



Periodic perturbations with rotational symmetry of planar systems driven by a central force

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Abstract

We consider periodic perturbations of a central force field having a rotational symmetry, and prove the existence of nearly circular periodic orbits. We thus generalize, in the planar case, some previous bifurcation results obtained by Ambrosetti and Coti Zelati in [1]. Our results apply, in particular, to the classical Kepler problem.

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1. Introduction

We study the planar system

$$\ddot{x} + g(|x|)x = \varepsilon \nabla_x V(t, x), \quad (S_\varepsilon)$$

where $g :]0, +\infty[\rightarrow \mathbb{R}$ is a continuously differentiable function and $V : \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) \rightarrow \mathbb{R}$ is continuous, T -periodic in its first variable, and twice continuously differentiable in its second variable. The real number ε will be considered as a small parameter.

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A particularly important special case of (S_ε) is illustrated by the perturbed Kepler problem

$$\ddot{x} + \frac{c}{|x|^3} x = \varepsilon \nabla_x V(t, x), \quad (K_\varepsilon)$$

where c is a positive constant. This kind of systems has been studied by plenty of authors, mainly with the aim of finding periodic solutions: see, e.g., [1,2,4–6,8–14,17,19]. Let us describe two of these results in detail.

In 1989, Ambrosetti and Coti Zelati [1] considered system (K_ε) in any dimension, and proved that periodic solutions of period $2T$ exist assuming the perturbing potential to be *even* in x , i.e., that

$$V(t, x) = V(t, -x), \quad \text{for every } t, x.$$

Their method of proof is variational, and provides a bifurcation result from a circular solution $x_*(t)$ of (K_0) having minimal period $\tau_* = 2T/n$, with n odd. (By *circular solution* we mean a solution whose orbit is a circle centered at the origin.) Indeed, they proved that at least three such solutions $x(t)$ exist, satisfying the symmetry property

$$x(t + T) = -x(t), \quad \text{for every } t.$$

Recently, the authors of the present paper considered in [10] the case when the perturbing force is *radially symmetric*, i.e., when $\nabla_x V(t, x) = p(t, |x|)x$, for some scalar function $p: \mathbb{R} \times]0, +\infty[\rightarrow \mathbb{R}$. It was proved that, if $x_*(t)$ is a circular solution having minimal period τ_* , and T/τ_* is a rational number n/m which is not an integer, then near this circular solution there are mT -periodic solutions $x(t)$ of system (K_ε) making exactly n rotations around the origin in their period time, provided that $|\varepsilon|$ is small enough. Moreover, $t \mapsto |x(t)|$ is T -periodic, and $t \mapsto \text{Rot}(x, [t, t + T])$ is constant.

Let us recall here that, when $\gamma: [\tau_1, \tau_2] \rightarrow \mathbb{R}^2$ is a curve such that $\dot{\gamma}(t) \neq 0$ for every t , writing $\gamma(t) = \rho(t)(\cos \theta(t), \sin \theta(t))$, all functions being continuous, the *rotation number* around the origin is defined as

$$\text{Rot}(\gamma; [\tau_1, \tau_2]) = \frac{\theta(\tau_2) - \theta(\tau_1)}{2\pi}.$$

In this paper we assume that the perturbing force has a *rotational symmetry*, i.e., there is a rotation around the origin \mathcal{R} , of angle

$$\vartheta = \frac{2\pi}{m},$$

with m being a positive integer, such that

$$\nabla_x V(t, \mathcal{R}x) = \mathcal{R} \nabla_x V(t, x), \quad \text{for every } t, x. \quad (1)$$

Notice that, if $m = 1$, the above assumption will be trivially satisfied.

Let $\rho_* > 0$ be such that $g(\rho_*) > 0$. Then, the unperturbed equation (S_0) has a circular solution $x_*(t)$, with $|x_*(t)| = \rho_*$, rotating counter-clockwise, having minimal period

$$\tau_* = \frac{2\pi}{\sqrt{g(\rho_*)}}. \tag{2}$$

We will ask the quotient T/τ_* to be a rational number; more precisely, we assume that there exist some positive integers ℓ, n for which

$$\frac{mT}{\tau_*} = \frac{n}{\ell}. \tag{3}$$

We then look for solutions $x : \mathbb{R} \rightarrow \mathbb{R}^2 \setminus \{0\}$ such that

$$x(t + \ell T) = \mathcal{R}^n x(t), \quad \text{for every } t; \tag{4}$$

these solutions are $m\ell T$ -periodic, and rotate n times around the origin in their period time. Notice that, in this case, $t \mapsto |x(t)|$ is ℓT -periodic and $t \mapsto \text{Rot}(x, [t, t + \ell T])$ is constant, precisely equal to n/m .

