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MULTIPLE PERIODIC SOLUTIONS OF INFINITE-DIMENSIONAL PENDULUM-LIKE EQUATIONS

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Abstract. We prove the multiplicity of periodic solutions for an equation in a separable Hilbert space H, with T-periodic dependence in time, of the type

$$\ddot{x} + \mathcal{A}x + \nabla_x V(t,x) = e(t)\,.$$

Here, $\mathcal{A}$ is a semi-negative definite bounded selfadjoint operator, with nontrivial null-space $\mathcal{N}(\mathcal{A})$, the function $V(t,x)$ is bounded above, periodic in x along a basis of $\mathcal{N}(\mathcal{A})$, with $\nabla_x V$ having its image in a compact set, and $e(t)$ has mean value in $\mathcal{N}(\mathcal{A})^\perp$. Our results generalize several well-known theorems in the finite-dimensional setting, as well as a recent existence result in [1].

1. Introduction

Motivated by the model of a periodically forced pendulum, the existence of *at least two* geometrically distinct T-periodic solutions for a scalar differential equation of the form

$$\ddot{x} + \partial_x V(t,x) = e(t)$$

was first proved in [16], using the direct method of the calculus of variations and the Mountain Pass Theorem, assuming $V(t,x)$ to be T-periodic with respect to t and τ-periodic with respect to x, and $e(t)$ to be T-periodic with zero mean, i.e.,

$$\int_0^T e(t)\,dt = 0\,. \tag{1.1}$$

This result extended an existence theorem first proved in [12], and later rediscovered independently in [5, 21].

Here, and in the sequel, for simplicity all functions will be assumed to be continuous. It can be seen that the multiplicity result in [16] is optimal if no further conditions are added. Different proofs have also been provided, e.g., in [8, 9, 11], by the use of some generalized versions of the Poincaré–Birkhoff theorem.

The result in [16] was later generalized in [17], through a similar approach, to the corresponding system in $\mathbb{R}^N$,

$$\ddot{x} + \nabla_x V(t,x) = e(t), \tag{1.2}$$

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when $V(t,x) = V(t,x_1,\dots,x_N)$ is T-periodic in t and τ_k-periodic in x_k, for every $k = 1,\dots,N$. The first aim of this paper is to extend such a result to an infinite-dimensional setting. So, let H be a separable Hilbert space, and let $(e_k)_{k\geq 1}$ be a Hilbert basis. We assume $V : \mathbb{R} \times H \to \mathbb{R}$ to be continuous, T-periodic with respect to its first variable t, and continuously differentiable with respect to its second variable x. Here is our result.

Theorem 1.1. *Assume that there exists a sequence of positive real numbers $(\tau_k)_{k\geq 1}$ such that*

$$V(t, x + \tau_k e_k) = V(t,x)\,, \quad \text{for every } (t,x) \in [0,T] \times H \text{ and } k = 1, 2, \dots \tag{1.3}$$

If $\sum_{k=1}^{\infty} \tau_k^2 < +\infty$, then equation (1.2) *has* at least two *geometrically distinct T-periodic solutions, for every $e(t)$ satisfying* (1.1).

The above theorem generalizes [1, Theorem 6], where further regularity assumptions were made on V, in order to obtain the existence of *at least one* T-periodic solution. A Galerkin-type argument was used there to reduce the problem to a sequence of finite-dimensional differential systems, to which a generalized version of the Poincaré–Birkhoff theorem applies (cf. [9]), followed by a limit process.

The proof of Theorem 1.1 will follow the same ideas introduced in [16, 17], taking advantage of the compactness of the Hilbert cube $\prod_{k=1}^{\infty}[0,\tau_k]$. The first solution will be obtained by minimization of the action functional, while the second one will be of mountain pass type.

Using the Lusternik–Schnirelmann theory, it was proved in [15] that, under the same assumptions, system (1.2) in $\mathbb{R}^N$ has indeed *at least* $N+1$ geometrically distinct T-periodic solutions, thus generalizing the result in [17]. (Notice that, when $N \geq 2$, the multiplicity result is not optimal, as shown by the four equilibria of a double pendulum.) Even more, a system of the type

$$\ddot{x} + \mathcal{A}x + \nabla_x V(t,x) = e(t) \tag{1.4}$$

was considered there, involving a symmetric matrix $\mathcal{A}$. Other results in this direction, including the case of Hamiltonian systems leading to a strongly indefinite action functional, were studied, e.g., in [3, 4, 6, 7, 9, 10, 13, 14, 20].

