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ORIGINAL ARTICLE

Radial periodic perturbations of the Kepler problem

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Abstract We consider radial periodic perturbations of a central force field and prove the existence of rotating periodic solutions, whose orbits are nearly circular. The proof is mainly based on the Implicit Function Theorem, and it permits to handle some small perturbations involving the velocity, as well. Our results apply, in particular, to the classical Kepler problem.

Keywords Kepler problem · Periodic solutions · Radially symmetric systems

Mathematics Subject Classification 34C25

1 Introduction

Let us start by considering the radially perturbed system

$$\ddot{x} + \frac{c}{|x|^3}\, x = p(t, |x|; \varepsilon)\, x. \tag{1}$$

Here, c is a positive constant, while $p : \mathbb{R}\times\,]0, +\infty[\, \times\mathbb{R} \to \mathbb{R}$ is a continuous function, T-periodic in its first variable, continuously differentiable in its second and third variables, with

$$p(t, \rho; 0) = 0, \quad \text{for every } (t, \rho) \in \mathbb{R}\times\,]0, +\infty[. \tag{2}$$

Notice that, when $\varepsilon = 0$, we have the classical equation for the Kepler problem

$$\ddot{x} = -\frac{c}{|x|^3}\, x. \tag{3}$$

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It is well known that the orbits of the solutions of (1) are planar: Hence, we will always assume that $x(t) \in \mathbb{R}^2$, for every $t \in \mathbb{R}$. We are looking for periodic solutions of (1) which never attain the singularity, usually named *collisionless solutions*, satisfying

$$x(t) \neq 0, \quad \text{for every } t \in \mathbb{R}.$$

Our first result is the following.

Theorem 1 *Let $x_*(t)$ be a circular solution of the unperturbed system* (3)*, having minimal period τ_*. Assume that*

$$\frac{T}{\tau_*} = \frac{n}{m} \in \mathbb{Q} \setminus \mathbb{N}. \tag{4}$$

Then, for any $\sigma > 0$ there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, Eq. (1) *has a mT-periodic solution $x(t)$ that makes exactly n revolutions around the origin in the period time mT, and such that*

$$|x(t) - x_*(t)| \leq \sigma, \text{ for every } t \in \mathbb{R}. \tag{5}$$

By *circular solution* we mean a solution whose orbit is a circle centered at the origin. Condition (4) says that the quotient T/τ_* has to be a rational number, but not an integer, a kind of nonresonance condition. However, since the minimal period of the circular orbits of the Kepler problem is strictly increasing with their radius, we can see that, taking *any* circular orbit of the unperturbed system (3), assumption (4) will be verified by infinitely many circular orbits which are arbitrarily near to it. As a consequence, we have the following result.

Theorem 2 *For any circular solution $x_*(t)$ of the unperturbed system* (3)*, for any positive integer N and any $\sigma > 0$, there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, Eq.* (1) *has at least N periodic solutions $x(t)$ satisfying* (5).

Our result can successfully be applied, e.g., to the equation

$$\ddot{x} = -\frac{c + \varepsilon e(t)}{|x|^3} x,$$

where $e : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function. This is a model of a Kepler problem with a periodically varying gravitational force.

The problem of the existence of periodic solutions to the Kepler problem has been studied by many authors. Even if it enters in the wide class of systems with a singularity, it has some peculiar degeneracies that makes it rather difficult to solve. Following a pioneering paper by Gordon (1975), variational methods have been used, e.g., in Ambrosetti and Coti Zelati (1989), Capozzi et al. (1991), Degiovanni and Giannoni (1989), Serra and Terracini (1994). In Ambrosetti and Coti Zelati (1989), the authors considered periodic perturbations of the Kepler problem, and they proved the existence of T-periodic solutions bifurcating from some circular solutions of the unperturbed problem. Similar bifurcation results were obtained in Boscaggin and Ortega (2016), Cabral and Vidal (2000), Vidal (2001). In Boscaggin and Ortega (2016), the stability of such solutions was also studied. In Fonda et al. (2012), a rotating perturbation was considered. We refer to the books Ambrosetti and Coti Zelati (1993), Cordani (2003), Torres (2015) for more information and references.

When comparing our theorems with the previous literature, we remark that our results are stated for the problem of *radial perturbations*. This fact will allow us to deal separately with the radial and the angular component of the solutions, a technique introduced in Fonda and Toader (2008) and already exploited in several papers (Fonda and Sfecci 2017; Fonda and Toader 2011a, b, 2012; Fonda et al. 2012; Fonda and Ureña 2011; Torres et al. 2013). However, our approach also permits to deal with small perturbations of the type $\kappa q(t)\dot{x}$, provided that $q(t)$ is T-periodic and has zero mean. Even if we cannot see in this additional perturbation a direct physical interpretation, we believe it worth to be included in our discussion. The details are provided below.