It will be useful to introduce the auxiliary function

$$h(\rho) = 4g(\rho) + g'(\rho)\rho.$$

Here is our main result.

Theorem 1. *In the above setting, assume that $g'(\rho_*) \neq 0$ and*

$$h(\rho_*) \notin \left\{ \left(\frac{2\pi k}{\ell T} \right)^2 : k \in \mathbb{N} \setminus \{0\} \right\}. \tag{5}$$

Then, for any $\sigma > 0$ there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, system (S_ε) has at least four geometrically distinct $m\ell T$ -periodic solutions $x(t)$ satisfying (4). Two of them rotate counter-clockwise, and are such that, for some θ_1, θ_2 in $[0, \tau[$,

$$|x(t) - x_*(t + \theta_i)| \leq \sigma, \quad \text{for every } t \in \mathbb{R}. \tag{6}$$

The other two solutions rotate clockwise, and are such that, for some θ_3, θ_4 in $[0, \tau[$,

$$|x(t) - x_*(-t + \theta_i)| \leq \sigma, \quad \text{for every } t \in \mathbb{R}. \tag{7}$$

When assuming (1), once a solution $x(t)$ has been found, also $\mathcal{R}^j x(t)$ is clearly a solution, with $j = 1, \dots, m - 1$. In the above statement, we say that the solutions are *geometrically distinct* if they are not related to each other in this way.

When dealing with the perturbed Kepler problem (K_ε) , the functions $h(\rho)$ and $g(\rho) = c\rho^{-3}$ coincide, and we have following.

Corollary 2. *If (3) is verified, and*

$$\frac{\ell T}{\tau_*} = \frac{n}{m} \notin \mathbb{N}, \tag{8}$$

then the same conclusion of Theorem 1 holds for system (K_ε) .

Indeed, in this case we have that $h(\rho_*) = (2\pi/\tau_*)^2$, so that (5) follows directly from (8). Notice that the case $\ell = 1$, $m = 2$, and n odd was treated in [1, Theorem 2.2], showing the existence of *at least three* $2T$ -periodic solutions near $x_*(t)$, a circular solution of the unperturbed problem having minimal period $\tau_* = 2T/n$, in an arbitrary dimensional space. In the planar case, there are at least four $m\ell T$ -periodic solutions because it is possible to distinguish between two of them rotating clockwise and two counter-clockwise. On the other hand, when considering the radially symmetric situation as in [10], the integer number $m \geq 2$ can be chosen arbitrarily. Theorem 1 thus extends [10, Corollary 4].

Now, since the minimal period of the circular orbits of the Kepler problem is strictly increasing with their radius, we may expect that, taking *any* circular orbit $x_*(t)$ of the unperturbed system (K_0) , the assumptions of Corollary 2 will be satisfied by many circular orbits which are arbitrarily near to it. This is indeed the case, and will be established in Theorem 6 below.

The paper is organized as follows. In Section 2 we provide the proof of Theorem 1 by the use of a bifurcation result, in a variational setting. At the end of the section we will make some remarks on our choice of dealing only with the problem in dimension 2. In Section 3 we consider a generalized Kepler problem, taking $g(\rho) = c\rho^{-\gamma}$, for some $\gamma > 0$, and we provide the existence of a myriad of periodic solutions near *any* circular solution of the unperturbed system. As an example of application we propose a gravitational model of a pulsating star having some rotational symmetry.

Let us also mention that all our results still hold when $V(t, x)$ is replaced by some $V(t, x, \varepsilon)$, with a smooth dependence on the parameter ε .

2. Proof of Theorem 1

Searching for $m\ell T$ -periodic solutions, we will deal with the Hilbert space

$$H_{m\ell T}^1 = \{x \in H^1(0, m\ell T) : x(0) = x(m\ell T)\},$$

equipped with the usual scalar product

$$\langle v, w \rangle = \int_0^{m\ell T} [\langle v(t), w(t) \rangle + \langle \dot{v}(t), \dot{w}(t) \rangle] dt.$$

The elements of this space will be identified to the $m\ell T$ -periodic continuous functions with locally square integrable weak derivative.

