



# Periodic solutions of weakly coupled superlinear systems <sup>☆</sup>

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## Abstract

By the use of a higher dimensional version of the Poincaré–Birkhoff theorem, we are able to generalize a result of Jacobowitz [15] and Hartman [14], thus proving the existence of infinitely many periodic solutions for a weakly coupled superlinear system.

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## 1. Introduction

In 1976, using the Poincaré–Birkhoff fixed point theorem, Jacobowitz [15] proved the existence of infinitely many periodic solutions for a superlinear second order scalar equation of the type

$$x'' + f(t, x) = 0. \quad (1)$$

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His result was refined one year later by Hartman [14], assuming the function  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  to be continuous,  $T$ -periodic in its first variable  $t$ , locally Lipschitz continuous in its second variable  $x$  (so to guarantee the uniqueness of solutions to the associated Cauchy problems), and satisfying the following two hypotheses:

- (i)  $\frac{f(t, x)}{x}$  is bounded near  $x = 0$ , uniformly in  $t \in [0, T]$ ;
- (ii)  $\lim_{|x| \rightarrow \infty} \frac{f(t, x)}{x} = +\infty$ , uniformly in  $t \in [0, T]$ .

Our aim here is to provide a generalization of this result for weakly coupled systems.

In order to simplify the statements of our results, let us consider a slightly less general situation, and write equation (1) as

$$-x'' = x h(t, x),$$

assuming  $h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  to be continuous. In such a way, condition (i) above is trivially satisfied, while (ii) becomes

$$(ii') \quad \lim_{|x| \rightarrow \infty} h(t, x) = +\infty, \text{ uniformly in } t \in [0, T].$$

We now consider a system of the type

$$(S) \quad \begin{cases} -x_1'' = x_1 [h_1(t, x_1) + p_1(t, x_1, \dots, x_N)] \\ \vdots \\ -x_N'' = x_N [h_N(t, x_N) + p_N(t, x_1, \dots, x_N)]. \end{cases}$$

Here, all the functions  $h_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous,  $T$ -periodic in their first variable  $t$ , and such that

$$\lim_{|x| \rightarrow \infty} h_i(t, x) = +\infty, \text{ uniformly in } t \in [0, T]. \tag{2}$$

The functions  $p_i : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  are continuous,  $T$ -periodic in their first variable  $t$ , and bounded. Hence, there is a constant  $D > 0$  for which

$$|p_i(t, x_1, \dots, x_N)| \leq D, \tag{3}$$

for every  $(t, x_1, \dots, x_N) \in [0, T] \times \mathbb{R}^N$  and  $i = 1, \dots, N$ .

Moreover, we require (S) to be a Hamiltonian system: hence, we assume the existence of a function  $\mathcal{U} : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that

$$\frac{\partial \mathcal{U}}{\partial x_i}(t, x_1, \dots, x_N) = x_i p_i(t, x_1, \dots, x_N),$$

for every  $(t, x_1, \dots, x_N) \in [0, T] \times \mathbb{R}^N$  and  $i = 1, \dots, N$ .

Simple examples of such functions can be given. For instance, we could have

$$\mathcal{U}(t, x_1, \dots, x_N) = \alpha(t) \prod_{i=1}^N G_i(x_i),$$

where the  $C^2$ -functions  $G_i : \mathbb{R} \rightarrow \mathbb{R}$  are bounded with their derivatives, satisfying  $G_i'(0) = 0$ . Or else, we could choose  $\mathcal{U}(t, x_1, \dots, x_N) = \tilde{\mathcal{U}}(t, r)$ , where  $r = x_1^2 + \dots + x_N^2$ , with a bounded derivative  $\partial \tilde{\mathcal{U}} / \partial r$ .

We will prove the following result.

**Theorem 1.** *There exists a positive integer  $K$  such that, for any choice of  $N$  integers  $K_1, \dots, K_N \geq K$ , there are  $N + 1$  solutions of system (S) which are  $T$ -periodic and such that, for every index  $i$ , the component  $x_i$  has exactly  $2K_i$  simple zeros in the interval  $[0, T[$ .*

In the above theorem, we only consider  $T$ -periodic solutions with all nontrivial components. Admitting that the solutions to system (S) can have trivial components, we can state the following.

