DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **37**, Number **3**, March **2017**

doi:10.3934/dcds.2017059

pp. **1425–1436**

MULTIPLE PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS CONFINED IN A BOX

Alessandro Fonda*

Dipartimento di Matematica e Geoscienze
Università degli Studi di Trieste
P.le Europa 1, I-34127 Trieste, Italy

Andrea Sfecci

Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche 12, I-60131 Ancona, Italy

(Communicated by Manuel del Pino)

Abstract. We consider a nonautonomous Hamiltonian system, T-periodic in time, possibly defined on a bounded space region, the boundary of which consists of singularity points which can never be attained. Assuming that the system has an interior equilibrium point, we prove the existence of infinitely many T-periodic solutions, by the use of a generalized version of the Poincaré–Birkhoff theorem.

1. **Introduction.**

Let us start by considering a planar Hamiltonian system

$$x' = \frac{\partial \mathcal{H}}{\partial y}(t,x,y)\,, \qquad y' = -\frac{\partial \mathcal{H}}{\partial x}(t,x,y)\,. \tag{1}$$

The Hamiltonian function $\mathcal{H}(t,x,y)$ is assumed to be continuous, T-periodic in t, and continuously differentiable in x and y. It is defined for (x,y) belonging to a "generalized rectangle"

$$\mathcal{R} = \,]a_{1,1}, a_{1,2}[\, \times \,]a_{2,1}, a_{2,2}[\,,$$

with $a_{i,j} \in \mathbb{R} \cup \{-\infty, +\infty\}$. This means that $\mathcal{R}$ can be either a rectangle or an unbounded set, like a quadrant, a half-plane, or even the whole plane $\mathbb{R}^2$.

We assume that the Hamiltonian function may be decomposed as

$$\mathcal{H}(t,x,y) = H_1(t,x) + H_2(t,y) + \mathcal{U}(t,x,y)\,,$$

2010 *Mathematics Subject Classification.* 34C25.
Key words and phrases. Periodic solutions, Poincaré-Birkhoff theorem, singularities, Hamiltonian systems, rotation number.
The authors were partially supported by INdAM-GNAMPA.
* Corresponding author: Alessandro Fonda.

all functions being T-periodic in t. Correspondingly, let us introduce the following three assumptions.

Assumption A1. *There is an equilibrium point (x_0, y_0) in $\mathcal{R}$, and there exist a constant $\alpha > 0$ and a neighborhood $\mathcal{V}$ of (x_0, y_0) such that*

$$\left|\frac{\partial H_1}{\partial x}(t,x)\right| + \left|\frac{\partial \mathcal{U}}{\partial x}(t,x,y)\right| \le \alpha |x - x_0|\,,$$

and

$$\left|\frac{\partial H_2}{\partial y}(t,y)\right| + \left|\frac{\partial \mathcal{U}}{\partial y}(t,x,y)\right| \le \alpha |y - y_0|\,,$$

for every $(t,x,y) \in [0,T] \times \mathcal{V}$.

Assumption A2. *There exist some continuous and increasing functions $\kappa_{i,\ell} :]a_{i,1}, a_{i,2}[\to \mathbb{R}$ such that*

$$\kappa_{1,1}(x) < \frac{\partial H_1}{\partial x}(t,x) < \kappa_{1,2}(x)\,, \qquad \kappa_{2,1}(y) < \frac{\partial H_2}{\partial y}(t,y) < \kappa_{2,2}(y)\,,$$

for every $(t,x,y) \in [0,T] \times \mathcal{R}$, with

$$\lim_{x \to a_{1,j}} \frac{\kappa_{1,\ell}(x)}{x - x_0} = +\infty \quad \text{and} \quad \lim_{y \to a_{2,j}} \frac{\kappa_{2,\ell}(y)}{y - y_0} = +\infty\,, \qquad j, \ell \in \{1,2\}\,, \tag{2}$$

and the primitive functions

$$\mathcal{K}_{1,\ell}(x) = \int_{x_0}^{x} \kappa_{1,\ell}(\sigma)\, d\sigma\,, \qquad \mathcal{K}_{2,\ell}(y) = \int_{y_0}^{y} \kappa_{2,\ell}(\sigma)\, d\sigma$$

satisfy

$$\lim_{x \to a_{1,j}} \mathcal{K}_{1,\ell}(x) = +\infty \quad \text{and} \quad \lim_{y \to a_{2,j}} \mathcal{K}_{2,\ell}(y) = +\infty\,, \qquad j, \ell \in \{1,2\}\,. \tag{3}$$

Assumption A3. *The function $\mathcal{U} : \mathbb{R} \times \overline{\mathcal{R}} \to \mathbb{R}$ is continuous and has a bounded continuous gradient with respect to (x,y) belonging to $\overline{\mathcal{R}}$.*

As a simple example of a system verifying the above assumptions, we propose, e.g.,

$$\begin{cases} x' = \alpha(t) \left[\dfrac{c_{2,2}}{(a_{2,2} - y)^2} - \dfrac{c_{2,1}}{(y - a_{2,1})^2} \right] \\[2ex] y' = \beta(t) \left[\dfrac{c_{1,1}}{(x - a_{1,1})^2} - \dfrac{c_{1,2}}{(a_{1,2} - x)^2} \right], \end{cases}$$

where the constants $c_{i,j}$ are positive, and the functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are continuous, positive, and T-periodic. Indeed, in this case, we can choose

$$H_1(t,x) = \beta(t) \left[\frac{c_{1,1}}{x - a_{1,1}} + \frac{c_{1,2}}{a_{1,2} - x} \right], \quad H_2(t,y) = \alpha(t) \left[\frac{c_{2,1}}{y - a_{2,1}} + \frac{c_{2,2}}{a_{2,2} - y} \right],$$

and $\mathcal{U}(t,x,y)$ identically equal to zero.

We are now ready to state our main result for the planar case.