In order to take into account the symmetry requirement (4), we will work on the Hilbert subspace

$$\mathcal{E} = \left\{ x \in H_{m\ell T}^1 : x(t + \ell T) = \mathcal{R}^n x(t), \text{ for every } t \in [0, m\ell T] \right\}.$$

Here, we recall, \mathcal{R} is a rotation around the origin of angle $2\pi/m$. Without loss of generality, we will assume it to be counter-clockwise, i.e., in complex notation, if $z \in \mathbb{R}^2$,

$$\mathcal{R}z = e^{\frac{2\pi i}{m}} z.$$

Moreover, since we are interested in solutions which never cross the origin, it will be worth introducing the open subset

$$\mathcal{E}_* = \{x \in \mathcal{E} : x(t) \neq 0, \text{ for every } t \in [0, m\ell T]\}.$$

We look for the critical points of the functional $\varphi_\varepsilon : \mathcal{E}_* \rightarrow \mathbb{R}$ defined as

$$\varphi_\varepsilon(x) = \int_0^{m\ell T} \left[\frac{1}{2} |\dot{x}(t)|^2 - F(|x(t)|) + \varepsilon V(t, x(t)) \right] dt,$$

where

$$F(\rho) = \int_0^\rho sg(s) ds.$$

Indeed, we have the following information.

Lemma 3. *Under the assumption (1), the critical points of the functional φ_ε correspond to $m\ell T$ -periodic solutions of (S_ε) satisfying (4).*

Proof. Let $x \in \mathcal{E}_*$ be a critical point of φ_ε , namely

$$\varphi'_\varepsilon(x)v = \int_0^{m\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt = 0,$$

for every $v \in \mathcal{E}$. Notice that, since both x and v belong to \mathcal{E} , using (1) we have that

$$\begin{aligned} & \int_0^{m\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt = \\ &= \sum_{j=0}^{m-1} \int_{j\ell T}^{(j+1)\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt \\ &= \sum_{j=0}^{m-1} \int_0^{\ell T} [\langle \dot{x}(t + j\ell T), \dot{v}(t + j\ell T) \rangle - g(|x(t + j\ell T)|) \langle x(t + j\ell T), v(t + j\ell T) \rangle \\ & \quad - \varepsilon \langle \nabla_x V(t + j\ell T, x(t + j\ell T)), v(t + j\ell T) \rangle] dt \end{aligned}$$

$$= m \int_0^{\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt.$$

Hence,

$$\int_0^{\ell T} [\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle] dt = 0,$$

for every $v \in H_{\ell T}^1$, since such a function $v(t)$ can be extended on $[0, m\ell T]$ to a function belonging to the space \mathcal{E} . We then deduce that $\dot{x}(t)$ has a weak derivative on $[0, \ell T]$, which almost everywhere coincides with $\psi(t) = -g(|x(t)|)x(t) + \varepsilon \nabla_x V(t, x(t))$, a continuous function. Therefore, $x(t)$ is twice continuously differentiable on $[0, \ell T]$, and it is a classical solution of the differential equation (S_ε) on $[0, \ell T]$. Since $x(t + \ell T) = \mathcal{R}^n x(t)$ and (1) holds, the same conclusion is reached on every interval $[j\ell T, (j+1)\ell T]$, with $j = 1, \dots, m-1$. Being $\psi(t)$ continuous on the whole real line, we see that $x(t)$ is indeed a classical $m\ell T$ -periodic solution of (S_ε) . \square

In order to state the abstract perturbation result, let us consider the symmetric bilinear form $\varphi_0''(x)$ which associates to every pair of vectors $v, w \in \mathcal{E}$ the real number

$$\begin{aligned} \varphi_0''(x)(v, w) &= \lim_{\lambda \rightarrow 0} \frac{\varphi_0'(x + \lambda w)v - \varphi_0'(x)v}{\lambda} \\ &= \int_0^{m\ell T} \left[\langle \dot{v}, \dot{w} \rangle - \frac{g'(|x|)}{|x|} \langle x, v \rangle \langle x, w \rangle - g(|x|) \langle v, w \rangle \right] dt \end{aligned}$$

By the Riesz representation theorem, we associate to this bilinear form a bounded selfadjoint operator in \mathcal{E} , still denoted by $\varphi_0''(x)$, such that, for every $v, w \in \mathcal{E}$,

$$\langle \varphi_0''(x)v, w \rangle = \varphi_0''(x)(v, w).$$

Let us now recall a well-established perturbation result which will be used in our proof (see, e.g., [3,7,15,16,18]).