**Corollary 2.** *There exists an integer  $K > 0$  such that, for every choice of  $N$  integers  $K_1, \dots, K_N \geq K$ , there exist  $2^{N-1}(N + 2)$  distinct  $T$ -periodic solutions to system (S) such that, for every index  $i$ , the component  $x_i$  is either identically zero, or has exactly  $2K_i$  zeros in the period  $[0, T[$ .*

Indeed, for every fixed  $k = 1, \dots, N$  one has  $\binom{N}{k}$  possibilities of having exactly  $k$  nontrivial components, and for each of these [Theorem 1](#) provides  $k + 1$  solutions. The result then follows from the formula

$$\sum_{k=0}^N \binom{N}{k} (k + 1) = 2^{N-1} (N + 2).$$

Notice that, in the case  $N = 1$ , [Corollary 2](#) gives us the existence of three  $T$ -periodic solutions for equation (1): the equilibrium  $x \equiv 0$  and the two nontrivial solutions having exactly  $2K_1$  zeros in the period interval  $[0, T[$ , given by the Poincaré–Birkhoff theorem.

As a direct corollary of [Theorem 1](#), we also find  $mT$ -periodic solutions, where  $m$  is any positive integer. Notice that, if at least one of the integers  $K_1, \dots, K_N$  is relatively prime with  $m$ , these solutions will not be  $\ell T$ -periodic, for any  $\ell = 1, \dots, m - 1$ . Such solutions are sometimes named *subharmonic solutions*. However, their minimal period could be small, as it happens for an autonomous scalar equation like  $x'' + x^3 = 0$ . On the other hand, for some particularly structured equations, the period of the solutions cannot be less than  $T$ : take, for instance, an equation like  $x'' + q(t)x^3 = 0$ , with  $q(t)$  having minimal period  $T$ . We thus see that [Theorem 1](#) can sometimes be used to provide subharmonic solutions having *minimal period*  $mT$ .

The proof of [Theorem 1](#) will be carried out using a higher dimensional version of the Poincaré–Birkhoff theorem for Poincaré maps of Hamiltonian systems, recently obtained by the first author and A.J. Ureña in [12]. So, as Jacobowitz and Hartman were able to prove their results for a *scalar* differential equation by the use of a suitably adapted version of the classical planar Poincaré–Birkhoff theorem, similarly we may deal here with a *system* of differential equations, since we are now equipped with a higher dimensional version of the same theorem.

Let us recall that, in our assumptions, the solutions to Cauchy problems could be neither unique, nor globally defined. For instance, it was shown in [7] that there are positive continuous functions  $q(t)$  such that the differential equation  $x'' + q(t)x^3 = 0$  has a solution which does not exist on  $[0, T]$ . So, the Poincaré map may not be well defined. To overcome this difficulty, besides assuming the uniqueness for the initial value problems, Jacobowitz [15] considered the *successor map*, instead of the Poincaré map. However, in order to have this map well defined, he needed to assume an additional sign condition on the nonlinearity. One year later, Hartman [14] was able to avoid this additional condition. His approach consists in modifying the nonlinearities, and making use of some *a priori estimates* for the solutions with a *prescribed number of rotations* in the phase plane (see also [18]). We will follow the same idea, without the need of assuming the uniqueness for the initial value problems, making use of [12, Theorem 8.2], where such an assumption is not necessary.

The global existence problem does not arise when the differential equation has some particular structure, e.g. when  $f(t, x)$  in (1) is of the type  $f(x) + e(t)$ . In this case, the existence of periodic solutions has been proved for superlinear scalar equations in [8,13,17] (see also [4], where a planar system is considered). However, even in such a situation, very few results are available for higher dimensional superlinear systems. In [6] some symmetry has been assumed so to reduce to a simpler two-point boundary value problem, while in [5,12] some higher dimensional version of the Poincaré–Birkhoff theorem has been used.

