Birkhäuser Advanced Texts
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Playing Around Resonance

An Invitation to the Search of Periodic Solutions for Second Order Ordinary Differential Equations

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Contents

Introduction

This book is an introduction to the problem of the existence of solutions to some type of semilinear boundary value problems. It arises from a series of courses which I have given to undergraduate and graduate students in the last few years.

The aim of the book is to give the possibility to any good student to reach a research level in this field, starting from the basic knowledge of mathematical analysis which is usually acquired before graduation. To this aim, I will develop some tools which could be used to attack many different boundary value problems, arising from ordinary or partial differential equations. However, I have chosen to deal mainly with the periodic problem for a second-order scalar ordinary differential equation. One reason for this choice is that this apparently simple model already shows so many different aspects, and can be approached by such different techniques, that it seems the ideal starting point to the further understanding of more technical boundary value problems. Another reason comes, of course, from its intrinsic importance in the applications.

So, I will be concerned with an equation of the type

$$x'' + g(t,x) = 0\,, \tag{1}$$

where $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, which is T-periodic in its first variable. The main problem will be to find some conditions on the function g which guarantee the existence of T-periodic solutions of Eq. (1).

More generally, we will deal with the problem

$$(P) \qquad \begin{cases} x'' + g(t,x) = 0\,, \\ x(0) = x(T)\,,\;\; x'(0) = x'(T)\,, \end{cases}$$

where $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Indeed, if $g(t,x)$ is defined on $\mathbb{R} \times \mathbb{R}$, and T-periodic in its first variable, it is easy to see that any solution $x(t)$ of problem (P) can be extended to the whole $\mathbb{R}$ as a T-periodic solution of Eq. (1).

What about the word *resonance* appearing in the title? When does resonance appear? Can it be precisely identified? Can it be related to the existence or nonexistence of solutions for problem (P)? Answering these questions is not a simple task, but I will say a few words, to clarify.

It is easier to explain the idea of resonance in the particular case when $g(t,x) = g(x) - e(t)$, so that Eq. (1) can be written as

$$x'' + g(x) = e(t)\,. \tag{2}$$

This equation can be seen as a model for the motion of a particle when subjected to a restoring force $g(x)$ and an external forcing $e(t)$, which we assume to be T-periodic. The simplest situation is when $g(x) = \lambda x$, the linear case, where λ is a positive constant. The equation $x'' + \lambda x = e(t)$ then models a *linear oscillator*, and it is well known what resonance means in this case: if the frequency of a Fourier component of the external forcing coincides with the natural frequency of the free oscillator, then the solutions will grow in amplitude as time goes on, without any possible bound.[1] This is the so-called linear resonance: we notice that, in this case, there are no periodic solutions.

When the restoring force $g(x)$ is not linear any more, the situation can be much more complicated. Even if it is not clear what resonance (or perhaps *nonlinear resonance*) would mean in the general case, one can expect that a phenomenon similar to linear resonance may appear in situations when the function $g(x)$ gives rise to free periodic oscillations whose frequencies interfere with those of the external forcing $e(t)$. To be more precise, assume, for example, that we are in a case when all the solutions of the differential equation $x'' + g(x) = 0$ are periodic, but not necessarily of the same period. Then, it is intuitive that, if we want to avoid a *nonlinear resonance phenomenon* to appear, the frequencies of $e(t)$ should not be approached too much by those of the large amplitude periodic solutions of the free oscillator. Otherwise, indeed, the solutions of the forced oscillator (2) could enter into a resonance-like situation, become larger and larger, and, in particular, there might be no periodic solutions.

So, it seems that the existence of periodic solutions to Eq. (2) can be a way to stay away from resonance: at least, this could be a starting point. However, it is known that nonlinear phenomena can be much more intricate: for example, in some situations, Eq. (2) may have a T-periodic solution, but at the same time some solutions may be unbounded in the future or in the past. Of course, similar considerations can be made for Eq. (1), as well.

In this book, I have not tried to give a general definition of *resonance*, except for the well-known linear case. On the other hand, I have used this word, or its counterpart, *nonresonance*, several times. They are often used to put in evidence the kind of behavior we expect for the solutions of the differential equation: is there

[1] This can be a very undesirable situation in mechanical systems, since large oscillations can lead to a collapse. On the other hand, it could be used on purpose, to amplify some physical phenomena.

some control in the oscillations, leading to the existence of periodic solutions? Are we near a resonance-type situation or far away from it? Following the tradition, I have thus adopted the practical use of the word, trying, however, not to mistreat it too much.

