)}\Big(t, \varphi_1\big(t + \tau_1^{(\lambda)}; e_1^{(\lambda)}\big)\Big)\Big\rangle\,. \tag{8.16}$$

The solution of the above system for the initial conditions $\tau_1^{(\lambda)}(0) = \tau_0$, $e_1^{(\lambda)}(0) = H_1(z_0)$, where τ_0 is such that $z_0 = \varphi_1(\tau_0, H_1(z_0))$, will be denoted by $\tau_1^{(\lambda)}(t; \tau_0, H_1(z_0))$, $e_1^{(\lambda)}(t; \tau_0, H_1(z_0))$.

Condition (8.13) will be satisfied unless, for some integer n (which may be positive, negative or zero), some $\tau_0 \in [0, T_1]$, some $\lambda \in [0, 1]$, we have

$$\begin{aligned} e_1^{(\lambda)}(2\pi; \tau_0, H_1(z_0)) &= H_1(z_0)\,, \\ 2\pi + \tau_1^{(\lambda)}(2\pi; \tau_0, H_1(z_0)) &= \tau_0 + n\,T_1\,. \end{aligned} \tag{8.17}$$

Using Lemma 8.1 with $H^* = H_1$, we see that n is the number of turns of the solution around the origin, counted positively in the clockwise sense. Indeed, condition (8.2) follows from hypothesis (8.7), for any $\lambda \in [0, 1]$. Using (8.9), we deduce from (8.15) that, given any $\eta > 0$, for sufficiently large values of E_0,

$${\tau_1}^{(\lambda)}(2\pi; \tau_0, H_1(z_0)) - \tau_0 \geq -\eta\,,$$

for any $z_0 \in H^{-1}(E_0)$ (remember that τ_0 depends on z_0). Hence, by (8.11), the periodicity condition (8.17) can hold only if $n > n_0$.

A similar argument, with the solutions of (8.4) considered as perturbations of the solutions of

$$Jz' = \nabla H_2(z)\,, \tag{8.18}$$

shows that $n < n_0 + 1$, so that (8.17) is impossible, n being an integer. □

We illustrate the above theorem by considering the simple case where H_1 and H_2 are positively homogeneous of degree 2, i.e., they satisfy (2.9). Due to the homogeneity property of H_1, H_2, as observed in Section 2.2, we can build $\varphi_1(;E)$, $\varphi_2(t;E)$ in such a way that

$$\frac{\partial \varphi_i}{\partial E}(t;E) = \frac{1}{2E}\,\varphi_i(t;E) \quad (i = 1,2)$$

(see (2.10) and (2.11)). Moreover, for $i = 1,2$, the gradients $\nabla H_i(\varphi_i(0;E))$, $i = 1,2$, are then positive multiples of $\varphi_i(0;E)$ so that conditions (8.7), (8.8) are satisfied if

$$\langle z, F^{(\lambda)}(t,z))\rangle > 0\,, \quad \textit{for } t \in [0,2\pi] \textit{ and } |z| \textit{ “large”.}$$

On the other hand, conditions (8.9), (8.10) hold if

$$1 \le \liminf_{|z|\to\infty} \frac{\langle F(t,z), z\rangle}{2\,H_1(z)}\,, \quad \limsup_{|z|\to\infty} \frac{\langle F(t,z), z\rangle}{2\,H_2(z)} \le 1\,, \quad \textit{uniformly in } t\,. \tag{8.19}$$

Notice that, by the homogeneity property of H_1, H_2, the conditions (8.19) imply that, for any $\varepsilon > 0$, we have $H_1(z) \le (1+\varepsilon)H_2(z)$, for $|z|$ sufficiently large. Using, for instance, the arguments of Section 4.1, this in turn entails that $T_2 \le T_1$. We can deduce the following corollary from Theorem 8.2.

Corollary 8.3. *Assume that the functions H, H_1, H_2 satisfy the assumptions A1 to A3, that ∇H is locally Lipschitz continuous, and that the functions H_1, H_2 satisfy* (2.9)*, and hence are isochronous, their respective minimal periods being denoted by T_1, T_2. Assume moreover that* (8.5) *is satisfied. If conditions* (8.19) *hold and if, for some integer n_0,*

$$\frac{2\pi}{n_0+1} < T_2 \le T_1 < \frac{2\pi}{n_0}\,, \tag{8.20}$$

then equation (8.1) *admits at least one 2π-periodic solution.*

Remark 8.4. When H_1 is positively homogeneous of degree 2, the computation of the “rotation number” associated with the auxiliary function H_1, as introduced, for instance, in [1, 33], is equivalent to the computation of $2\pi + \tau_1^{(\lambda)}(t;\tau_0, H_1(z_0)) - \tau_0$, with $\tau_1^{(\lambda)}$ being defined by (8.15).