In Sect. 2 of the paper we study a very general equation, trying to grasp the main ingredients needed to obtain our bifurcation results. The principal tool in the proof will be the Implicit Function Theorem, which will be used twice: a first time for obtaining T-periodic radial components, the second one to get subharmonic, or quasi-periodic angular components. Indeed, we will find solutions which in each time interval $[t, t+T]$ rotate of a given angle, precisely the one spanned in the same time interval by the circular solution of the unperturbed problem.

In Sect. 3 we illustrate some examples of applications. In particular, the gravitational model (1) will be modified so to allow the power 2 in the inverse square law to be replaced by any real number, with the exception of -1, when some linear resonance phenomena may appear. We will also deal with a perturbation of Levi-Civita equation approximating relativistic effects. Finally, we will present a model with two slightly oscillating masses, describing the motion of a particle on the plane orthogonal to the segment joining them.

2 The general problem

We want to deal with the more general equation

$$\ddot{x} + g(|x|)\, x = p(t, |x|; \varepsilon)\, x + \kappa\, q(t)\, \dot{x}, \tag{6}$$

where $g\ :]0, +\infty[\to\ \mathbb{R}$ is assumed to be continuously differentiable. The function $p : \mathbb{R}\times\,]0, +\infty[\, \times\mathbb{R} \to \mathbb{R}$ is continuous, T-periodic in its first variable, and continuously differentiable in its second and third variables, satisfying (2), while $q : \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic, and such that

$$\int_0^T q(t)\, \mathrm{d}t = 0. \tag{7}$$

First of all, we notice that the orbits arising from Eq. (6) are planar. To see this, assume for simplicity that we are in a three-dimensional space. Then, if $x(t)$ is a solution,

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) \wedge \dot{x}(t)) = \kappa\, q(t)\, (x(t) \wedge \dot{x}(t)),$$

hence,

$$x(t) \wedge \dot{x}(t) = e^{\kappa \int_0^t q(s)\, \mathrm{d}s}(x(0) \wedge \dot{x}(0)),$$

showing that the direction of $x(t) \wedge \dot{x}(t)$ is always the same. The proof is similar if we are in any higher dimension, cf. Fonda and Toader (2008, Appendix A). So, we will always assume that $x(t) \in \mathbb{R}^2$, for every $t \in \mathbb{R}$.

Before stating our main result, we need to recall the notion of *rotation number* around the origin for a planar curve. For $\tau_1 < \tau_2$, let $\gamma : [\tau_1, \tau_2] \to \mathbb{R}^2$ be continuous and such that $\gamma(t) \neq 0$, for every $t \in [\tau_1, \tau_2]$. Writing $\gamma(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, with continuous functions $\rho(t)$ and $\theta(t)$, the rotation number of γ around the origin is defined as

$$\mathrm{Rot}\,(\gamma; [\tau_1, \tau_2]) = \frac{\theta(\tau_2) - \theta(\tau_1)}{2\pi}.$$

A solution $x : \mathbb{R} \to \mathbb{R}^2$ is said to be *T-radially periodic* if the function $t \mapsto |x(t)|$ is T-periodic, and $t \mapsto \mathrm{Rot}\,(x; [t, t+T])$ is constant.

Let $\rho_* > 0$ be such that $g(\rho_*) > 0$. Then, the unperturbed equation

$$\ddot{x} = -g(|x|)\,x \tag{8}$$

has a circular solution $x_*(t)$, with $|x_*(t)| = \rho_*$, having minimal period

$$\tau_* = \frac{2\pi}{\sqrt{g(\rho_*)}}. \tag{9}$$

Notice that $\mathrm{Rot}(x_*, [0, T]) = \frac{T}{2\pi}\sqrt{g(\rho_*)}$. Here is our main result.

Theorem 3 *Assume that there is a* $\rho_* > 0$ *such that*

$$g(\rho_*) > 0, \qquad g'(\rho_*) \neq 0,$$

and

$$4g(\rho_*) + g'(\rho_*)\rho_* \notin \left\{ \left(\frac{2\pi k}{T}\right)^2 : k \in \mathbb{N} \setminus \{0\} \right\}. \tag{10}$$

Then, for any $\sigma > 0$ *there exist* $\bar{\varepsilon} > 0$ *and* $\bar{\kappa} > 0$ *such that, if* $|\varepsilon| \le \bar{\varepsilon}$ *and* $|\kappa| \le \bar{\kappa}$*, Eq. (6) has a T-radially periodic solution* $x(t)$*, with* $\mathrm{Rot}(x, [0, T]) = \frac{T}{2\pi}\sqrt{g(\rho_*)}$*, satisfying (5), i.e.,*

$$|x(t) - x_*(t)| \le \sigma, \quad \textit{for every } t \in \mathbb{R}.$$

Proof Let us define the functions

$$f(\rho) = g(\rho)\rho, \qquad \hat{p}(t, \rho; \varepsilon) = p(t, \rho; \varepsilon)\rho.$$