Theorem 1.1. *Let Assumptions A1, A2, and A3 be satisfied. Then, there exists an integer K_0 such that, for every integer $K > K_0$, the Hamiltonian system (1) has at least two T-periodic solutions performing exactly K clockwise rotations around (x_0, y_0) in the time interval $[0,T]$.*

The above theorem thus provides the existence of infinitely many T-periodic solutions of system (1). Its proof will be carried out in Section 2, by the use of a generalized version of the Poincaré–Birkhoff theorem [7, Theorem 1.2], recently obtained by the first author and A. J. Ureña., after a suitable modification of the Hamiltonian function, so to guarantee the global existence of the solutions to the associated initial value problems. Notice that we are not assuming the uniqueness of such solutions, so that the Poincaré map could be multivalued. The meaning of the conclusion concerning the K clockwise rotations of the T-periodic solutions around (x_0, y_0) will be clarified in Section 2, after introducing the so called *rotation number*.

Let us mention that conditions like (2) have been introduced in [4] in order to treat scalar second order equations with a nonlinearity having either one or two repulsive singularities, or a superlinear growth at infinity. It has been shown, indeed, that the singularities of this type provide a behavior of the solutions which resembles the situation encountered while dealing with superlinear systems (see, e.g. [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13]). In the same spirit, the existence of subharmonic solutions can also be easily obtained, but we will avoid such a discussion, for briefness.

The above situation will be generalized in Section 3 to a Hamiltonian system in $\mathbb{R}^{2N}$. Again, the proof will be based on the above mentioned higher dimensional generalized version of the Poincaré–Birkhoff theorem.

2. **The proof of Theorem 1.1.** There is no loss of generality in assuming that the equilibrium (x_0, y_0) coincides with the origin $(0,0)$. Moreover, by Assumption A3, we can assume that $\mathcal{U}$ is the restriction of a continuous function defined on the whole space $\mathbb{R}\times\mathbb{R}^2$, still having a bounded continuous gradient. Indeed, in the case when $\mathcal{R}$ is a strict subset of $\mathbb{R}^2$, we can choose a larger open set $\mathcal{S}$, containing the closure of $\mathcal{R}$, and extend $\mathcal{U}$ so that it vanishes outside $\mathbb{R}\times\mathcal{S}$, thus keeping the gradient bounded. Hence, let $C > 0$ be a constant for which

$$|\nabla_z \mathcal{U}(t,z)| \le C\,, \qquad \text{for every } (t,z)\in\mathbb{R}\times\mathbb{R}^2\,. \tag{4}$$

2.1. **The modified problem.** In order to ensure the existence of the solutions on the whole interval $[0,T]$, we are going to modify our system, and, whenever $\mathcal{R}$ does not already coincide with $\mathbb{R}^2$, extend it on the whole plane.

Define, for every $i,j\in\{1,2\}$ and every $p\in\,]0,1[$, the numbers

$$a^p_{i,j} = \begin{cases} -(1-p)^{-1} & \text{if } a_{i,j} = -\infty \\ p\,a_{i,j} & \text{if } a_{i,j}\in\mathbb{R} \\ (1-p)^{-1} & \text{if } a_{i,j} = +\infty\,, \end{cases}$$

and the rectangle

$$\mathcal{R}(p) = \,]a^p_{1,1}, a^p_{1,2}[\, \times \,]a^p_{2,1}, a^p_{2,2}[\,, \tag{5}$$

whose closure is contained in $\mathcal{R}$. We will be interested in taking p near to 1, so that the rectangle $\mathcal{R}(p)$ "approaches" $\mathcal{R}$, so to speak.

By Assumption A1, we can consider two auxiliary continuous functions $f_1 : \mathbb{R}\times\,]a_{1,1}, a_{1,2}[\,\to\mathbb{R}$ and $f_2 : \mathbb{R}\times\,]a_{2,1}, a_{2,2}[\,\to\mathbb{R}$, so that

$$\frac{\partial H_1}{\partial x}(t,x) = x f_1(t,x)\,, \qquad \frac{\partial H_2}{\partial y}(t,y) = y f_2(t,y)\,. \tag{6}$$

Define, for every $p \in]0,1[$ and $i \in \{1,2\}$,

$$f_i^p(t,\xi) = \begin{cases} f_i(t,a_{i,1}^p) & \text{if } \xi \in \,]-\infty, a_{i,1}^p[\\ f_i(t,\xi) & \text{if } \xi \in [\,a_{i,1}^p, a_{i,2}^p\,] \\ f_i(t,a_{i,2}^p) & \text{if } \xi \in \,]a_{i,2}^p, +\infty[\,, \end{cases}$$

and set

$$H_i^p(t,\xi) = H_i(t,0) + \int_0^\xi \sigma f_i^p(t,\sigma)\,d\sigma\,.$$

We thus obtain a new Hamiltonian function $\mathcal{H}^p : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$, defined as

$$\mathcal{H}^p(t,x,y) = H_1^p(t,x) + H_2^p(t,y) + \mathcal{U}(t,x,y)\,,$$

which coincides with $\mathcal{H}$ on $\mathbb{R}\times\mathcal{R}(p)$. We then consider the auxiliary Hamiltonian system

$$x' = \frac{\partial\mathcal{H}^p}{\partial y}(t,x,y)\,, \qquad y' = -\frac{\partial\mathcal{H}^p}{\partial x}(t,x,y)\,. \tag{7}$$

It will be useful to write the solutions $z(t) = \big(x(t),y(t)\big)$ of (7), whenever they do not attain the origin, using the polar coordinates

$$x(t) = \rho(t)\cos\vartheta(t)\,, \qquad y(t) = \rho(t)\sin\vartheta(t)\,. \tag{8}$$

Then, the radial velocity is given by

$$\rho'(t) = \frac{x(t)\,\dfrac{\partial\mathcal{H}^p}{\partial y}\big(t,x(t),y(t)\big) - y(t)\,\dfrac{\partial\mathcal{H}^p}{\partial x}\big(t,x(t),y(t)\big)}{\sqrt{x^2(t)+y^2(t)}}\,, \tag{9}$$

and the angular velocity by

$$-\vartheta'(t) = \frac{x(t)\,\dfrac{\partial\mathcal{H}^p}{\partial x}\big(t,x(t),y(t)\big) + y(t)\,\dfrac{\partial\mathcal{H}^p}{\partial y}\big(t,x(t),y(t)\big)}{x^2(t)+y^2(t)}\,. \tag{10}$$

If $z(t) \neq 0$ for every t in an interval $[\tau_0,\tau_1]$, the corresponding clockwise rotation number will be denoted by

$$\operatorname{rot}\big(z,[\tau_0,\tau_1]\big) = -\frac{1}{2\pi}\big(\vartheta(\tau_1)-\vartheta(\tau_0)\big)\,.$$

2.2. **The small amplitude solutions.** We first estimate the number of rotations of the solutions starting near the origin.