It is interesting to compare the above corollary to results of Section 4. We will do this for the equation

$$Jz' = \nabla H(z) + r(t)\,, \tag{8.21}$$

with r continuous and 2π-periodic, assuming that H is twice continuously differentiable, that H'' is globally bounded, and that

$$H_1(z) \le H(z) \le H_2(z)\,, \quad \textit{for } |z| \textit{ large,} \tag{8.22}$$

the Hamiltonians H_1 and H_2 satisfying (2.9). Notice that condition (8.5) is then automatically satisfied. With T_1, T_2 denoting the minimal periods of the nontrivial solutions of the systems associated with H_1, H_2, respectively, assume that (8.20) holds. The application of Corollary 8.3 requires that

$$1 \le \liminf_{|z|\to\infty} \frac{\langle \nabla H(z), z\rangle}{2\,H_1(z)}\,, \quad \limsup_{|z|\to\infty} \frac{\langle \nabla H(z), z\rangle}{2\,H_2(z)} \le 1\,,$$

whereas such conditions on ∇H are not needed in Corollary 4.2, the condition (8.22) combined with the hypotheses on T_1, T_2 being sufficient. This difference is explained by the fact that Corollary 8.3 does not exploit the Hamiltonian structure of the autonomous equation associated with (8.21). The application of the above corollary to equation (8.21) allows to recover results obtained long ago by Sędziwy [30].

8.2 The case when $n_0 = 0$

In the case when $n_0 = 0$, the function H_1 is superfluous, and the following result can be stated.

Theorem 8.5. *Assume that the functions H, H_2 satisfy the hypotheses A1 to A3, and that ∇H is locally Lipschitz continuous. Let $H(z)$, $F(t,z)$ be such that* (8.5) *holds. Assume that*

$$\langle \nabla H_2(z), F(t,z)\rangle > 0\,, \quad \langle \nabla H_2(z), \nabla H(z)\rangle > 0\,, \tag{8.23}$$

for $t \in [0, 2\pi]$, $|z|$ sufficiently large. Assume moreover that, for any given compact interval $I \subseteq \mathbb{R}$,

$$\limsup_{E\to+\infty} \left\langle \frac{\partial \varphi_2}{\partial E}(s;E), F(t, \varphi_2(s;E)) \right\rangle \le 1\,,$$

$$\limsup_{E\to+\infty} \left\langle \frac{\partial\varphi_2}{\partial E}(s;E), \nabla H(\varphi_2(s;E)) \right\rangle \leq 1\,,$$

uniformly for $(t,s) \in [0,2\pi]\times I$. *If*

$$\liminf_{E\to+\infty} T_2(E) > 2\pi\,,$$

then equation (8.1) *admits at least one* 2π*-periodic solution.*

The proof is analogous to the one of Theorem 8.2, hence we omit it, for brevity. Notice however the difference between conditions (8.23) and (8.8). The conditions (8.23) ensure that a possible periodic solution of (8.4) crosses all the gradient curves associated with H_2, whereas in Theorem 8.2, condition (8.8), which concerns only one particular gradient curve, suffices. Notice also that it is not necessary here to assume the Hamiltonian function H_2 to be isochronous.

8.3 Application to equations with separated variables

Consider the case where the variables are "separated" in the planar system (8.1), i.e., the right-hand side has the form

$$F(t,(x,y)) = (g(t,x), k(t,y))\,.$$

We will assume that g, k are continuous, 2π-periodic in t, locally Lipschitz continuous in x, y, and that, for some $p > 1$, $q > 1$ related by condition (2.13), we have

$$g(t,x) = O(|x|^{p-1})\,,\quad \textit{for } |x|\to\infty\,,\qquad k(t,y) = O(|y|^{q-1})\,,\quad \textit{for } |y|\to\infty\,,$$

uniformly in t. More precisely, we will assume that there exist numbers $\alpha_2 \geq \alpha_1 > 0$ and $\beta_2 \geq \beta_1 > 0$ such that

$$\alpha_1 \leq \liminf_{|x|\to+\infty} \frac{x\,g(t,x)}{|x|^p} \leq \limsup_{|x|\to+\infty} \frac{x\,g(t,x)}{|x|^p} \leq \alpha_2\,, \tag{8.24}$$

$$\beta_1 \leq \liminf_{|y|\to+\infty} \frac{y\,k(t,y)}{|y|^q} \leq \limsup_{|y|\to+\infty} \frac{y\,k(t,y)}{|y|^q} \leq \beta_2\,, \tag{8.25}$$

the limits being assumed to be uniform in t. Actually, we could write more general results by considering separately the limits for x going to $+\infty$ and for x going to $-\infty$, and analogously for y. This involves no particular difficulty, but makes the formulation of the hypotheses more involved.

