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Generalizing the Lusternik–Schnirelmann critical point theorem

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To the memory of François Munjamarere

Abstract

We provide a multiplicity result for critical points of a functional defined on the product of a compact manifold without boundary and a convex set, by assuming, for example, an *avoiding rays* condition at the boundary of that set. We then extend this result to an infinite-dimensional setting which well applies to the search of periodic solutions of pendulum-like equations.

1. *Introduction*

In their pioneering paper [**15**], Lusternik and Schnirelmann opened the way to the search of multiple critical points of regular functionals by exploiting the topological properties of their domain. Let us recall their result.

Theorem 1 (Lusternik–Schnirelmann). *Let $\mathcal{V}$ be an N-dimensional compact manifold of class $\mathcal{C}^2$ without boundary, and let $\varphi : \mathcal{V} \to \mathbb{R}$ be a continuously differentiable functional. Then, φ has at least* $\mathrm{cat}(\mathcal{V})$ *critical points.*

In the above statement, $\mathrm{cat}(\mathcal{V})$ stands for the *Lusternik–Schnirelmann category*, introduced for that purpose in [**15**]: it is the least number of closed contractible sets which are needed to cover $\mathcal{V}$. For example, if $\mathcal{V} = \mathbb{S}^N$, the N-dimensional sphere, we have $\mathrm{cat}(\mathbb{S}^N) = 2$, while if $\mathcal{V} = \mathbb{T}^N$, the N-dimensional torus, we have $\mathrm{cat}(\mathbb{T}^N) = N + 1$.

Different generalizations of the above theorem have been proposed in the case of a manifold $\mathcal{V}$ with boundary. Typically, as in [**1**], the gradient of φ is assumed to 'point outward' or 'inward' on the boundary. See also, for example, [**13**, **16**, **17**, **25**].

The aim of this paper is to obtain multiple critical points of a continuously differentiable functional $\varphi : \mathcal{V} \times D \to \mathbb{R}$, defined on the product of an N-dimensional compact manifold $\mathcal{V}$ of class $\mathcal{C}^2$ without boundary and an M-dimensional convex compact set D with nonempty interior. In this case, our assumptions on the direction of the gradient of φ on the boundary are indeed much weaker than the ones usually considered in literature.

In order to state our results, let $\nu_D(y)$ denote the unit outward normal to the boundary of D at some point $y \in \partial D$. Here is our first contribution.

Theorem 2. *Assume that D has a smooth boundary, and that*

$$\nabla_y \varphi(x, y) \notin \{\alpha\, \nu_D(y) : \alpha \geqslant 0\}\,, \qquad \text{for every } \ (x, y) \in \mathcal{V} \times \partial D. \tag{1}$$

Then, there are at least $\mathrm{cat}(\mathcal{V})$ *critical points of φ in $\mathcal{V} \times D$.*

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Note that Theorem 1 could be seen as a special case of Theorem 2, taking $M = 0$. The *avoiding outer rays condition* (1) recalls the usual assumption in the Poincaré–Bohl theorem. It can easily be replaced by the following *avoiding inner rays condition*

$$\nabla_y \varphi(x, y) \notin \{-\alpha\, \nu_D(y) : \alpha \geqslant 0\}\,, \qquad \text{for every } \ (x, y) \in \mathcal{V} \times \partial D. \tag{2}$$

We will also prove the following variant of Theorem 2, where the gradient of φ on the boundary is in some sense driven by a symmetric matrix. We assume here that D is *strongly convex*, meaning that, for any $y \in \partial D$, the height function $\eta \mapsto \langle \eta - y, -\nu_D(y)\rangle$ has a nondegenerate minimum at $\eta = y$.