Writing Eq. (6) in polar coordinates

$$x(t) = \rho(t)\big(\cos\theta(t), \sin\theta(t)\big), \tag{11}$$

a standard procedure [see, e.g., Arnold (1978)] leads to the system

$$\begin{cases} \ddot{\rho} - \rho\dot{\theta}^2 = -f(\rho) + \hat{p}(t, \rho; \varepsilon) + \kappa\, q(t)\dot{\rho}, \\ 2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = \kappa\, q(t)\rho\dot{\theta}. \end{cases} \tag{12}$$

Defining the scalar angular momentum

$$M(t) = \rho^2(t)\dot{\theta}(t),$$

we see that the second equation in (12) becomes

$$\dot{M} = \kappa\, q(t) M,$$

so that

$$M(t) = \mu e^{\kappa \int_0^t q(s)\,\mathrm{d}s},$$

for some constant $\mu \in \mathbb{R}$. We will consider only the case when $\mu > 0$, since the other case can be obtained symmetrically: This means that our solutions rotate counterclockwise. By assumption (7), the function $M(t)$ is T-periodic. Notice that, for the circular solution $x_*(t) = \big(\rho_* \cos\theta_*(t), \rho_* \sin\theta_*(t)\big)$, we have

$$\theta_*(t) = \theta_*(0) + \frac{2\pi}{\tau_*}\,t,$$

and its angular momentum is constantly equal to

$$\mu_* = \rho_*^2\sqrt{g(\rho_*)}. \tag{13}$$

We begin by proving the existence of a T-periodic solution $\rho(t)$ of the first equation in (12): Hence, we consider the problem

$$\begin{cases} \ddot{\rho} - \dfrac{\mu^2}{\rho^3}\, e^{2\kappa \int_0^t q(s)\, \mathrm{d}s} + f(\rho) = \hat{p}(t, \rho; \varepsilon) + \kappa\, q(t)\dot{\rho}, \\ \rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T). \end{cases} \tag{14}$$

We are looking for solutions $\rho(t)$ in a small neighborhood of ρ_*.

We fix $\sigma > 0$ and, without loss of generality, we can assume that $\sigma < \frac{1}{2}\rho_*$. Let us define the truncated continuous function $\hat{g} : \mathbb{R}^4 \to \mathbb{R}$ as

$$\hat{g}(t, \rho; \mu, \varepsilon) = \begin{cases} -\dfrac{\mu^2}{\rho^3}\, e^{2\kappa \int_0^t q(s)\, \mathrm{d}s} + f\,(\rho) - \hat{p}(t, \rho; \varepsilon), & \text{if } \rho \geq \dfrac{\rho_*}{2}, \\ -\dfrac{8\mu^2}{\rho_*^3}\, e^{2\kappa \int_0^t q(s)\, \mathrm{d}s} + f\left(\dfrac{\rho_*}{2}\right) - \hat{p}\left(t, \dfrac{\rho_*}{2}\,; \varepsilon\right), & \text{if } \rho \leq \dfrac{\rho_*}{2}, \end{cases}$$

and consider the modified problem

$$\begin{cases} \ddot{\rho} + \hat{g}(t, \rho; \mu, \varepsilon) = \kappa\, q(t)\dot{\rho}, \\ \rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T). \end{cases} \tag{15}$$

Let us introduce the following Banach spaces:

$$X = \mathcal{C}([0, T]), \quad Y = \mathcal{C}^1([0, T]), \quad Z = \mathcal{C}^2([0, T]).$$

We interpret problem (15) as

$$L\rho = N_{\mu,\varepsilon,\kappa}\rho, \tag{16}$$

where $L : D(L) \subseteq Y \to X$ is the linear operator defined as

$$D(L) = \{\rho \in Z : \rho(0) = \rho(T),\ \dot{\rho}(0) = \dot{\rho}(T)\},$$
$$L\rho = -\ddot{\rho},$$

and $N_{\mu,\varepsilon,\kappa} : Y \to X$ is the Nemytzkii operator:

$$(N_{\mu,\varepsilon,\kappa}\rho)(t) = \hat{g}(t, \rho(t); \mu, \varepsilon) - \kappa\, q(t)\dot{\rho}(t).$$

Let $P : X \to X$ be the projection associating to any function $\rho(t)$ the constant valued function

$$P\rho = \frac{1}{T}\int_0^T \rho(t)\, \mathrm{d}t,$$

and set $X_0 = \ker P$. Identifying the subspace of constant functions with $\mathbb{R}$, we have that $X = \mathbb{R} \oplus X_0$, and we will write each function $\rho \in X$ as

$$\rho(t) = \lambda + r(t),$$

with $\lambda = P\rho \in \mathbb{R}$ and $r = (I - P)\rho \in X_0$.