The second aim of this paper is to obtain multiplicity results for an infinite-dimensional system of the type (1.4), when $\mathcal{A} : H \to H$ is a semi-negative definite bounded selfadjoint operator, whose spectrum contains 0 as an isolated eigenvalue, $V(t,x)$ is bounded above and T-periodic in t, and the image of $\nabla_x V$ is contained in a compact set of H. Denoting by $\mathcal{N}(\mathcal{A})$ the null space of $\mathcal{A}$, we distinguish two cases.

If $\mathcal{N}(\mathcal{A})$ has finite dimension N and $V(t,x)$ satisfies a periodicity condition of the type (1.3), with the e_k replaced by the elements of an orthonormal basis of $\mathcal{N}(\mathcal{A})$, the existence of *at least* $N+1$ geometrically distinct T-periodic solutions is proved, when the mean value of $e(t)$ belongs to $\mathcal{N}(\mathcal{A})^{\perp}$. The precise statement will be given in Section 2. The proof, provided in Section 3, will be carried out by the use of an abstract theorem, given in [18] and inspired by [19], providing the multiplicity of critical points of some functionals in a Banach space X which are bounded below, invariant under the action of some discrete subgroups of X, and satisfy a suitable Palais–Smale condition.

If $\mathcal{N}(\mathcal{A})$ has infinite dimension, assuming in addition that $\sum_{k=1}^{\infty} \tau_k^2 < +\infty$, after finding the first solution by minimization of the action functional, a second one is provided by the Mountain Pass Theorem. We thus get, in this case, the existence of *at least two* geometrically distinct T-periodic solutions.

The paper ends with some examples and an open problem.

2. The main result

Let H be a separable Hilbert space, with scalar product $(\cdot,\cdot)$ and corresponding norm $|\cdot|$. In this space, we consider the equation

$$\ddot{x} + \mathcal{A}x + \nabla_x V(t,x) = e(t)\,, \tag{2.1}$$

where $\mathcal{A} \in \mathcal{L}(H)$ is a bounded selfadjoint operator, and $e : \mathbb{R} \to H$ is continuous and T-periodic. Concerning the function $V : \mathbb{R} \times H \to \mathbb{R}$, it is continuous, T-periodic in its first variable t, and differentiable with respect to its second variable x, with corresponding continuous gradient $\nabla_x V : \mathbb{R} \times H \to H$.

Let us introduce our assumptions. We denote by $\mathcal{N}(\mathcal{A})$ the null-space of $\mathcal{A}$, and by $\sigma(\mathcal{A})$ its spectrum. We take a Hilbert basis $(a_k)_k$ of $\mathcal{N}(\mathcal{A})$, considered as a subspace of H. If $\mathcal{N}(\mathcal{A})$ has a finite dimension, its basis will be given by $(a_1, \dots, a_N)$; if it is infinite-dimensional, we will have a sequence of vectors $(a_1, a_2, \dots)$.

A1. *The selfadjoint operator $\mathcal{A}$ is semi-negative definite, with $\mathcal{N}(\mathcal{A}) \neq \{0\}$, and*

$$\sup\big(\sigma(\mathcal{A}) \setminus \{0\}\big) < 0\,.$$

So, 0 is an isolated point of $\sigma(\mathcal{A})$.

A2. *The mean value of $e(t)$ is orthogonal to $\mathcal{N}(\mathcal{A})$, i.e.,*

$$\int_0^T e(t)\,dt \in \mathcal{N}(\mathcal{A})^\perp\,.$$

Then, we have that

$$\int_0^T (e(t), a_k)\,dt = 0\,, \quad \text{for every } k = 1, 2, \dots$$

A3. *There exists a sequence of positive real numbers $(\tau_k)_{k\geq 1}$ such that*

$$V(t, x + \tau_k a_k) = V(t,x)\,, \quad \text{for every } (t,x) \in [0,T] \times H \text{ and } k = 1, 2, \dots$$

A4. *There is a nonnegative constant C such that*

$$V(t,x) \leq C\,, \quad \text{for every } (t,x) \in [0,T] \times H\,.$$

A5. The set $\nabla_x V([0,T] \times H)$ is precompact in H.

In the above setting, we can now state the main result of this paper.

Theorem 2.1. *Assume that conditions* A1 *to* A5 *hold. If* $\mathcal{N}(\mathcal{A})$ *is finite-dimensional, then equation* (2.1) *has* at least $\dim \mathcal{N}(\mathcal{A}) + 1$ *geometrically distinct* T*-periodic solutions. On the other hand, if* $\mathcal{N}(\mathcal{A})$ *is infinite-dimensional and* $\sum_{k=1}^{\infty} \tau_k^2 < +\infty$, *then there are* at least two *of them.*

Notice that, once a T-periodic solution $x(t)$ has been found, any function obtained by adding to it some integer multiples of $\tau_k a_k$ is still a T-periodic solution. We say that two T-periodic solutions are *geometrically distinct* if they cannot be obtained one from the other in this way.