Theorem 3. *Assume that D is strongly convex, with a smooth boundary, and that there exists a regular symmetric $(M \times M)$-matrix $\mathbb{A}$ for which*

$$\langle \nabla_y \varphi(x, y),\, \mathbb{A}\nu_D(y)\rangle > 0\,, \qquad \textit{for every } \ (x, y) \in \mathcal{V} \times \partial D. \tag{3}$$

Then, there are at least $\mathrm{cl}(\mathcal{V}) + 1$ critical points of φ in $\mathcal{V} \times D$.

Here we have the entrance of $\mathrm{cl}(\mathcal{V})$, the *cup length* of $\mathcal{V}$. Denoting by $\check{H}^*(\mathcal{V})$ the Alexander–Spanier cohomology of $\mathcal{V}$ with coefficients in $\mathbb{R}$, and by $\smile$ the cup product in $\check{H}^*(\mathcal{V})$, we recall that $\mathrm{cl}(\mathcal{V})$ is a positive integer m if there are m elements $u_i \in \check{H}^{n_i}(\mathcal{V})$, with $n_i \geqslant 1$, such that $u_1 \smile \cdots \smile u_m \neq 0$, and m is maximal with respect to this property. Such an m always exists if $\mathcal{V}$ is compact, cf. [**24**]. It can be proved that $\mathrm{cat}(\mathcal{V}) \geqslant \mathrm{cl}(\mathcal{V}) + 1$, and in many interesting cases equality holds. For example, if $\mathcal{V} = \mathbb{S}^N$, the N-dimensional sphere, we have $\mathrm{cl}(\mathbb{S}^N) = 1$, while if $\mathcal{V} = \mathbb{T}^N$, the N-dimensional torus, we have $\mathrm{cl}(\mathbb{T}^N) = N$.

It is reasonable that Theorem 3 should provide at least $\mathrm{cat}(\mathcal{V})$ critical points of φ in $\mathcal{V} \times D$, but I have not been able to prove it. Moreover, I guess that (1), (2), and (3) could be replaced by a nonzero degree assumption, but this seems to be a rather difficult task. However, some more general *avoiding cones conditions* could be considered, as in [**6**].

The proof of Theorem 2 is provided in Section 2, making use of some ideas introduced in [**9**]. In Section 3, a more general infinite-dimensional setting is considered, so to extend a critical point theorem in [**26**]. In Section 4 we propose two possible corollaries, thus generalizing Theorem 3 above. Finally, in Section 5, we provide an application to pendulum-like systems.

2. *Proof of Theorem 2*

By some extension theorems going back to Whitney (see, for example, [**2**, **12**]), our functional can be extended to a continuously differentiable functional on $\mathcal{V} \times \mathbb{R}^M$, for which we keep the same notation φ.

Let $\pi_D y$ denote the projection on the convex set D of a point $y \in \mathbb{R}^M$. Choose some $\bar{r} \in]0, 1[$ and a $\mathcal{C}^\infty$-smooth cutoff function $a : \mathbb{R} \to \mathbb{R}$, with

$$a(s) = \begin{cases} 1\,, & \text{if } s \leqslant 0\,, \\ 0\,, & \text{il } s \geqslant \bar{r}\,. \end{cases}$$

After multiplying $\varphi(x, y)$ by $a(|y - \pi_D y|)$ we see that, for the sake of proving Theorem 2, there is no loss of generality in assuming that

$$\varphi(x, y) = 0\,, \qquad \text{if } \ \mathrm{dist}(y, D) \geqslant \bar{r}.$$

Then, there must exist a constant $\bar{c} > 0$ for which

$$|\nabla_y \varphi(x, y)| < \bar{c}\,, \qquad \text{for every } \ (x, y) \in \mathcal{V} \times \mathbb{R}^M. \tag{4}$$

We claim that there is a $\rho \in]0, \bar{r}[$ such that

$$\nabla_y \varphi(x,y) \notin \{\alpha\, \nu_D(\pi_D y) : \alpha \geqslant 0\}\,, \qquad \text{if } \ 0 < \operatorname{dist}(y,D) < \rho. \tag{5}$$