Let $Y_0 = Y \cap X_0$. For every $h \in X_0$, the equation $L\rho = h$ has a unique solution ρ in $D(L) \cap Y_0$, which we denote by Kh. Since, by the Ascoli–Arzelà Theorem, $D(L) \cap Y_0$ is compactly imbedded in Y_0, the linear operator $K : X_0 \to Y_0$ thus defined is compact.

Following a standard Lyapunov–Schmidt method, we can now project Eq. (16) by the use of P and $I - P$, respectively, so to obtain an equivalent system which, after inversion in the first equation, can be written as

$$\begin{cases} r = K(I-P)N_{\mu,\varepsilon,\kappa}(\lambda+r), \\ 0 = PN_{\mu,\varepsilon,\kappa}(\lambda+r). \end{cases}$$

We are going to apply the Implicit Function Theorem to the function $\mathcal{F} : (Y_0 \times \mathbb{R}) \times \mathbb{R}^3 \to Y_0 \times \mathbb{R}$, defined as

$$\mathcal{F}\big((r,\mu),(\lambda,\varepsilon,\kappa)\big) = \big(r - K(I-P)N_{\mu,\varepsilon,\kappa}(\lambda+r),\, PN_{\mu,\varepsilon,\kappa}(\lambda+r)\big).$$

Recalling (13), we have that $\big((0,\mu_*),(\rho_*,0,0)\big) \in (Y_0 \times \mathbb{R}) \times \mathbb{R}^3$ corresponds to a circular solution of the unperturbed problem (8) with radius ρ^* and angular momentum μ_*; hence,

$$\mathcal{F}\big((0,\mu_*),(\rho_*,0,0)\big) = (0,0).$$

We notice that $\mathcal{F}$ is a continuously differentiable function on the open set

$$\Omega = \big\{\big((r,\mu),(\lambda,\varepsilon,\kappa)\big) : \mu > 0,\ \lambda + r(t) > \tfrac{1}{2}\rho_*,\ \text{for every } t \in [0,T]\big\},$$

containing $\big((0,\mu_*),(\rho_*,0,0)\big)$, and we compute the partial differential of $\mathcal{F}$ with respect to (r,μ). If $(r,\mu) \in \Omega$ and $(h,s) \in Y_0 \times \mathbb{R}$, writing $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2)$, we have

$$\begin{aligned} \frac{\partial \mathcal{F}_1}{\partial(r,\mu)}((r,\mu),(\lambda,\varepsilon,\kappa))(h,s) = h - K(I-P)\Bigg[\left(\frac{3\mu^2 h}{(\lambda+r)^4} - \frac{2\mu s}{(\lambda+r)^3}\right) e^{2\kappa\int_0^{\cdot} q(s)\,\mathrm{d}s} \\ + f'(\lambda+r)h - \frac{\partial \hat{p}}{\partial \rho}(\cdot,\lambda+r;\varepsilon)h + \kappa q \dot{h}\Bigg], \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \mathcal{F}_2}{\partial(r,\mu)}((r,\mu),(\lambda,\varepsilon,\kappa))(h,s) = P\Bigg[\left(\frac{3\mu^2 h}{(\lambda+r)^4} - \frac{2\mu s}{(\lambda+r)^3}\right) e^{2\kappa\int_0^{\cdot} q(s)\,\mathrm{d}s} \\ + f'(\lambda+r)h - \frac{\partial \hat{p}}{\partial \rho}(\cdot,\lambda+r;\varepsilon)h + \kappa q \dot{h}\Bigg]. \end{aligned}$$

Let us prove that

$$\mathcal{A} := \frac{\partial \mathcal{F}}{\partial(r,\mu)}\big((0,\mu_*),(\rho_*,0,0)\big) \in \mathcal{L}(Y_0 \times \mathbb{R}) \quad \text{is an isomorphism.}$$

Indeed, for $(h,s) \in Y_0 \times \mathbb{R}$, we have

$$\mathcal{A}(h,s) = \left(h - \sigma_* K h,\, -\frac{2\mu_*}{\rho_*^3}s\right),$$

where

$$\sigma_* = \frac{3\mu_*^2}{\rho_*^4} + f'(\rho_*) = 4g(\rho_*) + g'(\rho_*)\rho_*.$$

Hence, $I - \mathcal{A}$ is a compact operator and, using assumption (10), we see that 0 is not an eigenvalue of $\mathcal{A}$. Therefore, $\mathcal{A}$ is invertible, with bounded inverse, and the claim is proved.