The methods introduced in the book are developed in full details. Each method has its own advantages in the applications to the existence or multiplicity of solutions to problem (P). However, I will not search the greatest generality in the applications. On the contrary, I will try not to make the computations too cumbersome, even if the results I have chosen represent some of the most advanced achievements in the field. Moreover, when necessary, further results and remarks are added at the end of the chapters, as a guide to the most recent references.

I will now briefly formally describe the contents of the book.

The first two chapters are devoted to an introduction to the theory of linear operators in Hilbert spaces. Indeed, problem (P) can be transformed into a *fixed point problem* in a function space, e.g., the Hilbert space $L^2(0,T)$. In Chap. 3, the main properties of the differential operator are analyzed, and the fixed point problem associated to (P) is attacked by the use of the contraction theorem, assuming some nonresonance conditions upon the function $g(t,x)$. The same approach then leads to the study of more general nonresonance conditions for abstract semilinear equations in Hilbert spaces.

A fundamental technique which has been extensively used to solve problem (P) is the *topological degree*. In Chap. 4, we develop the theory of both the Brouwer degree and the Leray–Schauder degree which will be needed in the subsequent two chapters, where they will be applied to problem (P). In Chap. 5, starting with the use of the Schauder fixed point theorem, we at first introduce the method of lower and upper solutions. Then, we develop the Leray–Schauder continuation principle, which will be the main tool to deal with many different kinds of symmetric or asymmetric-type nonlinearities. Chapter 5 mainly deals with nonresonant situations, with respect to the spectrum of the differential operator or, more generally, to the associated Fučík spectrum. Also, a multiplicity result of the Ambrosetti–Prodi type is presented. In Chap. 6, the more subtle conditions of Landesman–Lazer type are introduced, which permit a closer approach to resonance. Most of these problems can be reduced to a situation where the associated topological degree is equal to 1. At the end of the chapter, a different situation is analyzed, where the degree can also be an arbitrary negative number.

In Chap. 7, an introduction to *variational methods* is provided. Problem (P) is shown to be equivalent to the search of critical points of a functional defined on a well-chosen Hilbert space. In particular, we will be interested in finding minimum points or saddlelike points. The Ambrosetti–Rabinowitz mountain pass theorem and the Rabinowitz saddle point theorem are presented, as particular cases of a more general situation. The proof is based on the Ekeland variational principle. In Chap. 8, we will show how to apply these methods to deal with functions $g(t,x)$ satisfying the Ahmad–Lazer–Paul conditions, a still closer approach to resonance. As a final result, we present a multiplicity result for periodic solutions of pendulum-like equations due to Mawhin and Willem.

In Chap. 9, we explain the theory by Lusternik and Schnirelmann in the simple case of functionals defined on the product of a torus and a Hilbert space. Since we do not assume the reader to be familiar with calculus on manifolds, we tried to maintain the exposition at an as elementary as possible level. The notions of *category* and *relative category* are introduced, leading to some theorems on multiplicity of critical points.

In Chap. 10, we propose a version of the Poincaré–Birkhoff theorem which is well suited for *Hamiltonian systems* in the plane. This is a very recent result I have obtained in collaboration with Antonio J. Ureña [107], which extends also to higher dimensions. However, we only deal here with the planar case, for simplicity. In Chap. 11, we show the far-reaching consequences of the Poincaré–Birkhoff theorem and obtain the multiplicity of periodic solutions for equations either with asymmetric nonlinearities or with nonlinearities having a superlinear growth.

The remaining part of the book consists of three appendices.

In Appendix A, we recall the main properties of spaces of continuous functions which are used in the book. In particular, we state and prove the Ascoli–Arzelà theorem and the Stone–Weierstrass theorem.

In Appendix B, we provide the needed background for differential calculus in infinite dimensions. The Fréchet differential is introduced, and its main properties are analyzed. In particular, the implicit function theorem is reported here.

Since for the construction of the topological degree we use some properties of differential forms, Appendix C is meant to briefly collect some of their main features, including the Stokes–Cartan theorem. This appendix could also be useful for clarifying the notations used in the text. For a more complete treatment, we refer to the nice book by Spivak [209].

The choice of the results contained in this book has been greatly influenced by my own research interests. I hope that the reader will share my enthusiasm for the beauty of this theory, which in recent years shows a still growing interest, as can be seen from the large number of recent publications in specialized journals.

The list of references is by no means complete, and I apologize for this. However, I have included some very recent papers, and the references therein will help the interested reader to find an up-to-date picture of the present situation.

I wish to warmly thank all the students who, following my courses, have often given me hints on how to clarify the exposition of the arguments contained in the book. Without them I would not even have found the motivation to write it.