So, we fix a positive radius ε such that the ball B_ε, centered at the origin, is fully contained in $\mathcal{R}(\frac12)\cap\mathcal{V}$, where $\mathcal{V}$ is the neighborhood of the origin introduced in Assumption A1. The desired estimate is given by the following proposition.

Proposition 1. *There exist a positive integer K_0 and some positive constants η,δ, with $\eta<\delta<\varepsilon$, having the following property: if $z(t)=\big(x(t),y(t)\big)$ is a solution to system (7), with $p\in\,]\frac12,1[$, and $|z(t_0)|=\delta$ at a certain time t_0, then*

$$\eta < |z(t)| < \varepsilon\,, \qquad \text{for every } t\in[t_0,t_0+T]\,,$$

and

$$\operatorname{rot}\big(z,[t_0,t_0+T]\big) \le K_0\,.$$

Proof. Consider a solution $z(t) = \big(x(t), y(t)\big)$ to (7), with $p > \frac{1}{2}$, such that $0 < |z(t)| < \varepsilon$ in a certain time interval $\mathcal{I}$, and parametrize it in polar coordinates, as in (8). Recalling Assumption A1, since $\mathcal{H}^p = \mathcal{H}$ in $\mathcal{R}(p) \supset \mathcal{R}(\frac{1}{2})$, it is easy to see that, for every $t \in \mathcal{I}$, the radial velocity (9) satisfies the inequality

$$|\rho'(t)| \leq \alpha\rho(t)\,.$$

Choose $\delta < \varepsilon e^{-\alpha T}$ and $\eta < \delta e^{-\alpha T}$, and assume that $|z(t_0)| = \delta$ at a certain time $t_0 \in \mathcal{I}$. We can now take $\mathcal{I}$ as the maximal interval of time in which $0 < |z(t)| < \varepsilon$. By Gronwall's Lemma we thus have

$$\delta e^{-\alpha|t-t_0|} \leq |z(t)| \leq \delta e^{\alpha|t-t_0|}\,, \qquad \text{for every } t \in \mathcal{I}\,.$$

Hence, $[t_0, t_0 + T] \subseteq \mathcal{I}$, and

$$\eta < |z(t)| < \varepsilon, \qquad \text{for every } t \in [t_0, t_0 + T]\,.$$

At this point, the last part of the statement can be deduced by just noticing that the angular velocity (10) is a continuous function in the compact set $\overline{B}_\varepsilon \setminus B_\eta$. □

2.3. **The large amplitude solutions.** We now estimate the number of rotations of the solutions starting sufficiently far from the origin.

Proposition 2. *For every positive integer K, there exists a constant $p^K \in \,\big]\frac{1}{2}, 1\big[$ with the following property: if $z(t) = \big(x(t), y(t)\big)$ is a solution to system (7), with $p \in]p^K, 1[\,$, and $z(t_0) \notin \mathcal{R}(p^K)$ at a certain time t_0, then $z(t) \neq 0$ for every $t \in [t_0, t_0 + T]$, and* rot $\big(z, [t_0, t_0 + T]\big) > K$.

In order to prove the above proposition, we first need two lemmas.

Lemma 2.1. *For every $p \in \,\big]\frac{1}{2}, 1\big[\,$, there exists a constant $A = A(p) > 0$ such that*

$$\left|\frac{\partial H_1}{\partial x}(t, x)\right| \leq A|x|\,, \qquad \text{for every } t \in [0, T] \text{ and } x \in \big[\,a^p_{1,1}, a^p_{1,2}\,\big],$$

and

$$\left|\frac{\partial H_2}{\partial y}(t, y)\right| \leq A|y|\,, \qquad \text{for every } t \in [0, T] \text{ and } y \in \big[\,a^p_{2,1}, a^p_{2,2}\,\big].$$

Proof. It is an immediate consequence of Assumption A1, and the compactness of $\overline{\mathcal{R}(p)}$, as a subset of $\mathcal{R}$. □

Lemma 2.2. *For every $\beta > 0$, there exists $p = p(\beta) \in \,\big]\frac{1}{2}, 1\big[$ such that, if $q \in]p, 1[\,$,*

$$x\,\frac{\partial \mathcal{H}^q}{\partial x}(t, x, y) > \beta x^2, \qquad \text{for every } t \in [0, T]\,,\ x \notin \,\big]a^p_{1,1}, a^p_{1,2}\big[\text{ and } y \in \mathbb{R}\,,$$

and

$$y\,\frac{\partial \mathcal{H}^q}{\partial y}(t, x, y) > \beta y^2, \qquad \text{for every } t \in [0, T]\,,\ x \in \mathbb{R} \text{ and } y \notin \,\big]a^p_{2,1}, a^p_{2,2}\big[\,.$$