By the Implicit Function Theorem we can find an open neighborhood U of $(\rho_*,0,0)$ in $\mathbb{R}^3$, an open neighborhood V of $(0,\mu_*)$ in $Y_0 \times \mathbb{R}$, and a C^1-function $\eta : U \to V$ such that $V \times U \subseteq \Omega$ and, if $(\lambda,\varepsilon,\kappa) \in U$, $(r,\mu) \in V$, then

$$\begin{aligned} \mathcal{F}\big((r,\mu),(\lambda,\varepsilon,\kappa)\big) = (0,0) \quad &\Leftrightarrow \quad (r,\mu) = \eta(\lambda,\varepsilon,\kappa) \\ &\Leftrightarrow \quad \begin{cases} r = \eta_1(\lambda,\varepsilon,\kappa), \\ \mu = \eta_2(\lambda,\varepsilon,\kappa). \end{cases} \end{aligned}$$

The function $\rho(t) = \lambda + r(t)$ can now be extended by T-periodicity on the whole real line $\mathbb{R}$.

Now we are going to deal with the second equation in (12). We are looking for solutions with rotation number equal to T/τ_*, i.e., we want $\theta(t)$ to satisfy

$$\theta(t+T) - \theta(t) = \frac{2\pi T}{\tau_*} = T\sqrt{g(\rho_*)}. \tag{17}$$

Notice that $\theta(0)$ can be chosen arbitrarily; hence, we can take $\theta(0) = \theta_*(0)$. It is then easily seen that (17) is equivalent to

$$\begin{aligned} T\sqrt{g(\rho_*)} &= \int_t^{t+T} \dot{\theta}(s)\,\mathrm{d}s \\ &= \int_t^{t+T} \frac{M(s)}{\rho^2(s)}\,\mathrm{d}s \\ &= \int_0^T \frac{M(t)}{\rho^2(t)}\,\mathrm{d}t \\ &= \int_0^T \frac{\mu\, e^{\kappa \int_0^t q(s)\,\mathrm{d}s}}{\rho^2(t)}\,\mathrm{d}t \\ &= \int_0^T \frac{\eta_2(\lambda,\varepsilon,\kappa)\, e^{\kappa \int_0^t q(s)\,\mathrm{d}s}}{(\lambda + \eta_1(\lambda,\varepsilon,\kappa)(t))^2}\,\mathrm{d}t. \end{aligned}$$

We would like to apply the Implicit Function Theorem to the function $\mathcal{G} : U \to \mathbb{R}$ defined as

$$\mathcal{G}(\lambda,\varepsilon,\kappa) = \int_0^T \frac{\eta_2(\lambda,\varepsilon,\kappa)\, e^{\kappa \int_0^t q(s)\,\mathrm{d}s}}{(\lambda + \eta_1(\lambda,\varepsilon,\kappa)(t))^2}\,\mathrm{d}t - T\sqrt{g(\rho_*)}.$$

We notice that $\mathcal{G}$ is a continuously differentiable function. Let us determine $\mathcal{G}(\lambda, 0, 0)$. We know that, if $(\lambda, 0, 0) \in U$, the only solution $(r, \mu) \in V$ of $\mathcal{F}\big((r,\mu), (\lambda, 0, 0)\big) = (0, 0)$ is given by $(r, \mu) = \eta(\lambda, 0, 0)$, and that $\rho(t) = \lambda + r(t)$ solves (15) with $\varepsilon = 0$ and $\kappa = 0$: Hence, being $\rho(t) > \frac{1}{2}\rho_*$, it solves

$$\begin{cases} \ddot{r} - \dfrac{\mu^2}{(\lambda + r)^3} + f(\lambda + r) = 0, \\ r(0) = r(T), \quad \dot{r}(0) = \dot{r}(T). \end{cases}$$

But we see that a particular solution (r, μ) of such a problem is

$$r \equiv 0, \quad \mu = \sqrt{\lambda^3 f(\lambda)},$$

and it belongs to V if λ is near ρ_*. Therefore, by uniqueness, if λ is near ρ_*, it has to be

$$\eta_1(\lambda, 0, 0) \equiv 0, \quad \eta_2(\lambda, 0, 0) = \sqrt{\lambda^3 f(\lambda)},$$

and hence,

$$\mathcal{G}(\lambda, 0, 0) = \int_0^T \frac{\sqrt{\lambda^3 f(\lambda)}}{\lambda^2}\,\mathrm{d}t - T\sqrt{g(\rho_*)} = T\Big(\sqrt{g(\lambda)} - \sqrt{g(\rho_*)}\Big).$$

Then, since $g'(\rho_*) \neq 0$,

$$\mathcal{G}(\rho_*, 0, 0) = 0, \quad \text{and} \quad \frac{\partial \mathcal{G}}{\partial \lambda}(\rho_*, 0, 0) \neq 0.$$