Indeed, if not, there are a sequence $(x_n, y_n)_n$ in $\mathcal{V} \times \mathbb{R}^M$ and a sequence $(\alpha_n)_n$ of nonnegative real numbers such that

$$0 < \operatorname{dist}(y_n, D) \leqslant 1/n \quad \text{and} \quad \nabla_y \varphi(x_n, y_n) = \alpha_n \nu_D(\pi_D y_n),$$

for every n. By the compactness of $\mathcal{V} \times D$ and (4), there are two subsequences $(x_{n_k}, y_{n_k})_k$ and $(\alpha_{n_k})_k$ such that, for some $(\bar{x}, \bar{y}) \in \mathcal{V} \times \partial D$ and $\bar{\alpha} \geqslant 0$,

$$(x_{n_k}, y_{n_k}) \to (\bar{x}, \bar{y}) \quad \text{and} \quad \alpha_{n_k} \to \bar{\alpha}.$$

By continuity,

$$\nabla_y \varphi(\bar{x}, \bar{y}) = \lim_k \nabla_y \varphi(x_{n_k}, y_{n_k}) = \lim_k \alpha_{n_k} \nu_D(\pi_D y_{n_k}) = \bar{\alpha} \nu_D(\bar{y}),$$

in contradiction with (1).

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$\gamma(s) = \begin{cases} 0, & \text{if } s \leqslant 0\,, \\ s^2, & \text{if } s \geqslant 0\,. \end{cases}$$

We define the continuously differentiable functional $\widetilde{\varphi} : \mathcal{V} \times \mathbb{R}^M \to \mathbb{R}$ as

$$\widetilde{\varphi}(x,y) = \varphi(x,y) - \frac{\bar{c}}{2\rho}\, \gamma(|y - \pi_D y|).$$

We see that $\widetilde{\varphi}$ coincides with φ on the set $\mathcal{V} \times D$, and

$$\widetilde{\varphi}(x,y) = -\frac{\bar{c}}{2\rho}\, |y - \pi_D y|^2\,, \qquad \text{if } \ \operatorname{dist}(y,D) \geqslant \bar{r}.$$

This fact easily implies that $-\widetilde{\varphi}$ is bounded from below and satisfies the Palais–Smale condition, and we deduce that $\widetilde{\varphi}$ has at least $\operatorname{cat}(\mathcal{V})$ critical points, cf. [**21**–**23**]. We will now show that these critical points must belong to $\mathcal{V} \times D$, so that they are indeed critical points of φ. Hence, to conclude the proof, we only need to show that $\widetilde{\varphi}$ has no critical points outside $\mathcal{V} \times D$.

So, let (x,y) be such that $y \notin D$. We have

$$\nabla_y \widetilde{\varphi}(x,y) = \nabla_y \varphi(x,y) - \frac{\bar{c}}{\rho}\, |y - \pi_D y|\, \nu_D(\pi_D y).$$

We distinguish two cases: if $0 < \operatorname{dist}(y,D) < \rho$, then $\nabla_y \widetilde{\varphi}(x,y) \neq 0$ by (5). On the other hand, if $\operatorname{dist}(y,D) \geqslant \rho$, then, by (4),

$$|\nabla_y \varphi(x,y)| < \bar{c} \leqslant \frac{\bar{c}}{\rho} |y - \pi_D y|,$$

so that $\nabla_y \widetilde{\varphi}(x,y) \neq 0$ also in this case.

The proof is thus completed.

3. *An extension of Theorem* 2

Let H be a Hilbert space, Y a finite-dimensional subspace, and let $Z = Y^\perp$, so that $H = Y \oplus Z$. (In the following, we will often identify H with $Y \times Z$.) As before, $\mathcal{V}$ will be an N-dimensional compact manifold of class $\mathcal{C}^2$ without boundary.