Proof. We just prove the first inequality, the other one being analogous. Given $\beta > 0$, by (2) in Assumption A2 it is possible to find $p \in \,\big]\frac{1}{2}, 1\big[$ such that

$$x\,\frac{\partial H_1}{\partial x}(t, x) > \beta x^2 + C|x|\,, \quad \text{for every } t \in [0, T] \text{ and } x \notin]a^p_{1,1}, a^p_{1,2}[\,,$$

where $C > 0$ is the constant introduced in (4). Equivalently, recalling (6), we can write $f_1(t, x) > \beta + C/|x|$. Consequently, for every $q > p$, one has

$$f^q_1(t, x) > \beta + \frac{C}{|x|}\,, \quad \text{for every } t \in [0, T] \text{ and } x \notin]a^p_{1,1}, a^p_{1,2}[\,,$$

so that

$$x\,\frac{\partial \mathcal{H}^q}{\partial x}(t,x,y) = x^2 f_1^q(t,x) + x\,\frac{\partial \mathcal{U}}{\partial x}(t,x,y) > \beta x^2 + C|x| - C|x| = \beta x^2\,,$$

for every $t \in [0,T]$, $x \notin]a^p_{1,1}, a^p_{1,2}[$ and $y \in \mathbb{R}$, thus concluding the proof. □

We can now proceed with the proof of Proposition 2. Given the integer K, we fix a constant b, so to have

$$b > \frac{2\pi}{T} K\,. \tag{11}$$

Use Lemma 2.2, with $\beta = b$, to find the value $p_1 = p(b)$. Then use Lemma 2.1, with $p = p_1$, and find $A = A(p_1)$. Now, set

$$d = \max\left\{ \left|\frac{a^{p_1}_{1,j}}{a^{p_1}_{2,k}}\right| , \left|\frac{a^{p_1}_{1,j}}{a^{p_1}_{2,k}}\right|^{-1} \,:\, j,k \in \{1,2\} \right\},$$

$$\Delta = \min\left\{ \left|a^{p_1}_{i,j}\right| \,:\, i,j \in \{1,2\} \right\},$$

and define

$$b' = b(1+d^2) + d^2 A + \frac{Cd}{\Delta}\,,$$

where C is the constant introduced in (4). Notice that $b' > b$. Use again Lemma 2.2, with $\beta = b'$, in order to find $p_2 = p(b') > p_1$.

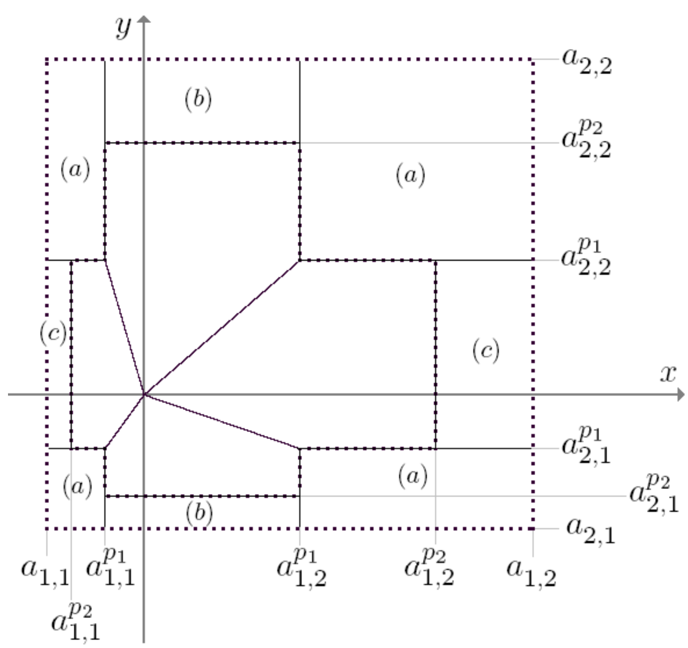

FIGURE 1. *The regions where we estimate the angular velocity of the solutions, in the three cases (a), (b) and (c).*

Let $z(t) = \big(x(t), y(t)\big)$ be a solution of (7), with $p > p_2$, which remains outside $\mathcal{R}(p_2)$ for some time. As long as t varies, $z(t)$ may cross different regions in the plane, and we distinguish three situations, as shown in Figure 1:

(a) If $x(t) \notin \,]a^{p_1}_{1,1}, a^{p_1}_{1,2}[$ and $y(t) \notin \,]a^{p_1}_{2,1}, a^{p_1}_{2,2}[$, it is easy to check that

$$-\vartheta'(t) > b\,.$$

(b) If $x(t) \in \big[a^{p_1}_{1,1}, a^{p_1}_{1,2}\big]$ and $y(t) \notin \,]a^{p_2}_{2,1}, a^{p_2}_{2,2}[$, then one has $|x(t)| < d\,|y(t)|$ and $|x(t)| < y^2(t)\,d/\Delta$, so that

$$\begin{aligned} -\vartheta'(t) &> \frac{1}{x^2(t)+y^2(t)}\big(-A\,x^2(t) - C\,|x(t)| + b'\,y^2(t)\big) \\ &> \frac{y^2(t)}{x^2(t)+y^2(t)}\left(-d^2 A - \frac{Cd}{\Delta} + b'\right) \\ &> \frac{b' - d^2 A - C\,d/\Delta}{1+d^2} = b\,. \end{aligned}$$

(c) If $x(t) \notin \,]a^{p_2}_{1,1}, a^{p_2}_{1,2}[$ and $y(t) \in \big[a^{p_1}_{2,1}, a^{p_1}_{2,2}\big]$, one has similarly $|y(t)| < d\,|x(t)|$ and $|y(t)| < x^2(t)\,d/\Delta$, so that

$$\begin{aligned} -\vartheta'(t) &> \frac{1}{x^2(t)+y^2(t)}\big(b'\,x^2(t) - A\,y^2(t) - C\,|y(t)|\big) \\ &> \frac{x^2(t)}{x^2(t)+y^2(t)}\left(b' - d^2 A - \frac{Cd}{\Delta}\right) \\ &> \frac{b' - d^2 A - C\,d/\Delta}{1+d^2} = b\,. \end{aligned}$$

As a consequence, every solution $z(t)$ to (7), with $p > p_2$, which remains outside $\mathcal{R}(p_2)$, has to rotate clockwise, with an angular velocity bounded below by b. In particular, by (11), if $z(t) \notin \mathcal{R}(p_2)$ for every $t \in [0,T]$, then $\operatorname{rot}\big(z,[0,T]\big) > K$.