By the Implicit Function Theorem there exist $\bar\varepsilon > 0$, $\bar\kappa > 0$, $\bar\delta > 0$ and a C^1-function $\nu :]-\bar\varepsilon, \bar\varepsilon[\times]-\bar\kappa, \bar\kappa[\to]\rho_* - \bar\delta, \rho_* + \bar\delta[$ such that, if $\varepsilon \in]-\bar\varepsilon, \bar\varepsilon[$, $\kappa \in]-\bar\kappa, \bar\kappa[$ and $\lambda \in]\rho_* - \bar\delta, \rho_* + \bar\delta[$,

$$\mathcal{G}(\lambda, \varepsilon, \kappa) = 0 \quad \Leftrightarrow \quad \lambda = \nu(\varepsilon, \kappa).$$

Therefore, if $|\varepsilon| \le \bar\varepsilon$ and $|\kappa| \le \bar\kappa$, we have found a solution $x(t)$ of (6) of the type (11), with $\rho = \rho_{\varepsilon,\kappa} = \nu(\varepsilon, \kappa) + \eta_1(\nu(\varepsilon, \kappa), \varepsilon, \kappa)$ and $\theta = \theta_{\varepsilon,\kappa}$ satisfying (17). Since the dependence on ε, κ is continuous, and when $\varepsilon = \kappa = 0$ we have the circular solution $x_*(t)$, we can take some smaller $\bar\varepsilon$ and $\bar\kappa$, if necessary, so that $\|x - x_*\|_\infty \le \sigma$, i.e., also (5) is verified. The proof is thus completed. □

We have the following immediate consequence of Theorem 3.

Corollary 4 *Under the assumptions of Theorem 3, if moreover*

$$\frac{T}{2\pi}\sqrt{g(\rho_*)} = \frac{n}{m} \in \mathbb{Q},$$

then the solution $x(t)$ of Eq. (6) provided by Theorem 3 is mT-periodic, and $\mathrm{Rot}(x, [0, mT]) = n$.

We thus also find a myriad of periodic solutions near the circular solution $x_*(t)$ of the unperturbed equation.

Corollary 5 *Under the assumptions of Theorem 3, for any positive integer N and any $\sigma > 0$, there exist $\bar\varepsilon > 0$ and $\bar\kappa > 0$ such that, if $|\varepsilon| \le \bar\varepsilon$ and $|\kappa| \le \bar\kappa$, Eq. (6) has at least N periodic solutions $x(t)$, which are T-radially periodic, and $\|x - x_*\|_\infty \le \sigma$.*

Proof Since $g'(\rho_*) \neq 0$, it is possible to find infinitely many values of ρ in $]\rho_* - \frac{1}{2}\sigma, \rho_* + \frac{1}{2}\sigma[$ for which the assumptions of Theorem 3 are satisfied, with $(T/2\pi)\sqrt{g(\rho)} \in \mathbb{Q}$. Take N of these, $\rho_1, \ldots, \rho_N$, with the corresponding rational numbers $n_1/m_1, \ldots, n_N/m_N$, respectively, and assume these numbers to be pairwise different. Let $x_*^1(t), \ldots, x_*^N(t)$ denote some circular solutions of (8) having those radii. By Corollary 4, we will find $2N$ positive numbers $\bar\varepsilon_1, \ldots, \bar\varepsilon_N$ and $\bar\kappa_1, \ldots, \bar\kappa_N$ such that, if $|\varepsilon| \le \bar\varepsilon_j$ and $|\kappa| \le \bar\kappa_j$, Eq. (1) has a m_jT-periodic solution $x_j(t)$ that makes exactly n_j revolutions around the origin, in the period time m_jT, and $\|x_j - x_*^j\|_\infty \le \frac{1}{2}\sigma$, for every $j = 1, \ldots, N$. Taking $\bar\varepsilon = \min\{\bar\varepsilon_1, \ldots, \bar\varepsilon_N\}$ and $\bar\kappa = \min\{\bar\kappa_1, \ldots, \bar\kappa_N\}$, we easily see that, if $|\varepsilon| \le \bar\varepsilon$ and $|\kappa| \le \bar\kappa$, the solutions $x_1(t), \ldots, x_N(t)$ are all different from one another, thus proving the result. □

We now look for a condition guaranteeing the assumptions of Theorem 3 to be satisfied by infinitely many values of ρ in a neighborhood of ρ_*.

Corollary 6 *Assume that there is a $\rho_* > 0$ such that $g(\rho_*) > 0$ and $g'(\rho_*) \neq 0$. If g is twice differentiable and*

$$5g'(\rho_*) + g''(\rho_*)\rho_* \neq 0, \tag{18}$$

then the same conclusion of Corollary 5 holds.