Existence results for some type of superlinear systems have also been obtained by the use of variational methods, see e.g. [1–3,9,16,19,20]. Usually, the main difficulty encountered in these papers is the verification of the Palais–Smale condition, for which some peculiar hypotheses must be assumed. Our method does not need such type of technical assumptions, but we are able to apply it only to systems which are weakly coupled. On the other hand, we get precise information about the nodal properties of the solutions. As a consequence of this fact, we have a clear picture of how multiplicity of  $T$ -periodic solutions arises, and we can identify the subharmonic solutions, as well.

## 2. Proof of Theorem 1

In this section, we provide a proof for Theorem 1, mainly based on the higher dimensional version of the Poincaré–Birkhoff theorem given in [12, Theorem 8.2].

In order to recover the existence of global solutions to the associated Cauchy problems, let us introduce the following truncations of the functions  $h_i$ . For every  $R > 1$ , let

$$h_i^R(t, x_i) = \begin{cases} h_i(t, -R), & \text{if } x_i < -R, \\ h_i(t, x_i), & \text{if } |x_i| \leq R, \\ h_i(t, R), & \text{if } x_i > R. \end{cases}$$

To simplify the notation, we will write the vector in  $\mathbb{R}^{N-1}$ , obtained removing from some  $x = (x_1, \dots, x_N)$  its  $i$ -th component, as follows:

$$\lambda^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N).$$

We define the functions  $g_i^R : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \rightarrow \mathbb{R}$  by

$$g_i^R(t, x_i, \lambda^i) = x_i \left[ h_i^R(t, x_i) + p_i(t, x_1, \dots, x_N) \right].$$

These functions are continuous, and satisfy, for every  $i = 1, \dots, N$  and  $R > 1$ ,

$$g_i^R(t, 0, \lambda^i) = 0, \text{ for every } t \in [0, T] \text{ and } \lambda^i \in \mathbb{R}^{N-1}. \tag{4}$$

Moreover, it is readily verified that, by continuity, there is a constant  $a > 0$  such that

$$\left| \frac{g_i^R(t, x_i, \lambda^i)}{x_i} \right| \leq a, \text{ when } 0 < |x_i| < 1, \tag{5}$$

for every  $t \in [0, T]$  and  $\lambda^i \in \mathbb{R}^{N-1}$ .

Let us now consider the modified system

$$(S_R) \quad \begin{cases} -x_1'' = g_1^R(t, x_1, \lambda^1) \\ \vdots \\ -x_N'' = g_N^R(t, x_N, \lambda^N). \end{cases}$$

Notice that, by construction, the solutions of the Cauchy problems associated to  $(S_R)$  are globally defined.

Let us write the  $2N$ -dimensional system

$$(P_R) \quad \begin{cases} x_i' = y_i \\ -y_i' = g_i^R(t, x_i, \lambda^i) \end{cases} \quad i = 1, \dots, N,$$

which is equivalent to  $(S_R)$ . In the following, we will denote by  $y$  the vector  $(y_1, \dots, y_N)$ , and we will write  $z = (x, y) \in \mathbb{R}^{2N}$ .

It will be useful to parametrize the component  $z_i = (x_i, y_i)$  of the solutions to  $(P_R)$ , whenever it does not attain the origin, in polar coordinates as

$$x_i(t) = \rho_i(t) \cos(\vartheta_i(t)), \quad y_i(t) = \rho_i(t) \sin(\vartheta_i(t)). \tag{6}$$

Hence, if  $z_i(t) \neq (0, 0)$  for every  $t$  in an interval  $[\tau_0, \tau_1]$ , we can define the *rotation number* of the  $i$ -th component of  $z = (x, y)$  along that interval as

$$\text{rot}^i(z, [\tau_0, \tau_1]) = -\frac{1}{2\pi}(\vartheta_i(\tau_1) - \vartheta_i(\tau_0)).$$

In the case when the interval  $[\tau_0, \tau_1]$  coincides with  $[0, T]$ , the rotation number will be simply denoted by  $\text{rot}^i(z)$ . In particular, if such solution is  $T$ -periodic, then  $\text{rot}^i(z)$  is an integer, and  $x_i$  has exactly  $2 \text{rot}^i(z)$  simple zeros in  $[0, T[$ . Indeed,  $\text{rot}^i(z)$  is the number of clockwise rotations performed by  $z_i = (x_i, y_i)$  around the origin, in the time interval  $[0, T]$ .