Theorem 4. *Let an open subset \mathcal{E}_* of a Hilbert space \mathcal{E} be the domain of a family of twice continuously differentiable functionals $\varphi_\varepsilon : \mathcal{E}_* \rightarrow \mathbb{R}$, depending smoothly on ε . Let $\mathcal{Z} \subseteq \mathcal{E}_*$ be a compact manifold without boundary such that*

- (i) $\varphi_0'(x) = 0$, for every $x \in \mathcal{Z}$;
- (ii) $\varphi_0''(x)$ is a Fredholm operator of index 0, for every $x \in \mathcal{Z}$;
- (iii) $T_x \mathcal{Z} = \ker \varphi_0''(x)$, for every $x \in \mathcal{Z}$.

Then, there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, the functional φ_ε has at least $\text{cat}(\mathcal{Z})$ critical points near \mathcal{Z} .

In the above statement, we have denoted by $\text{cat}(\mathcal{Z})$ the Lusternik–Schnirelmann category of the topological space \mathcal{Z} with respect to itself, namely the least integer k such that \mathcal{Z} can be covered by k closed subsets of \mathcal{Z} which are contractible to a point in \mathcal{Z} .

We will apply Theorem 4 separately for the clockwise and for the counter-clockwise rotating solutions. Let us deal with those last ones, taking as \mathcal{Z} the manifold of circular solutions with radius ρ_* of the unperturbed system (S_0) rotating counter-clockwise. (The other case is completely analogous.) Clearly enough, $\mathcal{Z} \subseteq \mathcal{E}_*$. Once we choose $x_*(t)$, one of such circular solutions, \mathcal{Z} can be parametrized as a curve $\gamma : [0, 2\pi] \rightarrow \mathcal{E}_*$, e.g.,

$$\gamma(\theta)(t) = x_*(t + \theta).$$

Hence, \mathcal{Z} is homeomorphic to S^1 , and $\text{cat}(\mathcal{Z}) = 2$. Let us now verify the assumptions of Theorem 4.

Condition (i) holds since every $x \in \mathcal{Z}$ is a solution of (S_0) , and hence $\varphi'_0(x) = 0$.

In order to prove condition (ii) notice that, taking $v, \xi \in \mathcal{E}$, one has $\varphi''_0(x)v = \xi$ if and only if

$$\int_0^{m\ell T} \left[\langle \dot{v} - \dot{\xi}, \dot{w} \rangle - \left\langle \frac{g'(|x|)}{|x|} \langle x, v \rangle x + g(|x|)v + \xi, w \right\rangle \right] dt = 0,$$

for every $w \in \mathcal{E}$. Setting $u = \xi - v$, with the same argument used to prove Lemma 3 we can then deduce that u is twice continuously differentiable, and satisfies the differential equation

$$\ddot{u} - u = \frac{g'(|x|)}{|x|} \langle x, v \rangle x + (g(|x|) + 1)v. \tag{9}$$

By the Ascoli–Arzelà Theorem it can be proved that the linear operator $P : \mathcal{E} \rightarrow \mathcal{E}$, taking every $v \in \mathcal{E}$ into the unique solution $u \in \mathcal{E}$ of (9), is a compact operator. Being $\varphi''_0(x) = \text{Id} + P$, the Fredholm Alternative Theorem guarantees that $\varphi''_0(x)$ has a finite dimensional kernel and a closed image. Since it is selfadjoint, $\ker \varphi''_0(x) = [\text{Im } \varphi''_0(x)]^\perp$, and we conclude that $\varphi''_0(x)$ is a Fredholm operator of index zero.

Finally, in order to verify (iii), we have to prove that $T_x \mathcal{Z} \supseteq \ker \varphi''_0(x)$ for every $x \in \mathcal{Z}$ (the other inclusion is always true because x is a critical point). So, we choose $x \in \mathcal{Z}$ and $v \in \ker \varphi''_0(x)$. Without loss of generality, we assume that $x(0) = (\rho_*, 0)$ so that, by (3), using the complex notation,

$$x(t) = \rho_* e^{\frac{2\pi i}{\tau_*} t} = \rho_* e^{\frac{2\pi n i}{m\ell T} t}.$$

We want to prove that $v \in T_x \mathcal{Z}$. Taking $\theta \in \mathbb{R}$ such that $x(t) = x_*(t + \theta)$, i.e., $x = \gamma(\theta)$, we notice that

$$T_x \mathcal{Z} = \{ \mu \gamma'(\theta) : \mu \in \mathbb{R} \} = \{ i \nu x : \nu \in \mathbb{R} \}.$$