In order to apply Theorem 8.2 or Theorem 8.5, we will use the following functions as references for the comparison:

$$H_1(x,y) = \alpha_1 \frac{|x|^p}{p} + \beta_1 \frac{|y|^q}{q}\,, \quad H_2(x,y) = \alpha_2 \frac{|x|^p}{p} + \beta_2 \frac{|y|^q}{q}\,. \tag{8.26}$$

We also need a Hamiltonian function H having the properties required for Theorem 8.2 and Theorem 8.5; it is possible to choose a function, whose gradient is locally Lipschitz continuous, and which is of the form

$$H(z) = \tfrac{1}{2}\left(H_1(z) + H_2(z)\right) + R(z)\,,$$

with $|\nabla R(z)| = o(|z|)$ for $|z| \to +\infty$. Denoting, as before, by $\varphi_1(t;E)$, $\varphi_2(t;E)$ the functions associated with the Hamiltonians H_1, H_2, and remembering (2.14), i.e.,

$$\frac{\partial \varphi_i}{\partial E}(t;E) = \frac{1}{E}\,\mathrm{diag}\left(\frac{1}{p}, \frac{1}{q}\right) \varphi_i(t;E) \quad (i = 1,2)\,,$$

it can be checked that all the conditions for the application of Theorem 8.2 or Theorem 8.5 are satisfied. Altogether, the following corollary is obtained.

Corollary 8.6. *Assume that, for some $p > 1$, $q > 1$ related by* (2.13)*, the conditions* (8.24), (8.25) *hold (with α_1, β_1 positive). Let the Hamiltonians H_1, H_2 then be defined by* (8.26)*, the minimal period of their nontrivial solutions being denoted by T_1, T_2, respectively. If either $T_2 > 2\pi$, or for some integer n_0, condition* (8.20) *holds, then the system*

$$x' = k(t,y)\,, \quad -y' = g(t,x)$$

admits a 2π-periodic solution.

A result close to the above corollary can be found in [6], for the case of a second order equation with a p-Laplacian operator, corresponding to the choice $k(t,y) = c\,|y|^q$, for some constant $c > 0$.

As already observed above in the case of systems with positively homogeneous Hamiltonians of degree 2, sharper results can be obtained for systems of the form

$$x' = k(y) + r_y(t)\,, \quad -y' = g(x) + r_x(t)\,,$$

by exploiting the Hamiltonian structure and resorting to the results of Section 4, the Hamiltonian being defined by $H(x,y) = G(x) + K(y)$, where G and K denote some primitives of g and k, respectively.

Acknowledgement. We would like to thank Andrea Sfecci and Fabio Zanolin for their helpful comments and suggestions.

The second author has been partially supported by the Italian research association GNAMPA–INdAM.

References

[1] A. Boscaggin and M. Garrione, Resonance and rotation numbers for planar Hamiltonian systems: multiplicity results via the Poincaré–Birkhoff theorem, *Nonlinear Anal.* 74 (2011), 4166–4185.

[2] A. Boscaggin and M. Garrione, Planar Hamiltonian systems at resonance: the Ahmad–Lazer–Paul condition, *Nonlinear Differ. Equ. Appl.* 20 (2013), 825–843.

[3] A. Capietto, J. Mawhin and F. Zanolin, A continuation theorem for periodic boundary value problems with oscillatory nonlinearities, *NoDEA Nonlinear Differential Equations Appl.* 2 (1995), 133–163.

[4] E. N. Dancer, Boundary-value problems for weakly nonlinear ordinary differential equations, *Bull. Austral. Math. Soc.* 15 (1976), 321–328.

[5] E. N. Dancer, Proofs of the results in “Boundary-value problems for weakly nonlinear ordinary differential equations”, *Rend. Istit. Mat. Univ. Trieste* 42 (2010), 31–57.

[6] M. del Pino, R. Manásevich and A. Murua, Existence and multiplicity of solutions with prescribed period for a second order quasilinear o.d.e., *Nonlinear Anal.* 18 (1992), 79–92.

[7] T. Ding, R. Iannacci and F. Zanolin, Existence and multiplicity results for periodic solutions of semilinear Duffing equations, *J. Differential Equations* 105 (1993), 364–409.

[8] P. Drabek and S. Invernizzi, On the periodic boundary value problem for forced Duffing equations with jumping nonlinearity, *Nonlinear Anal.* 10 (1986), 643–650.

[9] C. Fabry, Landesman–Lazer conditions for periodic boundary value problems with asymmetric nonlinearities, *J. Differential Equations* 116 (1995) 405–418.