Proof Let us introduce the function

$$h(\rho) = 4g(\rho) + g'(\rho)\rho,$$

so that condition (10) reads as

$$h(\rho_*) \notin \left\{ \left(\frac{2\pi k}{T}\right)^2 : k \in \mathbb{N} \setminus \{0\} \right\}.$$

We see that (18) implies

$$h'(\rho_*) = 5g'(\rho_*) + g''(\rho_*)\rho_* \neq 0.$$

Hence, arbitrarily near ρ_* there will be infinitely many values of ρ for which the assumptions of Theorem 3 hold true. □

Remark 7 All the results in this paper hold under some more general Carathéodory regularity assumptions. Indeed, we could assume that $q \in L^1(0, T)$, extended by T-periodicity, and

- $p(\cdot, \rho; \varepsilon)$ is measurable, for every $(\rho, \varepsilon) \in]0, +\infty[\, \times \mathbb{R}$,
- $p(t, \cdot, \cdot)$ is continuously differentiable, for almost every $t \in [0, T]$,
- for every $a, b \in \mathbb{R}$ such that $0 < a < b$ and every $c > 0$ there is a function $\ell \in L^1(0, T)$ such that, for almost every $t \in [0, T]$,

$$\big[\, a \leq \rho \leq b \;\text{ and }\; |\varepsilon| \leq c \,\big] \;\Rightarrow\; |p(t, \rho; \varepsilon)| + \left| \frac{\partial p}{\partial \rho}(t, \rho; \varepsilon) \right| + \left| \frac{\partial p}{\partial \varepsilon}(t, \rho; \varepsilon) \right| \leq \ell(t).$$

The only modification in the proof of Theorem 3 will be in the choice of the function spaces. A possibility could be, e.g., taking

$$X = L^1(0, T), \quad Y = W^{1,1}(0, T), \quad Z = W^{2,1}(0, T).$$

In this case, the solutions will not be twice continuously differentiable, but only once, with second derivative in $L^1(0, T)$, so that the differential equation will be satisfied for almost every $t \in \mathbb{R}$.

3 Some applications

As a particular case of (6), we consider the equation

$$\ddot{x} + \frac{c}{|x|^{\alpha+1}}\, x = p(t, |x|; \varepsilon)\, x + \kappa\, q(t)\, \dot{x}, \tag{19}$$

where c is a positive constant, and $\alpha \in \mathbb{R}$. The function $p : \mathbb{R} \times\,]0, +\infty[\, \times \mathbb{R} \to \mathbb{R}$ is as above, and we still assume (7), i.e., that $\int_0^T q(t)\, \mathrm{d}t = 0$. Moreover, we assume $\alpha \neq -1$. (The case $\alpha = -1$ may lead to some linear resonance phenomena and will not be stressed here.)

We fix a circular orbit $x_*(t)$ of the unperturbed system

$$\ddot{x} = -\frac{c}{|x|^{\alpha+1}}\, x, \tag{20}$$

with radius $\rho_* > 0$. By (9), the corresponding minimal period is then

$$\tau_* = 2\pi \sqrt{\frac{\rho_*^{\alpha+1}}{c}}. \tag{21}$$

Corollary 8 *Assume* $\alpha \neq -1$ *and*

$$(3 - \alpha) \left(\frac{T}{\tau_*} \right)^2 \notin \left\{ k^2 : k \in \mathbb{N} \setminus \{0\} \right\}. \tag{22}$$

Then, the same conclusion of Theorem 3 holds for Eq. (19). If, moreover,

$$\frac{T}{\tau_*} = \frac{n}{m} \in \mathbb{Q},$$

then, for any $\sigma > 0$ there exist $\bar{\varepsilon} > 0$ and $\bar{\kappa} > 0$ such that, if $|\varepsilon| \le \bar{\varepsilon}$ and $|\kappa| \le \bar{\kappa}$, Eq. (19) has a mT-periodic solution $x(t)$, which is T-radially periodic, $\mathrm{Rot}(x, [0, mT]) = n$*, and* $\|x - x_*\|_\infty \le \sigma$*.*

Proof Writing $g(\rho) = c\rho^{-\alpha-1}$, we see that $g(\rho_*) > 0$ and $g'(\rho_*) \neq 0$, while (22) is just a restatement of (10), so that the assumptions of Theorem 3 are verified. The second part of the statement is a direct consequence of Corollary 4. □

Notice that Theorem 1 follows immediately from Corollary 8, since in that case $\alpha = 2$; hence, condition (22) simply becomes $T/\tau_* \notin \mathbb{N}$.

On the other hand, as a consequence of Corollaries 5 and 6, we have the following rather surprising result.

