Compact Textbooks in Mathematics

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The Kurzweil-Henstock Integral for Undergraduates

A Promenade Along the Marvelous Theory of Integration

Birkhäuser

Introduction

This book is the outcome of the beginners' courses held over the past few years for my undergraduate students. The aim was to provide them with a general and sufficiently easy to grasp theory of the integral. The integral in question is indeed more general than Lebesgue's in $\mathbb{R}^N$, but its construction is rather simple, since it makes use of Riemann sums which, being geometrically viewable, are easily understandable.

This approach to the theory of the integral was developed independently by Jaroslav Kurzweil and Ralph Henstock since 1957 (cf. [5, 8]). A number of books are now available [1, 4, 6, 7, 9–13, 15–19, 21]. However, I feel that most of these monographs are addressed to an expert reader, rather than to a beginner student. This is why I wanted to maintain here the exposition at a very didactical level, trying to avoid as much as possible unnecessary technicalities.

The book is divided into three main chapters and five appendices, which I now briefly describe, mainly as a guide for the lecturer.

The first chapter outlines the theory for functions of one real variable. I have done my best to keep the explanation as simple as can be, following as far as possible the lines of the theory of the Riemann integral. However, there are some interesting peculiarities.

- The Fundamental Theorem of differential and integral calculus is very general and natural: one only has to assume the given function to be primitivable, i.e., to be the derivative of a differentiable function. The proof is simple and clearly shows the link between differentiability and integrability.
- The generalized integral, on a bounded but not compact interval, is indeed a standard integral: in fact, Hake's theorem shows that a function having a generalized integral on such an interval can be extended to a function which is integrable in the standard sense on the closure of its domain.
- Integrable functions according to Lebesgue are those functions which are integrable and whose absolute value is integrable, too.

In the second chapter, the theory is extended to real functions of several real variables. No difficulties are encountered while considering functions defined on rectangles. When the functions are defined on more general domains, however, an obstacle arises concerning the property of additivity on subdomains. It is then necessary to limit one's attention to functions which are integrable according to Lebesgue, after having introduced the concept of measurable set. On the other hand, for the Fubini Reduction Theorem there is no need to deal with Lebesgue integrable functions. It has a rather technical but conceptually simple proof, which only makes use of the

Kurzweil–Henstock definition. In the Theorem on the Change of Variables in the integral, once again complications may arise (see, e.g., [2]), so that I again decided to limit the discussion only to functions which are integrable according to Lebesgue. The same goes for functions which are defined on unbounded sets. These difficulties are intrinsic, not only at an expository level, and research on some of these issues is still being carried out.

The third chapter illustrates the theory of differential forms. The aim is to prove the classical theorems carrying the name of Stokes, and Poincaré's theorem on exact differential forms. Dimension 3 has been considered closely: indeed, the theorems by Stokes–Cartan and Poincaré are proved in this chapter only in this case, and the reader is referred to Appendix B for the general proof. Also, I opted to discuss only the theory for M-surfaces, without generalizing and extending it to more complex geometrical objects (see however Appendix C). In some parts of this chapter, the regularity assumptions could be weakened, but I did not want to enter into a topic touching a still ongoing research.

In Appendix A, the basic facts about differential calculus in $\mathbb{R}^N$ are reviewed.

In Appendix B, the theorems by Stokes–Cartan and Poincaré are proved. The proofs are rather technical but do not present great conceptual difficulties.

In Appendix C, one can find a brief introduction to the theory of differentiable manifolds, with particular emphasis on the corresponding version of the Stokes–Cartan theorem. I did not want to deal with this argument extensively, and the proofs are only sketched. For a more complete treatment, we refer to [20].

In Appendix D, one of the most surprising results of modern mathematics is reported, the so-called Banach–Tarski paradox. It states that a three-dimensional ball can be divided into a certain number of subsets which, after some well-chosen rotations and translations, finally give two identical copies of the starting ball. Why reporting on this in a book about integration? Well, the Banach–Tarski paradox shows the existence of sets which are not measurable (a rotation and a translation maintain the measure of a set, provided this set is measurable!), and it does this in a very spectacular way.

Appendix E entails a short historical note on the evolution of the concept of integral. This note is by no means complete. The aim is to give an idea of the role played by the Riemann sums in the different stages of the history of the integral.

Note A preliminary version of this book was published in Italian under the title *Lezioni sulla teoria dell'integrale*. It has been revised here, extending and improving most of the arguments.

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