Assume that $D \subseteq Y$ is a convex compact set with nonempty interior (in the topology of Y). We are interested in finding the critical points of a continuously differentiable functional

$$\varphi : \mathcal{V} \times D \times Z \to \mathbb{R},$$

having the property that

$$\varphi(x,y,z) = \tfrac{1}{2}\langle Lz, z\rangle + \psi(x,y,z), \tag{6}$$

where $L : Z \to Z$ is a bounded self-adjoint linear invertible operator, and $\psi : \mathcal{V} \times D \times Z \to \mathbb{R}$ is a continuously differentiable function, with a completely continuous and bounded gradient $\nabla\psi$. Recalling the usual definition of differentiability, we are thus assuming that, for some open neighborhood U of D, the functions ψ, φ are indeed defined on $\mathcal{V} \times U \times Z$, and continuously differentiable there.

Let $h : Y \to \mathbb{R}$ be a continuously differentiable function for which

$$D = \{y \in Y : \nabla h(y) = 0\}, \tag{7}$$

and assume that there are a constant $C > 0$ and an invertible linear operator $\mathbb{S} : Y \to Y$ such that

$$|\nabla h(y) - \mathbb{S}y| \leqslant C\,, \qquad \text{for every } y \in Y. \tag{8}$$

Without loss of generality, we can also assume that

$$h(y) = 0\,, \qquad \text{for every } y \in D.$$

Here is the main result of this section.

THEOREM 4. *In the above setting, assume that there is a constant $\rho > 0$ such that*

$$\nabla_y\varphi(x,y,z) \notin \{\alpha\nabla h(y) : \alpha \geqslant 0\}\,, \quad \text{if } 0 < \operatorname{dist}(y,D) < \rho. \tag{9}$$

Then, there are at least $\operatorname{cl}(\mathcal{V}) + 1$ *critical points of* φ *in* $\mathcal{V} \times D \times Z$.

Proof. Since the set $D \times Z$ is convex, we can use some theorems from [**2**, **12**] so to find a continuously differentiable extension of the functional ψ to the domain $\mathcal{V} \times Y \times Z$, for which we retain the same notation ψ, while keeping its gradient $\nabla\psi$ bounded and completely continuous. Let $\bar c > 0$ be such that

$$|\nabla_y\psi(x,y,z)| < \bar c\,, \qquad \text{for every } (x,y,z) \in \mathcal{V} \times Y \times Z. \tag{10}$$

We claim that there is a constant $\varepsilon \in]0,1[$ such that

$$0 < |\nabla h(y)| < \varepsilon \quad \Rightarrow \quad 0 < \operatorname{dist}(y,D) < \rho. \tag{11}$$

Indeed, assume by contradiction that for every positive integer n there is a $y_n \in Y$ such that $0 < |\nabla h(y_n)| < 1/n$ and $\operatorname{dist}(y_n, D) \geqslant \rho$. By (8), being $\mathbb{S}$ invertible, there is an $R > 0$ such that $|y_n| \leqslant R$, for every n. Hence, the sequence $(y_n)_n$ remains in a compact set, and there is a subsequence $(y_{n_k})_k$ such that $y_{n_k} \to \bar y$, for some $\bar y \notin D$. Then, by continuity, $\nabla h(\bar y) = \lim_k \nabla h(y_{n_k}) = 0$, in contradiction with (7).

We now define the bounded self-adjoint operator $\widetilde L : H \to H$ as

$$\widetilde L(y + z) = Lz - \frac{\bar c}{\varepsilon}\,\mathbb{S}y.$$

Note that, since L and $\mathbb{S}$ are invertible, also $\widetilde L$ is such. The continuously differentiable functional $\widetilde\varphi : \mathcal{V} \times H \to \mathbb{R}$, given by

$$\widetilde\varphi(x,y,z) = \tfrac{1}{2}\langle Lz, z\rangle + \psi(x,y,z) - \frac{\bar c}{\varepsilon}\,h(y),$$

coincides with φ on the set $\mathcal{V} \times D \times Z$, and satisfies

$$\widetilde\varphi(x,w) = \tfrac{1}{2}\langle \widetilde L w, w\rangle + \widetilde\psi(x,w),$$

where $\widetilde{\psi} : \mathcal{V} \times H \to \mathbb{R}$ is defined as

$$\widetilde{\psi}(x,y,z) = \psi(x,y,z) - \frac{\bar{c}}{\varepsilon}\left(h(y) - \tfrac{1}{2}\langle \mathbb{S}y, y\rangle\right).$$