Concerning assumption A5, we remark that it will surely be satisfied if the following holds.

A5′. *There exists a Hilbert basis* $(e_k)_{k\geq 1}$ *of* H *and a nonnegative sequence* $(M_k)_k$, *with* $\sum_{k=1}^{\infty} M_k^2 < +\infty$, *such that*

$$\left| \frac{\partial V}{\partial e_k}(t,x) \right| \leq M_k \,, \quad \text{for every } (t,x) \in [0,T] \times H \text{ and } k = 1,2,\dots$$

Indeed, A5′ implies that $\nabla_x V([0,T] \times H)$ is contained in a Hilbert cube, which is a compact set in H. In the above formula, we have used the notation

$$\frac{\partial V}{\partial e_k}(t,x) = \lim_{\tau \to 0} \frac{V(t, x + \tau e_k) - V(t,x)}{\tau} \,.$$

Notice that Theorem 1.1 is a direct consequence of Theorem 2.1, taking $\mathcal{A} = 0$ and $(a_k)_k = (e_k)_k$, a Hilbert basis of H. Indeed, the periodicity assumption in Theorem 1.1 and the compactness of the set $[0,T] \times \prod_{k=1}^{\infty} [0, \tau_k]$ show that A4 and A5 are surely satisfied.

In the proof of Theorem 2.1, we will need a result from [18], which we now recall, for the reader's convenience.

Let G be a discrete subgroup of a Banach space X and $\pi : X \to X/G$ be the canonical surjection. A subset S of X is G-invariant if $\pi^{-1}(\pi(S)) = S$, and a function f defined on X is G-invariant if $f(u+g) = f(u)$, for every $u \in X$ and every $g \in G$. If $\varphi \in C^1(X, \mathbb{R})$ is G-invariant, then φ' is also G-invariant, and if u is a critical point of φ, the same is true for $u + g$ for all $g \in G$. The corresponding set $\{u + g : g \in G\}$ is called a critical orbit of φ.

A G-invariant differentiable function $\varphi : X \to \mathbb{R}$ satisfies the $(PS)_G$ condition if, for every sequence $(u_n)_n$ in X such that $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$, the sequence $(\pi(u_n))_n$ contains a convergent subsequence.

The following multiplicity result for the critical points of G-invariant functionals is stated as Theorem 4.12 in [18].

Theorem 2.2. *Let* $\varphi \in C^1(X, \mathbb{R})$ *be a* G*-invariant functional satisfying the* $(PS)_G$ *condition. If* φ *is bounded from below and if the dimension* N *of the space generated by* G *is finite, then* φ *has at least* $N + 1$ *critical orbits.*

3. Proof of Theorem 2.1

Let $L^2([0,T],H)$ be the space of measurable functions $x:[0,T]\to H$ such that $|x|$ is square integrable. It is a Hilbert space equipped with the scalar product

$$\langle x,y\rangle_2 = \int_0^T (x(t),y(t))\,dt\,,$$

and corresponding norm

$$\|x\|_2 = \left(\int_0^T |x(t)|^2\,dt\right)^{\frac{1}{2}}.$$

We consider the space $H^1([0,T],H)$, made of those functions x belonging to $L^2([0,T],H)$ with weak derivative $\dot{x}$ also in $L^2([0,T],H)$. It is a Hilbert space, as well, with the scalar product

$$\langle x,y\rangle = \langle x,y\rangle_2 + \langle \dot{x},\dot{y}\rangle_2 = \int_0^T \left[(x(t),y(t)) + (\dot{x}(t),\dot{y}(t))\right]\,dt\,,$$

and corresponding norm

$$\|x\| = \left(\|x\|_2^2 + \|\dot{x}\|_2^2\right)^{\frac{1}{2}} = \left(\int_0^T \left[|x(t)|^2 + |\dot{x}(t)|^2\right]\,dt\right)^{\frac{1}{2}}.$$

Moreover, $H^1([0,T],H)$ is continuously embedded in $C([0,T],H)$, the space of continuous functions, with the usual norm

$$\|x\|_\infty = \max\{|x(t)| : t\in[0,T]\}\,.$$

(For further information on the space $H^1([0,T],H)$ we refer, e.g., to [2].)