By (5), it is possible to prove that solutions having a *small* nontrivial component cannot vanish in the interval  $[0, T]$ . The following lemma provides also an estimate of the rotation number for such solutions.

**Lemma 3.** *There exist a positive integer  $K$  and a positive constant  $\delta$  with the following property: if a solution  $z = (x, y)$  to system  $(P_R)$  satisfies  $0 < |z_i(t_0)| \leq \delta$ , for a certain index  $i$ , at a certain time  $t_0$ , then*

$$0 < |z_i(t)| < 1, \quad \text{for every } t \in [t_0, t_0 + T],$$

and

$$\text{rot}^i(z, [t_0, t_0 + T]) < K.$$

**Proof.** Consider a solution  $z$  to system  $(P_R)$  such that  $0 < |z_i(t)| < 1$  in a certain interval  $\mathcal{I}$ , and parametrize  $z_i = (x_i, y_i)$  in polar coordinates (6). For every  $t \in \mathcal{I}$ , one has, by (5),

$$|\rho'_i(t)| = \frac{|x'_i(t)x_i(t) + y'_i(t)y_i(t)|}{\sqrt{x_i(t)^2 + y_i(t)^2}} \leq (a + 1)\rho_i(t).$$

Choose  $\delta \in ]0, e^{-(a+1)T}[$ , and assume that  $0 < |z_i(t_0)| \leq \delta$ , for some  $t_0$ . We can now take  $\mathcal{I}$  as the maximal interval of time containing  $t_0$  in which  $0 < |z_i(t)| < 1$ . By Gronwall’s Lemma, we thus have

$$|z_i(t_0)|e^{-(a+1)|t-t_0|} \leq |z_i(t)| \leq |z_i(t_0)|e^{(a+1)|t-t_0|}, \quad \text{for every } t \in \mathcal{I}.$$

Hence,  $[t_0, t_0 + T] \subseteq \mathcal{I}$  and

$$0 < |z_i(t)| < 1, \quad \text{for every } t \in [t_0, t_0 + T].$$

So,  $\text{rot}^i(z, [t_0, t_0 + T])$  is well defined, and the following estimate holds for the angular velocity of  $z_i = (x_i, y_i)$ :

$$\begin{aligned} -\vartheta'_i(t) &= \frac{g_i^R(t, x_i(t), \lambda^i(t))x_i(t) + y_i(t)^2}{x_i(t)^2 + y_i(t)^2} \\ &< a \cos^2(\vartheta_i(t)) + \sin^2(\vartheta_i(t)). \end{aligned}$$

Thus, the time needed by  $z_i$  to perform a complete clockwise rotation around the origin must be greater than  $2\pi/\sqrt{a}$ . Hence, choosing the integer  $K$  satisfying

$$a < \left(\frac{2\pi}{T}K\right)^2,$$

one has that  $\text{rot}^i(z, [t_0, t_0 + T]) < K$ , thus completing the proof.  $\square$

2.1. The a priori bound

The following lemma gives us an a priori bound to the  $T$ -periodic solutions of  $(P_R)$  whose nontrivial components perform at most a fixed number of rotations around the origin. Notice that, by Lemma 3, if  $z = (x, y)$  is a solution to  $(P_R)$ , with a nontrivial component  $z_i = (x_i, y_i)$ , then  $z_i(t) \neq (0, 0)$ , for every  $t$ .