Hence, $\widetilde{\psi}$ has a bounded and completely continuous gradient. We can thus apply [**26**, Theorem 3.8] to deduce that $\widetilde{\varphi}$ has at least $\mathrm{cl}(\mathcal{V})+1$ critical points. We will now show that these critical points must belong to $\mathcal{V} \times D \times Z$, so that they are indeed critical points of φ. Hence, to conclude the proof, we only need to show that $\widetilde{\varphi}$ has no critical points (x,y,z) with $y \notin D$.

So, let $(x,y,z) \in \mathcal{V} \times Y \times Z$ be such that $y \notin D$. We will show that $\nabla_y \widetilde{\varphi}(x,y,z) \neq 0$, that is,

$$\nabla_y \psi(x,y,z) \neq \frac{\bar{c}}{\varepsilon}\, \nabla h(y). \tag{12}$$

We examine two cases: if $0 < |\nabla h(y)| < \varepsilon$, then (12) holds, by (11) and (9). On the other hand, if $|\nabla h(y)| \geqslant \varepsilon$, then, by (10),

$$|\nabla_y \psi(x,y,z)| < \bar{c} \leqslant \left| \frac{\bar{c}}{\varepsilon}\, \nabla h(y) \right|,$$

so that (12) holds, again. The proof is thus completed. □

4. *Some corollaries*

In this section we assume again that H is a Hilbert space, Y a finite-dimensional subspace, and $Z = Y^{\perp}$. As before, $\mathcal{V}$ is an N-dimensional compact manifold of class $\mathcal{C}^2$ without boundary, and $D \subseteq Y$ is a convex compact set with nonempty interior. The functional $\varphi : \mathcal{V} \times D \times Z \to \mathbb{R}$ is like in (6), that is,

$$\varphi(x,y,z) = \tfrac{1}{2}\langle Lz, z\rangle + \psi(x,y,z),$$

where $L : Z \to Z$ is a bounded self-adjoint linear invertible operator, and $\psi : \mathcal{V} \times D \times Z \to \mathbb{R}$ is a continuously differentiable function, with a completely continuous and bounded gradient $\nabla\psi$.

We will now state and prove two corollaries of Theorem 4, and finally give a proof of Theorem 3.

Corollary 5. *Assume that D has a smooth boundary, and that there is a constant $\rho > 0$ such that*

$$\nabla_y \varphi(x,y,z) \notin \{\alpha\, \nu_D(\pi_D y) : \alpha \geqslant 0\}, \quad \text{if } 0 < \mathrm{dist}(y, D) < \rho. \tag{13}$$

Then, there are at least $\mathrm{cl}(\mathcal{V})+1$ critical points of φ in $\mathcal{V} \times D \times Z$.

Proof. We need to consider a $\mathcal{C}^\infty$-smooth function $\sigma : \mathbb{R} \to \mathbb{R}$ such that

$$\sigma(s) = \begin{cases} 0\,, & \text{if } s \leqslant 0\,,\\ 1\,, & \text{if } s \geqslant 1\,, \end{cases} \qquad \sigma'(s) > 0\,, \text{ if } s \in \,]0,1[.$$