Let us now focus our attention on the solutions which enter the region $\mathcal{R}(p_2)$, coming from the outside, and remain inside this region in a time interval $[\tau_0,\tau_1]$, with $\tau_1 - \tau_0 \le T$. Applying Proposition 1, such solutions cannot reach the origin: more precisely, they remain outside the ball B_η, for every $t \in [\tau_0,\tau_1]$. Following one of these solutions, we see that, after entering the set $\mathcal{R}(p_2)$, it could perform a certain number of counter-clockwise rotations while remaining inside it. However, since $\overline{\mathcal{R}(p_2)} \setminus B_\eta$ is compact, the angular velocity will remain bounded. So, there exists a constant $D \ge 0$ such that, for every such solution $z(t)$ which remains inside $\mathcal{R}(p_2)$ in a time interval $[\tau_0,\tau_1]$, it has to be

$$\operatorname{rot}\big(z,[\tau_0,\tau_1]\big) \ge -D\,\frac{\tau_1-\tau_0}{T}\,.$$

Now, we will provide the existence of a guiding curve γ which controls the solutions of system (1). As a consequence, we will see that this curve guides also the solutions of system (7), when p is chosen large enough. By Assumption A2, we can find four functions $g_{i,\ell}$ such that

$$g_{1,1}(x) < \frac{\partial\mathcal{H}}{\partial x}(t,x,y) < g_{1,2}(x)\,, \qquad g_{2,1}(y) < \frac{\partial\mathcal{H}}{\partial y}(t,x,y) < g_{2,2}(y)\,,$$

for every $(t,x,y) \in [0,T]\times\mathcal{R}$, and it is not restrictive to assume these functions to be strictly increasing. Define, then, their primitives

$$G_{i,\ell}(\xi) = \int_0^\xi g_{i,\ell}(\sigma)\,d\sigma\,.$$

It is possible to choose all the functions $g_{i,\ell}$ in order to ensure that

$$\lim_{\xi \to a_{i,j}} \frac{g_{i,\ell}(\xi)}{\xi} = +\infty\,, \qquad i,j,\ell \in \{1,2\}\,, \tag{12}$$

$$\lim_{\xi \to a_{i,j}} G_{i,\ell}(\xi) = +\infty\,, \qquad i,j,\ell \in \{1,2\}\,. \tag{13}$$

Being the functions $g_{i,\ell}$ strictly increasing, there are $x_1 < 0 < x_2$ and $y_1 < 0 < y_2$ such that

$$g_{1,1}(x_2) = g_{1,2}(x_1) = g_{2,1}(y_2) = g_{2,2}(y_1) = 0\,,$$

and these points are unique. Define the four regions in $\mathcal{R}$, depicted in Figure 2,

$$\begin{aligned} \mathcal{R}_{1,1} &= \mathcal{R} \cap \{x \geq x_2\,,\, y \leq y_2\}\,, \\ \mathcal{R}_{1,2} &= \mathcal{R} \cap \{x \leq x_2\,,\, y \leq y_1\}\,, \\ \mathcal{R}_{2,1} &= \mathcal{R} \cap \{x \geq x_1\,,\, y \geq y_2\}\,, \\ \mathcal{R}_{2,2} &= \mathcal{R} \cap \{x \leq x_1\,,\, y \geq y_1\}\,, \end{aligned}$$

and the energy functions

$$E_{\mu,\nu}(x,y) = G_{1,\mu}(x) + G_{2,\nu}(y)\,,$$

where $\mu, \nu \in \{1,2\}$. Then, for every solution $z(t)$ to (1),

$$\frac{d}{dt} E_{\mu,\nu}\big(z(t)\big) > 0\,, \quad \text{if } z(t) \in \mathcal{R}_{\mu,\nu}\,.$$

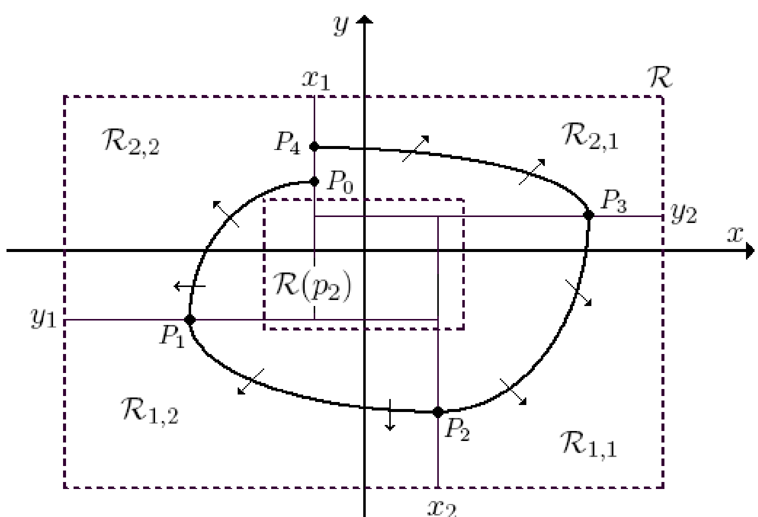

FIGURE 2. *The construction of the first lap of the guiding curve, outside the rectangle $\mathcal{R}(p_2)$, using the level curves of the energy functions.*