Corollary 9 *Assume $\alpha \neq -1$. Then, for any circular solution $x_*(t)$ of the unperturbed system (20), for any positive integer N and any $\sigma > 0$, there exist $\bar{\varepsilon} > 0$ and $\bar{\kappa} > 0$ such that, if $|\varepsilon| \le \bar{\varepsilon}$ and $|\kappa| \le \bar{\kappa}$, Eq. (19) has at least N periodic solutions $x(t)$, which are T-radially periodic, and $\|x - x_*\|_\infty \le \sigma$.*

Proof As seen above, writing $g(\rho) = c\rho^{-\alpha-1}$, we see that $g(\rho_*) > 0$ and $g'(\rho_*) \neq 0$. Concerning (18), it holds if and only if $\alpha \neq 3$. However, if $\alpha = 3$, we see that (10) holds. The conclusion then follows from Corollaries 5 and 6. □

Notice that Theorem 2 is a direct consequence of Corollary 9, since $\alpha = 2$ there.

Another interesting equation we can deal with is

$$\ddot{x} + \frac{c}{|x|^3}\,x + \frac{\mathrm{d}}{|x|^4}\,x = p(t, |x|; \varepsilon)\,x + \kappa\,q(t)\,\dot{x}, \tag{23}$$

where both c and d are positive, and (7) holds. This is a perturbation of Levi-Civita equation

$$\ddot{x} = -\left(\frac{c}{|x|^3} + \frac{\mathrm{d}}{|x|^4}\right)x, \tag{24}$$

which was introduced in Levi-Civita (1928) in order to approximate relativistic effects. The periodic problem for the perturbed equation has been faced, e.g., in Ambrosetti and Bessi (1990), by the use of a variational method. Setting $g(\rho) = c\rho^{-3} + d\rho^{-4}$, it can be seen that the assumptions of Corollary 6 are always satisfied, thus proving the following result.

Corollary 10 *For any circular solution $x_*(t)$ of the unperturbed system (24), for any positive integer N and any $\sigma > 0$, there exist $\bar{\varepsilon} > 0$ and $\bar{\kappa} > 0$ such that, if $|\varepsilon| \le \bar{\varepsilon}$ and $|\kappa| \le \bar{\kappa}$, Eq. (23) has at least N periodic solutions $x(t)$, which are T-radially periodic, and $\|x - x_*\|_\infty \le \sigma$.*

As a final example, let us illustrate the situation, depicted in Fig. 1, describing the motion of a particle gravitationally attracted by two masses, which slightly oscillate near two given points.

Consider at first two fixed unitary masses in the three-dimensional space. Without loss of generality, we can assume that the origin lies at the midpoint of the segment joining them. Consider the plane π, orthogonal to the line ℓ joining the two masses, containing the origin. If $2d_0$ denotes the distance between the two masses, we obtain the equation

$$\ddot{x} = -\frac{2}{[\,|x|^2 + d_0^2\,]^{3/2}}\,x. \tag{25}$$

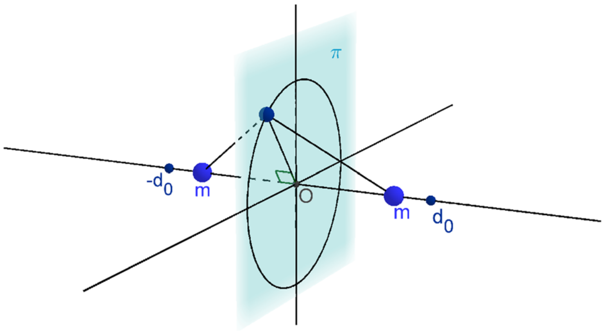

Fig. 1 The two oscillating masses problem

Assume now the two masses not to be fixed, but to display a slight periodic oscillation near the previously fixed points, in such a way that they remain on the line ℓ and they are always in a symmetric position with respect to the origin. If moreover there is a small perturbation of the type $\kappa\, q(t)\dot{x}$, the differential equation becomes

$$\ddot{x} = -\frac{2}{[\,|x|^2 + (d_0 + \varepsilon e(t))^2\,]^{3/2}}\, x + \kappa\, q(t)\dot{x}, \tag{26}$$

where $e(t)$ is a T-periodic function. As before, we assume that $q(t)$ has zero mean, i.e., that (7) holds. Setting $g(\rho) = 2[\,\rho^2 + d_0^2\,]^{-3/2}$, the assumptions of Corollary 6 are again satisfied, so that we have the following result.

Corollary 11 *For any circular solution $x_*(t)$ of the unperturbed system (25), for any positive integer N and any $\sigma > 0$, there exist $\bar{\varepsilon} > 0$ and $\bar{\kappa} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$ and $|\kappa| \leq \bar{\kappa}$, Eq. (26) has at least N periodic solutions $x(t)$, which are T-radially periodic, and $\|x - x_*\|_\infty \leq \sigma$.*

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