We define the function $h : Y \to \mathbb{R}$ by

$$h(y) = \xi(y)|y - \pi_D y|^2,$$

where

$$\xi(y) = \begin{cases} 0\,, & \text{if } y \in D\,,\\ \tfrac{1}{2}\sigma(|y - \pi_D y|)\,, & \text{if } y \in Y \setminus D\,. \end{cases} \tag{14}$$

Note that

$$\nabla\xi(y) = \frac{\sigma'(|y-\pi_D y|)}{2|y-\pi_D y|}(y-\pi_D y)\,, \qquad \text{for every } y \in Y \setminus D. \tag{15}$$

Then, if $y \in Y \setminus D$,

$$\nabla h(y) = \left[\tfrac{1}{2}\sigma'(|y-\pi_D y|)|y-\pi_D y| + \sigma(|y-\pi_D y|)\right](y-\pi_D y),$$

hence (7) and (8) hold, with $\mathbb{S} = \mathbb{I}$. Moreover, since $\nabla h(y)$ has the same direction as $\nu_D(\pi_D y)$, for every $y \in Y \setminus D$, we see that (13) is equivalent to (9), and the result follows from Theorem 4. □

REMARK 6. Note that assumption (13) can be replaced by

$$\nabla_y\varphi(x,y,z) \notin \{-\alpha\,\nu_D(\pi_D y) : \alpha \geqslant 0\}\,, \quad \text{if } 0 < \operatorname{dist}(y,D) < \rho. \tag{16}$$

In the proof, it is sufficient to take $h(y) = -\xi(y)|y-\pi_D y|^2$, and the result follows in a similar way.

Here is our second corollary.

COROLLARY 7. *Assume that D is strongly convex, with a smooth boundary, and that there exist a symmetric invertible linear operator $\mathbb{A} : Y \to Y$ and a constant $\rho > 0$ for which*

$$\langle\nabla_y\varphi(x,y,z),\, \mathbb{A}\nu_D(\pi_D y)\rangle > 0\,, \quad \text{if } 0 < \operatorname{dist}(y,D) < \rho. \tag{17}$$

Then, there are at least $\operatorname{cl}(\mathcal{V})+1$ critical points of φ in $\mathcal{V}\times D\times Z$.

Proof. We consider the $\mathcal{C}^\infty$-smooth function $\xi : Y \to \mathbb{R}$ introduced in the proof of Corollary 5, and define $h : Y \to \mathbb{R}$ by

$$h(y) = -\xi(y)\langle\mathbb{A}(y-\pi_D y), y-\pi_D y\rangle.$$

By the chain rule, if $y \in Y \setminus D$,

$$\nabla h(y) = -\langle\mathbb{A}(y-\pi_D y), y-\pi_D y\rangle\nabla\xi(y) - 2\xi(y)(\mathrm{Id}-\pi_D'(y))^*\mathbb{A}(y-\pi_D y).$$

For $|y|$ large enough, since $\xi(y) = \frac{1}{2}$ and $\nabla\xi(y) = 0$, we have

$$\begin{aligned} |\nabla h(y) + \mathbb{A}y| &= |\mathbb{A}\pi_D y + \pi_D'(y)^*\mathbb{A}(y-\pi_D y)| \\ &\leqslant |\mathbb{A}\pi_D y| + \|\pi_D'(y)^*\|\,\|\mathbb{A}\|\,|y-\pi_D y|. \end{aligned}$$

Since D is strongly convex, by [**9**, Lemma 2.2] there is a constant $c > 0$ such that

$$\|\pi_D'(y)\|\,|y-\pi_D y| \leqslant c\,, \qquad \text{for every } y \in Y \setminus D,$$

hence (8) holds, with $\mathbb{S} = -\mathbb{A}$. Moreover, if $y \in Y \setminus D$,

$$\begin{aligned} \langle\nabla h(y), -\mathbb{A}\nu_D(\pi_D y)\rangle = &\ \langle\mathbb{A}(y-\pi_D y), y-\pi_D y\rangle\langle\nabla\xi(y), \mathbb{A}\nu(\pi_D y)\rangle + \\ &+2\xi(y)\langle(\mathrm{Id}-\pi_D'(y))^*\mathbb{A}(y-\pi_D y), \mathbb{A}\nu(\pi_D y)\rangle. \end{aligned}$$