Let

$$H_T^1 = \{x\in H^1([0,T],H) : x(0) = x(T)\}\,,$$

and define the functional $\varphi : H_T^1 \to \mathbb{R}$ as

$$\varphi(x) = \int_0^T \left[\tfrac{1}{2}|\dot{x}(t)|^2 - \tfrac{1}{2}(\mathcal{A}x(t),x(t)) - V(t,x(t)) + (e(t),x(t))\right]\,dt\,.$$

It is continuously differentiable, and its critical points correspond to the T-periodic solutions of (2.1). Moreover, by A2 and A3,

$$\varphi(x+\tau_k a_k) = \varphi(x)\,,\quad \text{for every } x\in H_T^1 \text{ and } k\geq 1\,. \tag{3.1}$$

As usual, we identify the constant functions with their constant value. So, having identified H with the space of constant functions, it will be a subspace of H_T^1. Hence, we can write

$$H_T^1 = H\oplus W = \mathcal{N}(\mathcal{A})\oplus\mathcal{N}(\mathcal{A})^\perp\oplus W = \mathcal{N}(\mathcal{A})\oplus\widetilde{W}\,.$$

Here, W is the orthogonal space to H in H_T^1, $\mathcal{N}(\mathcal{A})^\perp$ is the orthogonal to $\mathcal{N}(\mathcal{A})$ in H, and $\widetilde{W} = \mathcal{N}(\mathcal{A})^\perp\oplus W$. Correspondingly, we will write each $x\in H_T^1$ as

$x(t) = \bar{x} + \tilde{x}(t)$, with $\bar{x} \in \mathcal{N}(\mathcal{A})$ and $\tilde{x} \in \widetilde{W}$. Moreover, we will write $\tilde{x}(t) = \hat{x} + \check{x}(t)$, with $\hat{x} \in \mathcal{N}(\mathcal{A})^\perp$ and $\check{x} \in W$. Notice that, for any $x \in H^1_T$,

$$[x] := \frac{1}{T}\int_0^T x(t)\,dt = \bar{x} + \hat{x}\,, \quad \frac{1}{T}\int_0^T \check{x}(t)\,dt = 0\,. \tag{3.2}$$

Proposition 3.1. *For every $x \in H^1_T$, one has*

$$\|\check{x}\|_\infty \le \sqrt{T}\,\|\dot{x}\|_2\,. \tag{3.3}$$

Proof. Let $(e_k)_{k\ge1}$ be a Hilbert basis of H. Then, for any function $x \in H^1_T$, we may write

$$\check{x}(t) = \sum_{k=1}^\infty (\check{x}(t), e_k)\,e_k = \sum_{k=1}^\infty \check{x}_k(t)\,e_k\,.$$

Being $\check{x}_k$ continuous, T-periodic with zero mean, there is a $t_k \in [0,T]$ for which $\check{x}_k(t_k) = 0$, hence

$$|\check{x}_k(t)| = \left|\check{x}_k(t_k) + \int_{t_k}^t \dot{x}_k(s)\,ds\right| \le \int_0^T |\dot{x}_k(s)|\,ds \le \sqrt{T}\left(\int_0^T |\dot{x}_k(s)|^2\,ds\right)^{\frac12},$$

for every $t \in [0,T]$. As a consequence,

$$|\check{x}(t)|^2 = \sum_{k=1}^\infty |\check{x}_k(t)|^2 \le T\int_0^T \sum_{k=1}^\infty |\dot{x}_k(s)|^2\,ds = T\int_0^T |\dot{x}(s)|^2\,ds\,,$$

for every $t \in [0,T]$, whence the conclusion. ☐

By A1, A2, A4, (3.2) and (3.3), setting $\delta := -\sup(\sigma(\mathcal{A}) \setminus \{0\})$,

$$\begin{aligned}
\varphi(x) &= \int_0^T \left[\tfrac12|\dot{x}(t)|^2 - \tfrac12(\mathcal{A}\tilde{x}(t),\tilde{x}(t)) - V(t,x(t)) + (e(t),\tilde{x}(t))\right]dt \\
&\ge \int_0^T \left[\tfrac12|\dot{x}(t)|^2 - \tfrac12(\mathcal{A}\hat{x},\hat{x}) - \tfrac12(\mathcal{A}\check{x}(t),\check{x}(t))\right]dt - CT - T\|e\|_\infty\|\tilde{x}\|_\infty \\
&\ge \int_0^T \tfrac12|\dot{x}(t)|^2dt - \tfrac12 T(\mathcal{A}\hat{x},\hat{x}) - CT - T\|e\|_\infty(|\hat{x}| + \|\check{x}\|_\infty) \\
&\ge \tfrac12\|\dot{x}\|_2^2 + \tfrac12 T\delta|\hat{x}|^2 - CT - T^{\frac32}\|e\|_\infty\|\dot{x}\|_2 - T\|e\|_\infty|\hat{x}|\,.
\end{aligned}$$

Hence, since $\delta > 0$, there are two positive constants $c > 0$ and $c' > 0$ for which

$$\varphi(x) \ge c\left(\|\dot{x}\|_2^2 + |\hat{x}|^2\right) - c'\,, \tag{3.4}$$

and the functional φ is bounded below.