Now, starting from a point not belonging to $\mathcal{R}(p_2)$, we construct a guiding curve γ, glueing together different branches of the level sets of the energy functions $E_{\mu,\nu}$, in the corresponding regions $\mathcal{R}_{\mu,\nu}$. Precisely, such a curve starts from a point P_0, belonging to the segment $\mathcal{R}_{2,1} \cap \mathcal{R}_{2,2}$, and proceeds counter-clockwise, as shown in Figure 2, remaining in the rectangle $\mathcal{R}_{2,2}$ until it reaches the point P_1, on the segment $\mathcal{R}_{2,2} \cap \mathcal{R}_{1,2}$. Then, it enters and remains in $\mathcal{R}_{1,2}$ till it reaches the point P_2 on $\mathcal{R}_{1,2} \cap \mathcal{R}_{1,1}$. And so on, crossing the four regions, and thus completing a first lap at the point P_4, which again belongs to the segment $\mathcal{R}_{2,1} \cap \mathcal{R}_{2,2}$. Since

$$\begin{aligned} E_{2,2}(P_0) &= E_{2,2}(P_1) < E_{1,2}(P_1) = E_{1,2}(P_2) \\ &< E_{1,1}(P_2) = E_{1,1}(P_3) < E_{2,1}(P_3) = E_{2,1}(P_4) < E_{2,2}(P_4)\,, \end{aligned}$$

we have that P_4 necessarily lies above P_0.

Iterating such a construction, we can obtain a curve having the shape of a spiral, which rotates counter-clockwise around the origin while becoming larger and larger, having the following property: if a solution $z(t)$ to system (1) intersects the guiding curve γ at a certain time, then it must cross γ from the inner part to the outer part. As a consequence we have that, roughly speaking, a solution coming from some point near the boundary of $\mathcal{R}$, and approaching $\mathcal{R}(p_2)$, will have to rotate clockwise several times, guided by the curve γ.

It is possible to verify that the curve γ, so to speak, gets nearer and nearer the boundary of $\mathcal{R}$ while it rotates counter-clockwise in the plane. This is certainly true when $\mathcal{R}$ is a bounded rectangle. If not, in the case when $\mathcal{R}$ is unbounded in some directions, the curve will become larger and larger in those directions. Moreover, we can choose the starting point of the curve in order that the first lap, and hence the whole curve, lays outside $\mathcal{R}(p_2)$. Hence, we can draw $K+D+1$ laps of the curve γ, starting outside the region $\mathcal{R}(p_2)$, and then find some $p^K \in]p_2, 1[$ such that all the $K+D+1$ laps of γ are contained in the set $\mathcal{R}(p^K)$. Notice that, being $\mathcal{H}^p = \mathcal{H}$ in $\mathcal{R}(p)$, all the previous considerations on the guiding curve γ still hold for every solution to system (7), when p is chosen greater than p^K.

Let us show that this constant p^K verifies the property that, if $z(t)$ is a solution to (7), with $p > p^K$, and $z(t_0) \notin \mathcal{R}(p^K)$ at a certain time t_0, then $\mathrm{rot}\,\big(z, [t_0, t_0 + T]\big) > K$. By the above computation on the angular velocity, we only need to examine the case when there exists a time $t_1 \in [t_0, t_0 + T]$ at which the solution enters the rectangle $\mathcal{R}(p_2)$, since otherwise the solution would perform more than K clockwise rotations around the origin in a period time T. Therefore, in some interval $[t_0, t_1]$, the solution goes from outside $\mathcal{R}(p^K)$ to inside $\mathcal{R}(p_2)$, guided by the curve γ, thus performing at least $K+D+1$ clockwise rotations around the origin. Recalling that, as long as $z(t)$ remains inside $\mathcal{R}(p_2)$ in a time interval $[\tau_0, \tau_1]$, with $\tau_1 - \tau_0 \le T$, it never attains the origin and $\mathrm{rot}\,\big(z, [\tau_0, \tau_1]\big) \ge -D(\tau_1 - \tau_0)/T$, we finally deduce that

$$\mathrm{rot}\,\big(z, [t_0, t_0 + T]\big) \ge (K + D + 1) - D > K\,.$$

The proof of Proposition 2 is thus completed.

2.4. **The Poincaré – Birkhoff setting.** The proof of Theorem 1.1 will now be concluded by the use of a generalized version of the Poincaré – Birkhoff theorem provided in [7], which does not require uniqueness for the Cauchy problems associated with our equations.

Let K_0 and δ be given by Proposition 1, and fix an integer $K > K_0$. Once K has been fixed, let $p^K \in \big]\frac{1}{2}, 1\big[$ be given by Proposition 2, and fix a $p \in]p^K, 1[$. Notice that, necessarily, the closed ball $\overline{B}_\delta$ is contained in $\mathcal{R}(p^K)$. We want to apply [7, Theorem 1.2] to system (7), with respect to the annulus

$$\mathcal{A} = \overline{\mathcal{R}(p^K)} \setminus B_\delta\,.$$

Let us check that the twist condition holds. With this aim, let $z(t)$ be a solution of (7). If $|z(0)| = \delta$, then Proposition 1 guarantees that $\mathrm{rot}\,\big(z,[0,T]\big) \leq K_0 < K$. On the other hand, if $z(0) \in \partial\mathcal{R}(p^K)$, then Proposition 2 tells us that $\mathrm{rot}\,\big(z,[0,T]\big) > K$. The twist condition is thus verified.

Hence, by [7, Theorem 1.2], system (7) has at least two T-periodic solutions $z(t), \tilde z(t)$, with $\mathrm{rot}\,\big(z,[0,T]\big) = \mathrm{rot}\,\big(\tilde z,[0,T]\big) = K$. By Proposition 2 again, it has to be that $z(t), \tilde z(t) \in \mathcal{R}(p^K)$, for every $t \in [0,T]$, so that these are indeed solutions of the original system (1). The proof of Theorem 1.1 is thus completed.

Remark 1. In the case when the rectangle $\mathcal{R}$ reduces to a strip, like e.g. $]a,b[\,\times\mathbb{R}$, with a and b real numbers, the superlinear growth assumption in the y variable can be weakended, in the spirit of [4]. Indeed, being $b-a$ finite, the time needed for a large amplitude solution to go from one side to the other will be small, even if the angular speed does not necessarily approach infinity.