**Lemma 4** (A priori bound). *For every positive integer  $K_0$ , there exists a constant  $\mathcal{R} = \mathcal{R}(K_0) > 1$  with the following property: if  $z = (x, y)$  is a  $T$ -periodic solution to  $(P_R)$ , with  $R > \mathcal{R}$ , such that  $\text{rot}^i(z) \leq K_0$  for every nontrivial component  $z_i$ , then*

$$|z_i(t)| < \mathcal{R}, \text{ for every } t \in [0, T] \text{ and } i = 1, \dots, N.$$

**Proof.** Given  $K_0$ , fix a constant  $b > 1$  such that

$$b > 2 \left( \frac{2\pi}{T} K_0 \right)^2.$$

By (2) and (3), there exist two positive constants  $c$  and  $\bar{R}$ , with  $1 < c < \bar{R}$ , such that, for every  $R \geq \bar{R}$  and every index  $i$ ,

$$\begin{aligned} \frac{g_i^R(t, x_i, \lambda^i)}{x_i} &= h_i^R(t, x_i) + p_i(t, x_i, \lambda^i) > h_i^R(t, x_i) - D > b, \\ &\text{for every } x_i \text{ with } |x_i| \geq c, t \in [0, T], \text{ and } \lambda^i \in \mathbb{R}^{N-1}. \end{aligned} \tag{7}$$

Moreover, by continuity, it is possible to find a constant  $M > 0$  such that, for every index  $i$ ,

$$\begin{aligned} |g_i^R(t, x_i, \lambda^i)| &< M, \\ &\text{for every } x_i \text{ with } |x_i| \leq c, t \in [0, T], \text{ and } \lambda^i \in \mathbb{R}^{N-1}. \end{aligned} \tag{8}$$

Then, choose  $d > 1$  large enough to have  $d^2 > \max\{4Mc, 4c^2b\}$ , and define

$$\mathcal{A}_0 = [-c, c] \times [-d, d].$$

Consider now a  $T$ -periodic solution  $(x, y)$  to  $(P_R)$ , with  $R > \bar{R}$ , such that, for some index  $i_0$ ,

$$(x_{i_0}(t), y_{i_0}(t)) \notin \mathcal{A}_0, \text{ for every } t \in [0, T]. \tag{9}$$

We will now show that, in such a case,  $(x_{i_0}, y_{i_0})$  has to perform more than  $K_0$  rotations around the origin in the time interval  $[0, T]$ . Indeed, if  $|x_{i_0}(t)| \leq c$ , then  $|y_{i_0}(t)| \geq d$ , so one has

$$\begin{aligned} -\vartheta'_{i_0}(t) &= \frac{g_{i_0}^R(t, x_{i_0}(t), \lambda^{i_0}(t)) x_{i_0}(t) + y_{i_0}(t)^2}{x_{i_0}(t)^2 + y_{i_0}(t)^2} \\ &\geq \frac{-Mc + y_{i_0}(t)^2}{x_{i_0}(t)^2 + y_{i_0}(t)^2} \end{aligned}$$

$$\begin{aligned}
 &> \left(1 - \frac{Mc}{d^2}\right) \sin^2(\vartheta_{i_0}(t)) \\
 &> \frac{3}{4} \sin^2(\vartheta_{i_0}(t)) \\
 &> b \cos^2(\vartheta_{i_0}(t)) + \frac{1}{2} \sin^2(\vartheta_{i_0}(t)),
 \end{aligned}$$

and, if  $|x_{i_0}(t)| \geq c$ , one has

$$\begin{aligned}
 -\vartheta'_{i_0}(t) &= \frac{g_{i_0}^R(t, x_{i_0}(t), \lambda^{i_0}(t)) x_{i_0}(t) + y_{i_0}(t)^2}{x_{i_0}(t)^2 + y_{i_0}(t)^2} \\
 &> b \cos^2(\vartheta_{i_0}(t)) + \sin^2(\vartheta_{i_0}(t)) \\
 &> b \cos^2(\vartheta_{i_0}(t)) + \frac{1}{2} \sin^2(\vartheta_{i_0}(t)).
 \end{aligned}$$

Consequently, every solution to  $(P_R)$ , with  $R > \bar{R}$ , satisfying (9) is such that the component  $(x_{i_0}, y_{i_0})$  takes a time  $\tau$  to perform a rotation around the origin estimated by

$$\tau < \int_0^{2\pi} \frac{d\vartheta}{b \cos^2 \vartheta + \frac{1}{2} \sin^2 \vartheta} = \frac{2\pi}{\sqrt{b/2}} < \frac{T}{K_0}.$$