For $u \in C([0,T],H)$, we denote by Pu the indefinite integral defined on $[0,T]$ by

$$Pu(t) = \int_0^t u(s)\,ds\,.$$

Lemma 3.2. *Let $E \subseteq C([0,T],H)$ be such that $A := \{u([0,T]) : u \in E\}$ is precompact in H. Then:*

(a) the set $B := \{\int_0^T u(t)\,dt : u \in E\}$ is precompact in H;
(b) the set $S := \{Pu : u \in E\}$ is precompact in $C([0,T],H)$.

Proof. (a) Let $\varepsilon > 0$. There exists a finite sequence $(v_1, \ldots, v_n)$ in H such that, denoting by $B(u, \rho)$ any open ball of center u and radius ρ,

$$A \subseteq \bigcup_{k=1}^{n} B(v_k, \varepsilon)\,.$$

We denote by Q_0 the orthogonal projection from H to the space V generated by $(v_1, \ldots, v_n)$. The set

$$C = \left\{ \int_0^T Q_0 u(t)\, dt : u \in E \right\}$$

is bounded in V, hence precompact in V. This implies the existence of a finite sequence $(w_1, \ldots, w_m)$ in V such that

$$C \subseteq \bigcup_{k=1}^{m} B(w_k, \varepsilon)\,.$$

For every $u \in E$, we have

$$\left| \int_0^T u(t)\, dt - \int_0^T Q_0 u(t)\, dt \right| \le \int_0^T |u(t) - Q_0 u(t)|\, dt \le \varepsilon T\,.$$

It follows that

$$B \subseteq \bigcup_{k=1}^{m} B(w_k, (1+T)\varepsilon)\,.$$

Since ε is arbitrary, B is precompact in H.

(b) Let us define

$$R := \{P(Q_0 u) : u \in E\}\,.$$

The set $\{P(Q_0 u)(t) : t \in [0, T]\,, u \in E\}$ is bounded in V, hence precompact in V. For $0 \le t_1 \le t_2 \le T$, we have

$$|P(Q_0 u)(t_2) - P(Q_0 u)(t_1)| = \left| \int_{t_1}^{t_2} Q_0(u)(s)\, ds \right| \le c(t_2 - t_1)\,,$$

for some $c > 0$. By the Ascoli–Arzelá theorem, the set R is precompact in $C([0, T], V)$. This implies, for any $\varepsilon > 0$, the existence of a finite sequence $(f_1, \ldots, f_N)$ in $C([0, T], V)$ such that

$$R \subseteq \bigcup_{k=1}^{N} B(f_k, \varepsilon)\,.$$

Since, for every $u \in E$ and $t \in [0, T]$, we have

$$|Pu(t) - P(Q_0 u)(t)| \le \int_0^t |u(s) - Q_0 u(s)|\, ds \le T\varepsilon\,,$$

we conclude that

$$S \subseteq \bigcup_{k=1}^{N} B(f_k, (1+T)\varepsilon)\,.$$

Since $\varepsilon > 0$ is arbitrary, S is precompact in $C([0, T], H)$. □

We now prove the following.

Proposition 3.3. *If $(x^n)_n$ is a sequence in H^1_T such that $(\varphi(x^n))_n$ is bounded and $\nabla\varphi(x^n) \to 0$, then $(\tilde{x}^n)_n$ has a convergent subsequence.*

Proof. Since $(\varphi(x^n))_n$ is bounded, by (3.3) and (3.4) we have that $(\tilde{x}^n)_n$ is bounded in H^1_T. On the other hand, we can modify $\bar{x}^n$ into some $\bar{z}^n$ such that the scalar product $(\bar{z}^n, a_k)$ belongs to $[0, \tau_k]$, for every k, and $(\bar{z}^n, a_k) \equiv (\bar{x}^n, a_k) \bmod \tau_k$. Defining $z^n = \bar{z}^n + \tilde{x}^n$, we have a new sequence for which $\varphi(z^n) = \varphi(x^n)$ and $\nabla\varphi(z^n) = \nabla\varphi(x^n)$, by (3.1). Moreover, $(z^n)_n$ is bounded, hence there is a subsequence, still denoted by $(z^n)_n$, which weakly converges to some $z^* \in H^1_T$. We want to show that $(z^n)_n$ strongly converges to z^* in H^1_T.