Particularizing equation (9) with $\xi = 0$, we see that any $v \in \ker \varphi''_0(x)$ satisfies the differential equation

$$-\ddot{v} = g(|x|)v + \frac{g'(|x|)}{|x|} \langle v, x \rangle x. \quad (10)$$

Here, as well, using the complex notation we write $v : \mathbb{R} \rightarrow \mathbb{C}$, and since we look for $m\ell T$ -periodic solutions, we have the corresponding Fourier series

$$v(t) \sim \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t},$$

with $c_k \in \mathbb{C}$. Moreover, since $v(t + \ell T) = \mathcal{R}^n v(t)$, it has to be

$$\sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} (t + \ell T)} = e^{\frac{2\pi n i}{m}} \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t},$$

so that

$$c_k e^{\frac{2\pi k i}{m}} = c_k e^{\frac{2\pi n i}{m}}, \quad \text{for every } k \in \mathbb{Z},$$

hence

$$\text{either } c_k = 0, \text{ or } k \in n + m\mathbb{Z}. \quad (11)$$

Now, we write equation (10) as

$$\sum_{k \in \mathbb{Z}} \left(\frac{2\pi k}{m\ell T} \right)^2 c_k e^{\frac{2\pi k i}{m\ell T} t} = g(\rho_*) \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t} + g'(\rho_*) \left\langle \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t}, \rho_* e^{\frac{2\pi n i}{m\ell T} t} \right\rangle e^{\frac{2\pi n i}{m\ell T} t}.$$

Recalling that, by (3),

$$g(\rho_*) = \left(\frac{2\pi}{\tau_*} \right)^2 = \left(\frac{2\pi n}{m\ell T} \right)^2,$$

and setting

$$\alpha = \frac{g'(\rho_*)\rho_*}{2} \left(\frac{m\ell T}{2\pi} \right)^2,$$

we get

$$\sum_{k \in \mathbb{Z}} (k^2 - n^2) c_k e^{\frac{2\pi k i}{m\ell T} t} = 2\alpha \left\langle \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t}, e^{\frac{2\pi n i}{m\ell T} t} \right\rangle e^{\frac{2\pi n i}{m\ell T} t}.$$

Remark that, having assumed $g'(\rho_*) \neq 0$, it follows that $\alpha \neq 0$, as well. Since $\langle z_1, z_2 \rangle = \operatorname{Re}(z_1 z_2^*)$ and $\operatorname{Re}(z) = \frac{1}{2}(z + z^*)$, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} (k^2 - n^2) c_k e^{\frac{2\pi ki}{m\ell T} t} &= 2\alpha \operatorname{Re} \left(\sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi(k-n)i}{m\ell T} t} \right) e^{\frac{2\pi ni}{m\ell T} t} \\ &= \alpha \sum_{k \in \mathbb{Z}} \left[c_k e^{\frac{2\pi ki}{m\ell T} t} + c_k^* e^{\frac{2\pi(2n-k)i}{m\ell T} t} \right]. \end{aligned}$$

We then get

$$\sum_{k \in \mathbb{Z}} (k^2 - n^2 - \alpha) c_k e^{\frac{2\pi ki}{m\ell T} t} = \alpha \sum_{k \in \mathbb{Z}} c_k^* e^{\frac{2\pi(2n-k)i}{m\ell T} t},$$

which, after a shift in the indices, becomes

$$\sum_{k \in \mathbb{Z}} (k^2 - n^2 - \alpha) c_k e^{\frac{2\pi ki}{m\ell T} t} = \alpha \sum_{k \in \mathbb{Z}} c_{2n-k}^* e^{\frac{2\pi ki}{m\ell T} t}.$$

Hence, we must have that

$$(k^2 - n^2 - \alpha) c_k = \alpha c_{2n-k}^*, \quad \text{for every } k \in \mathbb{Z}.$$

Taking $k = n$, we immediately observe that $c_n = -c_n^*$, i.e., $\operatorname{Re}(c_n) = 0$. On the other hand, by a change of indices, we have that

$$\left((n + j)^2 - n^2 - \alpha \right) c_{n+j} = \alpha c_{n-j}^*, \quad \text{for every } j \in \mathbb{Z}. \tag{12}$$

So, also

$$\left((n - j)^2 - n^2 - \alpha \right) c_{n-j} = \alpha c_{n+j}^*, \quad \text{for every } j \in \mathbb{Z},$$

which is equivalent to

$$\alpha c_{n+j} = \left((n - j)^2 - n^2 - \alpha \right) c_{n-j}^*, \quad \text{for every } j \in \mathbb{Z}. \tag{13}$$