[10] C. Fabry and D. Fayyad, Periodic solutions of second order differential equations with a p-Laplacian and asymmetric nonlinearities, *Rend. Istit. Mat. Univ. Trieste* 24 (1992), 207–227.

[11] C. Fabry and A. Fonda, Periodic solutions of nonlinear differential equations with double resonance, *Ann. Mat. Pura Appl.* 157 (1990), 99–116.

[12] L. Fernandes and F. Zanolin, Periodic solutions of a second order differential equation with one-sided growth restriction on the restoring term, *Arch. Math. (Basel)* 51 (1988), 151–163.

[13] A. Fonda, *Playing Around Resonance. An Invitation to the Search of Periodic Solutions for Second Order Ordinary Differential Equations*, Birkhäuser, Basel, 2016.

[14] A. Fonda and M. Garrione, Double resonance with Landesman–Lazer conditions for planar systems of ordinary differential equations, *J. Differential Equations* 250 (2011), 1052–1082.

[15] A. Fonda and M. Garrione, A Landesman–Lazer type condition for asymptotically linear second order equations with a singularity, *Proc. Royal Society Edinburgh* 142 (2012), 1263–1277.

[16] A. Fonda, M. Sabatini and F. Zanolin, Periodic solutions of perturbed Hamiltonian systems in the plane by the use of the Poincaré–Birkhoff Theorem, *Topol. Methods Nonlinear Anal.* 40 (2012), 29-52

[17] A. Fonda and A. Sfecci, A general method for the existence of periodic solutions of differential systems in the plane, *J. Differential Equations* 252 (2012), 1369–1391.

[18] A. Fonda and A.J. Ureña, A higher dimensional Poincaré–Birkhoff theorem for Hamiltonian flows, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34 (2017), 679–698.

[19] A. Fonda and F. Zanolin, On the use of time-maps for the solvability of nonlinear boundary value problems, *Arch. Math. (Basel)* 59 (1992), 245–259.

[20] M. Garrione, A note on a nonresonance condition at zero for first-order planar systems, *Electron. J. Differential Equations* 2015, No. 21, 17 pp.

[21] D. Hao and S. Ma, Semilinear Duffing equations crossing resonance points, *J. Differential Equations* 133 (1997), 98–116.

[22] M. Y. Jiang, A Landesman–Lazer type theorem for periodic solutions of the resonant asymmetric p-Laplacian equation, *Acta Math. Sinica* 21 (2005), 1219–1228.

[23] E. Landesman and A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.* 19 (1970), 609–623.

[24] P. Lindqvist, Some remarkable sine and cosine functions, *Ricerche Mat.* 44 (1995), 269–290.

[25] J. Mawhin, Resonance and nonlinearity: A survey, *Ukrainian Math. J.* 59 (2007), 197–214.

[26] P. Omari and F. Zanolin, A note on nonlinear oscillations at resonance, *Acta Math. Sinica* 3 (1987), 351–361.

[27] Perko L., *Differential Equations and Dynamical Systems*, Texts in Applied Mathematics 7, Springer, Berlin, 2001.

[28] C. Rebelo, A note on uniqueness of Cauchy problems associated to planar Hamiltonian systems, *Portugal. Math.* 57 (2000), 415–419.

[29] C. Rebelo, A note on the Poincar'e–Birkhoff fixed point theorem and periodic solutions of planar systems, *Nonlinear Anal.* 29 (1997), 291-311.

[30] S. Sędziwy, On periodic solutions of systems of two differential equations, *Zeszyty Nauk. Uniw. Jagiellon. Prace Mat.* 10 (1965), 79–88.

[31] A. Sfecci, Double resonance for one-sided superlinear or singular nonlinearities, *Ann. Mat. Pura Appl.* 195 (2016), 2007–2025.

[32] Z. Wang and T. Ma, Periodic solutions of planar Hamiltonian systems with asymmetric nonlinearities, *Bound. Value Probl.* (2017), Paper No. 46, 16 pp.

[33] F. Zanolin, Continuation theorems for the periodic problem via the translation operator, *Rend. Sem. Mat. Univ. Politec. Torino* 54 (1996), 1–23.

Authors' addresses:

Alessandro Fonda
Dipartimento di Matematica e Geoscienze
Università di Trieste
P.le Europa 1
I-34127 Trieste
Italy
e-mail: a.fonda@units.it

Christian Fabry
Prof. emeritus Université cath. de Louvain
B-1348 Louvain-la-Neuve
Belgium
e-mail: christian.fabry@uclouvain.be

Mathematics Subject Classification: 34C25

Keywords: Hamiltonian systems; resonance; periodic solutions; topological degree.