Since $\nabla\varphi(z^n) \to 0$ and $(z^n)_n$ weakly converges to z^*, we have that

$$\langle \nabla\varphi(z^n) - \nabla\varphi(z^*), z^n - z^* \rangle \to 0\,,$$

i.e.,

$$\begin{aligned} \lim_n \int_0^T \Big[|\dot{z}^n(t) - \dot{z}^*(t)|^2 - (\mathcal{A}(z^n(t) - z^*(t)), z^n(t) - z^*(t)) - \\ -(\nabla_x V(t, z^n(t)) - \nabla_x V(t, z^*(t)), z^n(t) - z^*(t)) \Big]\, dt = 0\,. \end{aligned} \tag{3.5}$$

Since $(z^n)_n$ weakly converges to z^* in $L^2([0,T], H)$,

$$\lim_n \int_0^T (\nabla_x V(t, z^*(t)), z^n(t) - z^*(t))\, dt = 0\,. \tag{3.6}$$

Claim. Up to a subsequence,

$$\lim_n \int_0^T (\nabla_x V(t, z^n(t)), z^n(t) - z^*(t))\, dt = 0\,. \tag{3.7}$$

Proof of the Claim. Define on $[0,T]$ the continuous functions

$$w^n(t) = \nabla_x V(t, z^n(t))\,, \quad y^n(t) = z^n(t) - z^*(t)\,,$$

having values in H. Using the notation in (3.2), we have

$$\begin{aligned} \int_0^T (w^n(t), y^n(t))\, dt &= \int_0^T \Big([w^n] + \breve{w}^n(t), [y^n] + \breve{y}^n(t) \Big)\, dt \\ &= T([w^n], [y^n]) + \int_0^T (\breve{w}^n(t), \breve{y}^n(t))\, dt\,. \end{aligned}$$

Since $(y^n)_n$ weakly converges to 0 in $L^2([0,T], H)$, we see that $([y^n])_n$ weakly converges to 0 in H. Indeed, for every $\eta \in H$, considering it as a constant function in $L^2([0,T], H)$, we have that

$$([y^n], \eta) = \Big(\frac{1}{T} \int_0^T y^n(t)\, dt\,, \eta \Big) = \frac{1}{T} \int_0^T (y^n(t)\,, \eta)\, dt \to 0\,.$$

Moreover, by A5, the set $\{w^n(t) : t \in [0,T], n \in \mathbb{N}\}$ is precompact in H. Hence, by Lemma 3.2(a), the sequence $([w^n])_n$ is contained in a compact subset of H. Then, up to a subsequence,

$$\lim_n ([w^n], [y^n]) = 0\,.$$

On the other hand, defining

$$\xi^n(t) = \int_0^t \breve{w}^n(s)\,ds = (P\breve{w}^n)(t)\,,$$

we have that $\xi^n(T) = \xi^n(0)$, and recalling that $\breve{y}^n(t)$ and $y^n(t)$ differ by a constant, integrating by parts we have

$$\int_0^T (\breve{w}^n(t), \breve{y}^n(t))\,dt = -\int_0^T (\xi^n(t), \dot{y}^n(t))\,dt\,.$$

We know that $(\dot{y}^n)_n$ weakly converges to 0 in $L^2([0,T],H)$. Moreover, since $\{w^n(t) : t \in [0,T], n \in \mathbb{N}\}$ is precompact in H, by Lemma 3.2(b), the sequence $(\xi^n)_n$ is contained in a compact subset of $C([0,T],H)$ and hence, up to a subsequence,

$$\lim_n \int_0^T (\xi^n(t), \dot{y}^n(t))\,dt = 0\,,$$

thus proving (3.7). The Claim is thus proved. □

Going back to (3.5), by (3.6) and (3.7), we get

$$\lim_n \int_0^T \left[|\dot{z}^n(t) - \dot{z}^*(t)|^2 - (\mathcal{A}(z^n(t) - z^*(t)), z^n(t) - z^*(t))\right]\,dt = 0\,.$$