We now solve the system (12)–(13). Since $\alpha \neq 0$, substitution yields

$$(j^2 - \alpha + 2nj) (j^2 - \alpha - 2nj) c_{n-j}^* = \alpha^2 c_{n-j}^*,$$

hence

$$\left[(j^2 - \alpha)^2 - 4n^2 j^2 - \alpha^2 \right] c_{n-j}^* = 0,$$

and finally

$$j^2 [j^2 - 2\alpha - 4n^2] c_{n-j}^* = 0, \quad \text{for every } j \in \mathbb{Z}. \tag{14}$$

From (11) we know that $c_{n-j} = 0$ when $j \notin m\mathbb{Z}$. Let now $j = mp$, for some $p \in (\mathbb{Z} \setminus \{0\})$; since

$$2\alpha + 4n^2 = \left(\frac{m\ell T}{2\pi}\right)^2 (4g(\rho_*) + g'(\rho_*)\rho_*) = \left(\frac{m\ell T}{2\pi}\right)^2 h(\rho_*),$$

from (14) we deduce that

$$\left[p^2 - \left(\frac{\ell T}{2\pi}\right)^2 h(\rho_*)\right]c_{n-mp}^* = 0,$$

and assumption (5) leads to $c_{n-mp} = 0$. We then conclude that

$$c_k = 0, \quad \text{for every } k \neq n.$$

Therefore we have proved that, if $v \in \ker \varphi_0''(x)$, it has the form

$$v(t) = c_n e^{\frac{2\pi i n}{m\ell T} t},$$

where c_n is a pure imaginary complex number, hence it can be written as $c_n = i b_n$ with $b_n \in \mathbb{R}$. Therefore,

$$v(t) = i b_n e^{\frac{2\pi i n}{m\ell T} t} = i \frac{b_n}{\rho_*} x(t).$$

Hence, $v \in T_x \mathcal{Z}$, and this concludes the proof that (iii) holds true.

Theorem 1 is thus completely proved. \square

To end this section, let us make some considerations on our choice of restricting the problem to the dimension 2. If one considers system (S_ε) in a higher dimension $d \geq 3$, a natural generalization of our symmetry condition (1) would be to assume the existence of a linear transformation $\mathcal{S} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ for which

$$\nabla_x V(t, \mathcal{S}x) = \mathcal{S} \nabla_x V(t, x), \quad \text{for every } t, x. \tag{15}$$

When looking for bifurcations from the manifold of the circular solutions of (S_0) having a prescribed period τ_* , we may consider the functional $\varphi_\varepsilon : \mathcal{E}_* \rightarrow \mathbb{R}$, as in the proof above, adapting the situation to the new space

$$\mathcal{E} = \left\{ x \in H_{m\ell T}^1 : x(t + \ell T) = \mathcal{S}x(t), \text{ for every } t \in [0, m\ell T] \right\}.$$

Clearly, we need the whole manifold of circular solutions with minimal period τ_* to be contained in \mathcal{E} . We claim that, in this case, \mathcal{S} has to be either the identity, or minus the identity.

Indeed, take a circular solution in a plane generated by two orthonormal vectors \vec{e}_1 and \vec{e}_2 , e.g.,

$$x(t) = \rho_* \left(\left(\cos \frac{2\pi}{\tau_*} t \right) \vec{e}_1 + \left(\sin \frac{2\pi}{\tau_*} t \right) \vec{e}_2 \right).$$

It has to belong to \mathcal{E} , hence

$$\left(\cos \frac{2\pi}{\tau_*}(t + \ell T)\right)\bar{e}_1 + \left(\sin \frac{2\pi}{\tau_*}(t + \ell T)\right)\bar{e}_2 = \left(\cos \frac{2\pi}{\tau_*}t\right)\mathcal{S}\bar{e}_1 + \left(\sin \frac{2\pi}{\tau_*}t\right)\mathcal{S}\bar{e}_2,$$

for every t . We thus see that the vectors $\mathcal{S}\bar{e}_1$ and $\mathcal{S}\bar{e}_2$ must lie on the subspace generated by \bar{e}_1 and \bar{e}_2 . Moreover, $|\mathcal{S}\bar{e}_1| = |\mathcal{S}\bar{e}_2| = 1$.