3. **Higher dimensional Hamiltonian systems.** Let us consider a higher dimensional system of the type

$$x' = \nabla_y \mathcal{H}(t,x,y)\,, \qquad y' = -\nabla_x \mathcal{H}(t,x,y)\,. \tag{14}$$

We are dealing with a Hamiltonian function $\mathcal{H} : \mathbb{R}\times\mathcal{R} \to \mathbb{R}$ which is T-periodic in its first variable and continuously differentiable in x and y. Here, $\mathcal{R}$ is a subset of $\mathbb{R}^{2N}$ of the type

$$\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_N\,,$$

where the sets

$$\mathcal{R}_m =]a^m_{1,1}, a^m_{1,2}[\,\times\,]a^m_{2,1}, a^m_{2,2}[$$

are some "generalized rectangles", in the above sense. A solution $z(t) = \big(x(t), y(t)\big)$ of system (14) is such that

$$x(t) = \big(x_1(t),\dots,x_N(t)\big)\,, \qquad y(t) = \big(y_1(t),\dots,y_N(t)\big)\,,$$

with the component $z_m(t) = \big(x_m(t), y_m(t)\big)$ varying in $\mathcal{R}_m$, for every $m = 1,\dots,N$. We will assume that the Hamiltonian function can be decomposed as

$$\mathcal{H}(t,x,y) = \sum_{m=1}^{N} \Big(H_{1,m}(t,x_m) + H_{2,m}(t,y_m)\Big) + \mathcal{U}(t,x,y)\,.$$

Let us introduce the analogues of Assumptions A1, A2, and A3.

Assumption A1′. *There exists an equilibrium point (x_0,y_0) in $\mathcal{R}$, with $x_0 = (x^0_1,\dots,x^0_N)$ and $y_0 = (y^0_1,\dots,y^0_N)$, and there exist a constant $\alpha > 0$ and a neighborhood $\mathcal{V}$ of (x_0,y_0) such that*

$$\left|\frac{\partial H_{1,m}}{\partial x_m}(t,x_m)\right| + \left|\frac{\partial \mathcal{U}}{\partial x_m}(t,x,y)\right| \leq \alpha |x_m - x^0_m|\,,$$

and

$$\left|\frac{\partial H_{2,m}}{\partial y_m}(t,y_m)\right| + \left|\frac{\partial \mathcal{U}}{\partial y_m}(t,x,y)\right| \leq \alpha |y_m - y^0_m|\,,$$

for every $(t,x,y) \in [0,T]\times\mathcal{V}$, and $m = 1,\dots,N$.

Assumption A2′. *There exist some continuous and increasing functions $\kappa^m_{i,\ell} :]a^m_{i,1}, a^m_{i,2}[\to \mathbb{R}$ such that*

$$\kappa^m_{1,1}(\xi) < \frac{\partial H_{1,m}}{\partial x_m}(t,\xi) < \kappa^m_{1,2}(\xi)\,, \quad \kappa^m_{2,1}(\upsilon) < \frac{\partial H_{2,m}}{\partial y_m}(t,\upsilon) < \kappa^m_{2,2}(\upsilon)\,, \tag{15}$$

for every $(t,\xi,\upsilon) \in [0,T] \times \mathcal{R}_m$, *with*

$$\lim_{\xi \to a^m_{1,j}} \frac{\kappa^m_{1,\ell}(\xi)}{\xi - x^0_m} = +\infty \quad \text{and} \quad \lim_{\upsilon \to a^m_{2,j}} \frac{\kappa^m_{2,\ell}(\upsilon)}{\upsilon - y^0_m} = +\infty \,, \qquad j,\ell \in \{1,2\}\,, \tag{16}$$

and their primitive functions

$$\mathcal{K}^m_{1,\ell}(\xi) = \int_{x^0_m}^{\xi} \kappa^m_{1,\ell}(\sigma)\, d\sigma \,, \qquad \mathcal{K}^m_{2,\ell}(\upsilon) = \int_{y^0_m}^{\upsilon} \kappa^m_{2,\ell}(\sigma)\, d\sigma$$

satisfy

$$\lim_{\xi \to a^m_{1,j}} \mathcal{K}^m_{1,\ell}(\xi) = +\infty \quad \text{and} \quad \lim_{\upsilon \to a^m_{2,j}} \mathcal{K}^m_{2,\ell}(\upsilon) = +\infty \,, \qquad j,\ell \in \{1,2\}\,, \tag{17}$$

for every $m = 1,\ldots,N$.

Assumption A3′. T*he function* $\mathcal{U} : \mathbb{R} \times \overline{\mathcal{R}} \to \mathbb{R}$ *is continuous and has a bounded continuous gradient with respect to* (x,y) *belonging to* $\overline{\mathcal{R}}$.

Theorem 3.1. *Let Assumptions A1′, A2′, and A3′ be satisfied. Then, there exists a positive integer* K_0 *such that, for any choice of* N *integers* $K_1,\ldots, K_N \ge K_0$*, the Hamiltonian system* (14) *has at least* $N+1$ *distinct* T*-periodic solutions* $\big(x(t),y(t)\big)$*, such that, for every index* $m = 1,\ldots,N$*, the component* $\big(x_m(t),y_m(t)\big)$ *performs exactly* K_m *clockwise rotations around* (x^0_m,y^0_m) *in the time interval* $[0,T]$.

Proof. We follow the lines of the proof of Theorem 1.1, working on each component (x_m,y_m) separately. It is not restrictive to assume that $(x_0,y_0) = (0,0)$, and we can extend the function $\mathcal{U}$ to the whole space $\mathbb{R} \times \mathbb{R}^{2N}$, preserving the boundedness of its gradient. We define, for every $p \in]0,1[$,

$$\mathcal{R}(p) = \mathcal{R}_1(p) \times \cdots \times \mathcal{R}_N(p)\,,$$

where the rectangles $\mathcal{R}_m(p)$ are defined as in (5). Then, following the procedure described in Section 2.1, we accordingly modify the functions $H_{1,m}$ and $H_{2,m}$, so to obtain the new Hamiltonian function $\mathcal{H}^p : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, defined as

$$\mathcal{H}^p(t,x,y) = \sum_{m=1}^{N} \Big(H^p_{1,m}(t,x_m) + H^p_{2,m}(t,y_m) \Big) + \mathcal{U}(t,x,y)\,.$$

Consider now the modified system

$$x' = \nabla_y \mathcal{H}^p(t,x,y)\,, \qquad y' = -\nabla_x \mathcal{H}^p(t,x,y)\,. \tag{18}$$

The following are the analogues of Propositions 1 and 2.