Now, in view of (15), $\nabla\xi(y)$ has the same direction as $y-\pi_D y$. Since $y-\pi_D y = \operatorname{dist}(y,\partial D)\nu(\pi_D y)$, the first term in the right-hand side of the equality is nonnegative. On the other hand, by [**9**, Lemma 2.2], we have that $(\mathrm{Id}-\pi_D'(y))^*$ is positive definite, for any $y \in Y \setminus D$, and the second term in the right-hand side of the equality is positive. Therefore,

$$\langle\nabla h(y), \mathbb{A}\nu_D(\pi_D y)\rangle < 0\,, \quad \text{for every } y \in Y \setminus D. \tag{18}$$

This implies (7), and we see that (17) and (18) imply (9), hence the result follows from Theorem 4. ☐

We finish this section showing how Theorem 3 follows from Corollary 7.

Proof of Theorem 3. Let $H = Y = \mathbb{R}^M$ (so that the space Z is reduced to $\{0\}$). Assume by contradiction that (17) does not hold. Then, there is a sequence $(x_n, y_n)_n$ in $\mathcal{V} \times Y$ such that

$$0 < \operatorname{dist}(y_n, D) \leqslant \frac{1}{n} \quad \text{and} \quad \langle \nabla_y \varphi(x_n, y_n),\, \mathbb{A}\nu_D(\pi_D y_n)\rangle \leqslant 0,$$

for every n. By the compactness of $\mathcal{V} \times D$, there is subsequence $(x_{n_k}, y_{n_k})_k$ which converges to some $(\bar{x}, \bar{y}) \in \mathcal{V} \times \partial D$. By continuity,

$$\langle \nabla_y \varphi(\bar{x}, \bar{y}),\, \mathbb{A}\nu_D(\bar{y})\rangle = \lim_k \langle \nabla_y \varphi(x_{n_k}, y_{n_k}),\, \mathbb{A}\nu_D(\pi_D y_{n_k})\rangle \leqslant 0,$$

in contradiction with (3). ☐

5. *An example of application*

Let us start from the periodically forced pendulum equation

$$\ddot{q} + a \sin q = e(t),$$

where $e : \mathbb{R} \to \mathbb{R}$ is a locally integrable T-periodic function with zero mean, that is,

$$\frac{1}{T} \int_0^T e(t)\, dt = 0. \tag{19}$$

Setting $E(t) = \int_0^t e(s)\, ds$, we can write the equivalent Hamiltonian system

$$\dot{q} = p + E(t)\,, \qquad -\dot{p} = a \sin q. \tag{20}$$

We now work on the Hilbert space $H^{1/2}([0,T], \mathbb{R}^2)$, cf. [**11**, Section 3.3]. Set

$$x = \frac{1}{T} \int_0^T q(t)\, dt\,, \qquad y = \frac{1}{T} \int_0^T p(t)\, dt,$$

and define the zero-mean functions $u, v : \mathbb{R} \to \mathbb{R}$ such that

$$q(t) = x + u(t)\,, \qquad p(t) = y + v(t).$$

Finally, let $z : \mathbb{R} \to \mathbb{R}^2$ be the vector-valued function

$$z(t) = (u(t), v(t)).$$

Since the Hamiltonian function is T-periodic in x, we will consider x as varying in the manifold S^1. Denote by Y the one-dimensional space of the constants y, and let Z be the space of those functions $z = (u, v)$ having zero-mean. Finally, let $H = Y \oplus Z$. Define the bounded self-adjoint operator $L : Z \to Z$ formally as follows: writing $z = (u, v)$ and $w = (\hat{u}, \hat{v})$,

$$\langle Lz, w\rangle = \int_0^T [\dot{u}(t)\hat{v}(t) - \dot{v}(t)\hat{u}(t) + v(t)\hat{v}(t)]\, dt.$$