As a consequence of the above argument, the linear function \mathcal{S} is an isometry, and it transforms every 2-dimensional subspace into itself. It then follows by linearity that also every 1-dimensional subspace, being the intersection of two 2-dimensional subspaces, is transformed into itself. (We have used here the fact that $d \geq 3$.) Then, for every $x \in \mathbb{R}^d$, it has to be $\mathcal{S}x = \pm x$.

Let now $\{\bar{e}_1, \dots, \bar{e}_d\}$ be the canonical basis of \mathbb{R}^d . By the above, the matrix associated to \mathcal{S} is diagonal, with entries ± 1 . Assume by contradiction, e.g., that $\mathcal{S}\bar{e}_1 = \bar{e}_1$ and $\mathcal{S}\bar{e}_2 = -\bar{e}_2$. Then $\mathcal{S}(\bar{e}_1 + \bar{e}_2) = \bar{e}_1 - \bar{e}_2$, while it should be $\mathcal{S}(\bar{e}_1 + \bar{e}_2) = \pm(\bar{e}_1 + \bar{e}_2)$. We thus conclude that either all diagonal entries are equal to 1, or they are all equal to -1 . The claim is thus proved.

We remark once more that Ambrosetti and Coti Zelati in [1] have indeed considered perturbations of Keplerian type problems in the d -dimensional space assuming $\mathcal{S} = -\text{Id}$, hence requiring the potential $V(t, x)$ to be an even function of x .

3. A myriad of periodic solutions

As a variant of system (K_ε) , we may consider the generalized Kepler problem

$$\ddot{x} + \frac{c}{|x|^\gamma} x = \varepsilon \nabla_x V(t, x), \tag{K̃_\varepsilon}$$

where c is a positive constant, and $\gamma \neq 0$. We assume again that (1) holds, where \mathcal{R} is a rotation around the origin of angle $2\pi/m$. We fix a circular orbit $x_*(t)$ of the unperturbed system (\tilde{K}_0) , with radius ρ_* and minimal period τ_* , and we assume (3), for some positive integers ℓ, n . In this case, the function $h(\rho)$ coincides with $(4 - \gamma)g(\rho) = (4 - \gamma)c\rho^{-\gamma}$, and we have the following immediate consequence of Theorem 1.

Corollary 5. *If (3) is verified, and*

$$(4 - \gamma) \left(\frac{\ell T}{\tau_*}\right)^2 \notin \left\{k^2 : k \in \mathbb{N} \setminus \{0\}\right\}, \tag{16}$$

then the same conclusion of Theorem 1 holds for system (\tilde{K}_ε) .

Notice that Corollary 2 is a direct consequence of Corollary 5. However, condition (16) could be verified even if $m = 1$, in which case no symmetry is required on the perturbing term. This surely happens, e.g., if $\gamma \geq 4$ or, assuming $\gamma < 4$ and recalling (3), when $\sqrt{4 - \gamma}$ is an irrational number.

One could ask whether a condition like (16) is really needed. Indeed, such a condition is not found in the paper by Ambrosetti and Coti Zelati [1]. However, there is an unclear point in their proof, when they deduce $(1.14)_k$ from (1.13), claiming that $n \pm n(4 - \gamma)^{1/2}$ cannot be an integer when $\gamma \neq 3$. It could be possible that a weaker nondegenerate condition would guarantee the existence of periodic solutions, but we will not investigate further this point.

Nevertheless, we will now prove the following rather surprising result.

Theorem 6. Assume $\gamma > 0$ and, whenever $\gamma = 3$, then $m \geq 2$. For any circular solution $x_*(t)$ of the unperturbed system (\tilde{K}_0) , for any positive integer N and any $\sigma > 0$, there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, equation (\tilde{K}_ε) has at least $4N$ periodic solutions $x(t)$, whose orbits are near the one of $x_*(t)$, in the sense of (6) or (7).

Proof. Making ρ vary near ρ_* , we find infinitely many of its values for which $mT/\tau(\rho)$ is a rational number, i.e.,

$$\frac{mT}{\tau(\rho)} = \frac{n(\rho)}{\ell(\rho)}, \quad (17)$$

for some positive integers $\ell(\rho)$, $n(\rho)$. We would like to apply Corollary 5 when ρ_* and τ_* are replaced by ρ , $\tau(\rho)$, respectively. To this aim, we need to prove that condition (16) is satisfied, i.e., that

$$(4 - \gamma) \left(\frac{\ell(\rho)T}{\tau(\rho)} \right)^2 \notin \left\{ k^2 : k \in \mathbb{N} \setminus \{0\} \right\}, \quad (18)$$

for several values of ρ near ρ_* .