By A1, being $\mathcal{A}$ semi-negative definite, we deduce that

$$\lim_n \int_0^T |\dot{z}^n(t) - \dot{z}^*(t)|^2\,dt = 0\,,$$

and

$$\lim_n \int_0^T (\mathcal{A}(z^n(t) - z^*(t)), z^n(t) - z^*(t))\,dt = 0\,,$$

i.e.,

$$\lim_n \int_0^T [(\mathcal{A}(\hat{z}^n - \hat{z}^*), \hat{z}^n - \hat{z}^*) + (\mathcal{A}(\breve{z}^n(t) - \breve{z}^*(t)), \breve{z}^n(t) - \breve{z}^*(t))]\,dt = 0\,,$$

Hence, $\dot{z}^n \to \dot{z}^*$ in $L^2([0,T],H)$, and, by A1, also $\hat{z}^n \to \hat{z}^*$. By Proposition 3.1, $\breve{z}^n \to \breve{z}^*$ so that, being $\tilde{z}^n = \hat{z}^n + \breve{z}^n$, we have proved that $(\tilde{z}^n)_n$ converges in H^1_T. This fact leads to the conclusion of the proof. □

We now distinguish the two cases. If $\mathcal{N}(\mathcal{A})$ has finite dimension N, then Theorem 2.2 applies, because Proposition 3.3 provides the $(PS)_G$ condition for

$$G = \left\{ \sum_{k=1}^N m_k \tau_k a_k : m_k \in \mathbb{Z} \right\}\,,$$

which is a subgroup of H^1_T, and φ is G-invariant. We thus get $N+1$ critical orbits of φ.

Assume now that $\mathcal{N}(\mathcal{A})$ is infinite-dimensional. We first prove that φ has a minimum. To this aim, let $(x^n)_n$ be a sequence in H^1_T such that $\varphi(x^n) \to \iota := \inf \varphi(H^1_T)$. By the Ekeland Principle, there is a sequence $(y^n)_n$ such that

$$\|x^n - y^n\| \to 0\,, \quad \varphi(y^n) \to \iota\,, \quad \nabla\varphi(y^n) \to 0\,.$$

Moreover, by (3.1), we can argue as in beginning of the proof of Proposition 3.3 and assume without loss of generality that

$$\bar{y}^n \in K := \left\{ y = \sum_{k=1}^{\infty} y_k a_k : y_k \in [0, \tau_k] \text{ for } k = 1, 2 \dots \right\},$$

for every n. The set K is compact, being isometric to the Hilbert cube $\prod_{k=1}^{\infty}[0,\tau_k]$ in ℓ^2, since $\sum_{k=1}^{\infty} \tau_k^2 < +\infty$. Using this and Proposition 3.3, there is a subsequence of $(y^n)_n$ converging to some $y^* \in H^1_T$. Then, $\varphi(y^*) = \iota$, and $\nabla\varphi(y^*) = 0$. We have thus found a minimum point for the functional φ.

If y^* is not an isolated minimum point, then there are infinitely many minimum points near y^*. In this case, then, there are infinitely many geometrically distinct critical points of φ.

Otherwise, if y^* is an isolated minimum point, there is a constant $r > 0$ such that

$$\varphi(u) > \min \varphi\,, \quad \text{for every } u \in \overline{B}(y^*, r) \setminus \{y^*\}\,.$$

(We denote by $B(y^*, r)$ the open ball centered at y^*, with radius $r > 0$, and by $\overline{B}(y^*, r)$ its closure.) Let us prove that

$$\inf_{\partial B(y^*,r)} \varphi > \min \varphi\,.$$

By contradiction, assume that there is a sequence $(\xi^n)_n$ in $\partial B(y^*, r)$ such that $\varphi(\xi^n) \to \min\varphi$. Using the Ekeland Principle, it is possible to find a sequence $(\eta^n)_n$ in H^1_T such that $\varphi(\eta^n) \to \min\varphi$, $\|\eta^n - \xi^n\| \to 0$ and $\nabla\varphi(\eta^n) \to 0$. By (3.1), we can assume without loss of generality that $\bar{\eta}^n \in K$, for every n. Then, by Proposition 3.3, there is a subsequence of $(\eta^n)_n$ which converges to some y in H^1_T. Being $\partial B(x, r)$ a closed set, we have that $y \in \partial B(y^*, r)$, and by continuity $\varphi(y) = \min\varphi$, a contradiction.