Note that L is invertible. Setting $D = [d_-, d_+]$, with

$$d_- < \frac{1}{T} \int_0^T E(t)\, dt < d_+,$$

we consider the functional $\psi : S^1 \times D \times Z \to \mathbb{R}$, defined as

$$\psi(x, y, z) = \int_0^T \left[\tfrac{1}{2}\, y^2 - a\cos(x + u(t)) + E(t)(y + v(t))\right] dt.$$

It has a completely continuous and bounded gradient (since D is bounded). The T-periodic solutions of our system can be obtained as critical points of the functional $\varphi : S^1 \times D \times Z \to \mathbb{R}$ given by (6). Note that this functional is indeed defined on $S^1 \times \mathbb{R} \times Z$. Being

$$\partial_y \varphi(x, y, z) = Ty + \int_0^T E(t)\, dt,$$

it is easily seen that (16) holds, that is, for $\rho > 0$,

$$\partial_y \varphi(x, y, z) \begin{cases} < 0\,, & \text{if } y \in\,]d_- - \rho, d_-[\,, \\ > 0\,, & \text{if } y \in\,]d_+, d_+ + \rho[\,. \end{cases} \tag{21}$$

Then, by Remark 6, we can apply Corollary 5, which provides us the existence of at least $\mathrm{cl}(S^1) + 1 = 2$ critical points of φ, corresponding to two geometrically distinct T-periodic solutions of system (20). We thus recover a classical result by Mawhin and Willem [**19**]. Note that an assumption like (21), reminiscent of the Landesman–Lazer condition, has been already considered in [**8**, Theorem 5.1] for pendulum-like equations.

The above argument can be extended to systems of the type

$$\ddot{q} + \nabla_q V(t, q) = e(t).$$

Here $V : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is assumed to be continuously differentiable, T-periodic in t and 2π-periodic in each component of $q = (q_1, \dots, q_N)$. The function $e : \mathbb{R} \to \mathbb{R}^N$ is locally integrable, T-periodic, and has a zero (vector) mean, that is, (19) holds. The equivalent system now reads as

$$\dot{q} = p + E(t), \quad -\dot{p} = \nabla_q V(t, q), \tag{22}$$

where $E(t) = \int_0^t e(s)\, ds$. This time, x varies in the N-dimensional torus $\mathbb{T}^N$, and taking as D a closed ball centered at the origin, with a sufficiently large radius $R > 0$, we see that there is some $\rho > 0$ for which both (13) and (17) hold, with $\mathbb{A} = \mathrm{Id}$. In this case, Corollaries 5 and 7 give us at least $\mathrm{cl}(\mathbb{T}^N) + 1 = N + 1$ critical points of φ, corresponding to $N + 1$ geometrically distinct T-periodic solutions of system (22). This result, first proved in [**20**], has been further extended in [**3–5**, **7**, **10**, **14**, **18**, **26**]; we do not enter into details, for briefness.

This situation can be generalized. Let Π be the projection defined as

$$\Pi\, p = \frac{1}{T} \int_0^T p(s)\, ds.$$

Clearly, $p = \Pi\, p + (\mathrm{Id} - \Pi)p$. We may then consider the Hamiltonian system

$$\dot{q} = \nabla\phi(\Pi\, p) + (\mathrm{Id} - \Pi)p + E(t), \quad -\dot{p} = \nabla_q V(t, q), \tag{23}$$

where $\phi : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable function such that

$$-\nabla\phi(y) \notin \{\alpha y + \tfrac{1}{T} \textstyle\int_0^T E(t)\, dt : \alpha \geqslant 0\}\,, \quad \text{if } |y| > R,$$

and we get the same conclusion, by Corollary 5 and Remark 6.

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