If $\gamma \geq 4$, then condition (18) is surely satisfied. Assume then $\gamma < 4$, and consider the number $\kappa = \sqrt{4 - \gamma}/m$. Then, by (17), condition (18) is equivalent to

$$\kappa \frac{\ell(\rho)mT}{\tau(\rho)} = \kappa n(\rho) \notin \mathbb{N}. \quad (19)$$

If κ is not rational, condition (19) is surely satisfied. Otherwise, let $\kappa = p/q \in \mathbb{Q}$, for some relatively prime positive integers p, q . We claim that q cannot be equal to 1. Indeed, if it were $q = 1$, i.e., $\gamma = 4 - (mp)^2$, since m, p are positive integers, we would have two possibilities: if $m = p = 1$, then $\gamma = 3$, in which case we contradict the assumption that $m \geq 2$; on the other hand, if $m \neq 1$ or $p \neq 1$, then $\gamma \leq 0$, contradicting the assumption that $\gamma > 0$.

Now, among all the fractions in (17) we can select only those for which $n(\rho)$ is relatively prime with q . There are infinitely many of them, and

$$\kappa n(\rho) = p \frac{n(\rho)}{q} \notin \mathbb{N}.$$

We have then proved that condition (19) is verified by infinitely many values of ρ near ρ_* , so that the assumptions of Corollary 5 are satisfied.

Take N of these values of ρ satisfying the conditions (17) and (18) above, denote them by ρ_1, \dots, ρ_N , and let $x_*^1(t), \dots, x_*^N(t)$ be some circular solutions of (\tilde{K}_0) having those radii. By Corollary 5, there are N positive numbers $\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_N$ such that, if $|\varepsilon| \leq \bar{\varepsilon}_j$, equation (\tilde{K}_ε) has at least four periodic solutions $x_j(t)$ whose orbit is near the one of $x_*^j(t)$. If $|\varepsilon|$ is small enough, all these solutions must be different from one another, thus proving the result. \square

As an illustrative example of application of Corollary 5 and Theorem 6, we consider the motion of a body having zero mass under the gravitational attraction of $m + 1$ bodies, in the following configuration. The first of them is fixed at the origin, and has a mass equal to M_0 . The

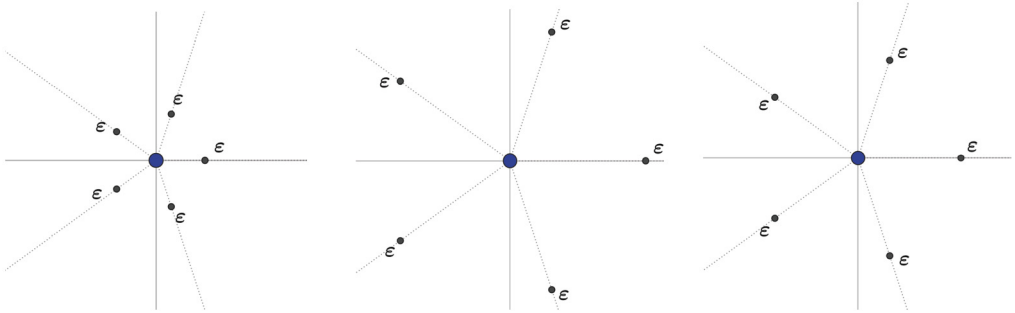


Fig. 1. The case $m = 5$, at three different times.

other m bodies move along the lines passing through the origin with angular directions $2\pi j/m$, with $j = 1, \dots, m$, and have small masses equal to the same value ε . We assume that these m small masses move symmetrically and periodically in time (see Fig. 1).

Precisely, let $p : \mathbb{R} \rightarrow [0, +\infty[$ be a T -periodic function, and denote by $P : \mathbb{R} \rightarrow \mathbb{R}^2$ the vector-valued function $P(t) = (p(t), 0)$. The equation of motion for our zero-mass body will then be of the type (K_ε) , with $c = GM_0$ (here, G is the universal gravitational constant), and

$$V(t, x) = \sum_{j=1}^m \frac{G}{|x - \mathcal{R}^j P(t)|}.$$

As usual, \mathcal{R} denotes the rotation of angle $2\pi/m$ about the origin. It is easily seen that (1) is satisfied, so that Corollary 5 (or Corollary 2) and Theorem 6 can be applied, providing the existence of plenty of periodic orbits.

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