Choosing, e.g., $y^{**} = y^* + \tau_1 a_1$, if $r > 0$ small enough we have that $y^{**} \notin B(y^*, r)$, and

$$\varphi(y^{**}) = \varphi(y^*) < \inf_{\partial B(y^*,r)} \varphi\,.$$

So, the Mountain Pass Theorem applies: setting

$$\Gamma = \{\gamma \in C([0,1], H^1_T) : \gamma(0) = y^*\,, \gamma(1) = y^{**}\}\,,$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t))\,,$$

there is a sequence $(u^n)_n$ in H^1_T such that

$$\lim_n \varphi(u^n) = c\,, \qquad \lim_n \nabla\varphi(u^n) = 0\,.$$

Moreover, $\varphi(y^*) < c$. Proceeding as in the first part of the proof, we can assume without loss of generality that $\bar{u}^n \in K$, and we can find a subsequence of $(u^n)_n$ which converges to some $u^* \in H^1_T$, so that $\varphi(u^*) = c$ and $\nabla\varphi(u^*) = 0$. Since $\varphi(y^*) < \varphi(u^*)$, we have thus found two critical points, y^* and u^*, which are geometrically distinct. □

4. Some examples and an open problem

Let $(e_k)_k$ be an orthonormal basis in H, and assume that the periodicity condition (1.3) holds. Assume moreover (1.1), i.e., that $e(t)$ has a zero mean. Defining, for every $N \geq 1$, the projection

$$P_N : H \to H\,, \quad x = \sum_{k=1}^{\infty} x_k e_k \mapsto \sum_{k=N+1}^{\infty} x_k e_k\,,$$

we have that

$$\mathcal{N}(P_N) = \operatorname{span}\{e_1, \dots, e_N\}\,.$$

Then, taking $\mathcal{A} = -P_N$, Theorem 2.1 applies to the system

$$\ddot{x} - P_N x + \nabla_x V(t,x) = e(t)\,,$$

and provides us with at least $N+1$ geometrically distinct T-periodic solutions.

Notice that the number of T-periodic solutions increases indefinitely together with N. However, passing to the limit on N, the system becomes

$$\ddot{x} + \nabla_x V(t,x) = e(t)\,,$$

to which Theorem 2.1 still applies, but guarantees *only two* T-periodic solutions. It is an open problem to know if, in this last case, the existence of more than two T-periodic solutions can be proved.

As a first example of application, we consider the space $H = \ell^2$ and the function

$$V(t,x) = -\sum_{k=1}^{+\infty} \frac{c_k}{\omega_k} \cos(\omega_k x_k)\cos(\omega_{k+1} x_{k+1}),$$

with $c_k > 0$ and $\omega_k > 0$, for every $k \geq 1$. We have the cyclically coupled system

$$x_k'' + \left[\frac{c_{k-1}\omega_k}{\omega_{k-1}}\cos(\omega_{k-1}x_{k-1}) + c_k\cos(\omega_{k+1}x_{k+1})\right]\sin(\omega_k x_k) = e_k(t),\ k = 1,2,\dots$$

where we have formally set $c_0 = 0$ and $\omega_0 = 1$. Assuming that the sequences

$$(c_k)_k, \quad \left(\frac{1}{\omega_k}\right)_k, \quad \left(\frac{c_{k-1}\omega_k}{\omega_{k-1}}\right)_k$$

all belong to ℓ^2 (e.g., we could take $c_k = 1/k$ and $\omega_k = k$), we can apply Theorem 2.1, so that at least two T-periodic solutions exist.

Another example can be obtained if we now identify ℓ^2 with the space of sequences $(\xi_k)_k$ where k ranges from $-\infty$ to $+\infty$, i.e., with $\ell^2(\mathbb{Z})$. Defining

$$\mathcal{V}(t,x) = -\sum_{k=-\infty}^{+\infty} \frac{1}{\omega_k}\cos(\omega_k x_k)\Big(c_k'\cos(\omega_{k-1}x_{k-1}) + c_k''\cos(\omega_{k+1}x_{k+1})\Big),$$

with $c'_k, c''_k > 0$ and $\omega_k > 0$ for every integer k, we have the system

$$x''_k + [\alpha_k \cos(\omega_{k-1} x_{k-1}) + \beta_k \cos(\omega_{k+1} x_{k+1})] \sin(\omega_k x_k) = e_k(t), \quad k \in \mathbb{Z},$$

where

$$\alpha_k = \frac{c'_k \omega_{k-1} + c''_{k-1} \omega_k}{\omega_{k-1}}, \qquad \beta_k = \frac{c''_k \omega_{k+1} + c'_{k+1} \omega_k}{\omega_{k+1}}.$$

If we assume that all the sequences $(c_k)_k$, $(\omega_k^{-1})_k$, $(\alpha_k)_k$, $(\beta_k)_k$ belong to $\ell^2(\mathbb{Z})$ (e.g., taking $c'_k = c''_k = (|k|+1)^{-1}$ and $\omega_k = |k|+1$), by Theorem 2.1 we conclude that at least two T-periodic solutions must exist.

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