Proposition 3. *There exist a positive integer* K_0 *and some positive constants* η,δ,ε*, with* $\eta < \delta < \varepsilon$*, with the following property: if* $z(t) = \big(x(t),y(t)\big)$ *is a solution to system* (18)*, with* $p \in \left]\frac{1}{2},1\right[$*, and* $|z_m(t_0)| = \delta$ *at a certain time* t_0*, for some* $m \in \{1,\ldots,N\}$*, then*

$$\eta < |z_m(t)| < \varepsilon\,, \qquad \text{for every } t \in [t_0, t_0+T]\,,$$

and

$$\operatorname{rot}\big(z_m, [t_0,t_0+T]\big) \le K_0\,.$$

Proof. Just follow the lines of the proof of Proposition 1. □

Proposition 4. *For every positive integer K, there exists a constant $p^K \in \left]\frac{1}{2}, 1\right[$ with the following property: if $z(t) = \big(x(t), y(t)\big)$ is a solution to system (18), with $p \in]p^K, 1[$, and $z_m(t_0) \notin \mathcal{R}_m(p^K)$ at a certain time t_0, for some $m \in \{1, \dots, N\}$, then $z_m(t) \neq 0$ for every $t \in [t_0, t_0 + T]$, and $\operatorname{rot}\big(z_m, [t_0, t_0 + T]\big) > K$.*

Proof. The main difference with the proof of Proposition 2 is the fact that we now need to construct N planar spirals, in $\mathcal{R}_1, \dots, \mathcal{R}_N$, respectively (see [5] for a similar approach). The construction of these curves is the same as the one performed in Section 2.3, so we avoid it, for briefness. $\square$

Let us now conclude the proof of Theorem 3.1. Given some integers $K_1, \dots, K_N$, greater than K_0, let us define $K = \max\{K_1, \dots, K_N\}$, and consider the generalized annulus

$$\mathcal{A} = \Big(\overline{\mathcal{R}_1(p^K)} \setminus B_\delta\Big) \times \cdots \times \Big(\overline{\mathcal{R}_N(p^K)} \setminus B_\delta\Big).$$

Let $z(t)$ be a solution of (18). If $|z_m(0)| = \delta$, for some $m \in \{1, \dots, N\}$, then Proposition 3 guarantees that $\operatorname{rot}\big(z_m, [0, T]\big) < K_m$. On the other hand, if $z_m(0) \in \partial\mathcal{R}_m(p^K)$, then Proposition 4 tells us that $\operatorname{rot}\big(z_m, [0, T]\big) > K_m$. Hence, by [7, Theorem 1.2], system (18) has at least $N + 1$ distinct T-periodic solutions, say $z^{(1)}(t), \dots, z^{(N+1)}(t)$, with $\operatorname{rot}\big(z_m^{(n)}, [0, T]\big) = K_m$. By Proposition 4 again, it has to be that $z_m^{(n)}(t) \in \mathcal{R}_m(p^K)$, for every $t \in [0, T]$, so that these are indeed solutions of the original system (14). The proof of Theorem 3.1 is thus completed. $\square$

REFERENCES

[1] A. Boscaggin and R. Ortega, Monotone twist maps and periodic solutions of systems of Duffing type, *Math. Proc. Cambridge Philos. Soc.*, **157** (2014), 279–296.

[2] A. Castro and A. C. Lazer, On periodic solutions of weakly coupled systems of differential equations, *Boll. Un. Mat. Ital.*, **18** (1981), 733–742.

[3] T. R. Ding and F. Zanolin, Periodic solutions of Duffing's equations with superquadratic potential, *J. Differential Equations*, **97** (1992), 328–378.

[4] A. Fonda, R. Manasevich and F. Zanolin, Subharmonic solutions for some second-order differential equations with singularities, *SIAM J. Math. Anal.*, **24** (1993), 1294–1311.

[5] A. Fonda and A. Sfecci, Periodic solutions of a system of coupled oscillators with one-sided superlinear retraction forces, *Differential Integral Equations*, **25** (2012), 993–1010.

[6] A. Fonda and A. Sfecci, Periodic solutions of weakly coupled superlinear systems, *J. Differential Equations,* **260** (2016), 2150–2162.

[7] A. Fonda and A. J. Ureña, A higher dimensional Poincaré – Birkhoff theorem for Hamiltonian flows, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, online first.

[8] S. Fučík and V. Lovicar, Periodic solutions of the equation $x''(t) + g(x(t)) = p(t)$, *Časopis Pěst. Mat.*, **100** (1975), 160–175.

[9] Ph. Hartman, On boundary value problems for superlinear second order differential equations, *J. Differential Equations*, **26** (1977), 37–53.

[10] H. Jacobowitz, Periodic solutions of $x'' + f(x, t) = 0$ via the Poincaré – Birkhoff theorem, *J. Differential Equations*, **20** (1976), 37–52.

[11] G. R. Morris, An infinite class of periodic solutions of $x'' + 2x^3 = p(t)$, *Proc. Cambridge Philos. Soc.*, **61** (1965), 157–164.

[12] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems: A survey, *SIAM J. Math. Anal.*, **13** (1982), 343–352.

[13] A. Sfecci, Positive periodic solutions for planar differential systems with repulsive singularities on the axes, *J. Math. Anal. Appl.*, **415** (2014), 110–120.

Received June 2015; revised October 2016.

E-mail address: a.fonda@units.it
E-mail address: sfecci@dipmat.univpm.it