Hence, every  $T$ -periodic solution  $z = (x, y)$  to  $(P_R)$ , with  $R > \bar{R}$ , satisfying (9) is such that

$$\text{rot}^{i_0}(z) > K_0.$$

Now, in order to find the needed a priori bound, we will use the so-called *admissible spiral* method, a tool introduced by the authors in [10,11]. We recall that, roughly speaking, an admissible spiral for a system in  $\mathbb{R}^2$  is a curve which guides the solutions in the phase plane, forcing them to rotate around the origin when they increase in norm: the solutions can never cross the spiral curve from the inner to the outer part. The generalization to systems in  $\mathbb{R}^{2N}$  of the type

$$\begin{cases} z'_1 = F_1(t, z_1, \dots, z_N) \\ \vdots \\ z'_N = F_N(t, z_1, \dots, z_N), \end{cases}$$

with  $z_i \in \mathbb{R}^2$ , has been explained in [11], and is depicted in Fig. 1.

To be more precise, for each index  $i$ , an  $i$ -admissible spiral is a continuous and injective curve  $\gamma^i : [0, +\infty[ \rightarrow \mathbb{R}^2$ , satisfying the following properties:

1. There exists an unlimited strictly increasing sequence

$$0 = \sigma_0^i < \sigma_1^i < \sigma_2^i < \dots < \sigma_k^i < \sigma_{k+1}^i < \dots$$



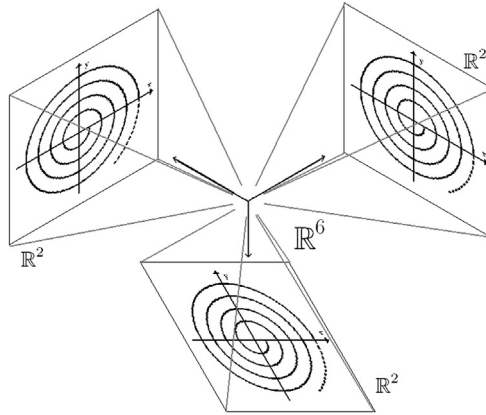


Fig. 1. An artistic view for the admissible spirals when  $N = 3$ .

such that the restriction of  $\gamma^i$  to every closed interval  $[\sigma_k^i, \sigma_{k+1}^i]$  is continuously differentiable, and

$$\langle J\dot{\gamma}^i(s), \gamma^i(s) \rangle > 0, \quad \text{for every } s \in [\sigma_k^i, \sigma_{k+1}^i].$$

Here,  $J$  denotes the standard symplectic  $2 \times 2$  matrix:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- 2. The curve grows in norm to infinity:

$$\lim_{s \rightarrow +\infty} |\gamma^i(s)| = +\infty.$$

- 3. The curve rotates clockwise infinitely many times:

$$\int_0^{+\infty} \frac{\langle J\dot{\gamma}^i(s), \gamma^i(s) \rangle}{|\gamma^i(s)|^2} ds = +\infty.$$

- 4. When restricted to any subinterval  $[\sigma_k^i, \sigma_{k+1}^i]$ , it satisfies

$$\langle J\dot{\gamma}^i(s), F_i(t, z_1, \dots, z_{i-1}, \gamma^i(s), z_{i+1}, \dots, z_N) \rangle < 0,$$

for every  $t \in [0, T], s \in [\sigma_k^i, \sigma_{k+1}^i]$ , and  $z_j \in \mathbb{R}^2$  with  $j \neq i$ .

We need to construct, for every index  $i = 1, \dots, N$ , an  $i$ -admissible spiral  $\gamma^i$ . The construction will be essentially the same for every index  $i$ , and we will briefly follow the ideas explained in [10,11] (see, in particular, [10, Theorem 3.10] and [11, Theorem 4.1]). Let  $\eta \in ]0, 1[$  be a fixed constant. The spirals will be defined piecewise in the following regions (see Fig. 2):

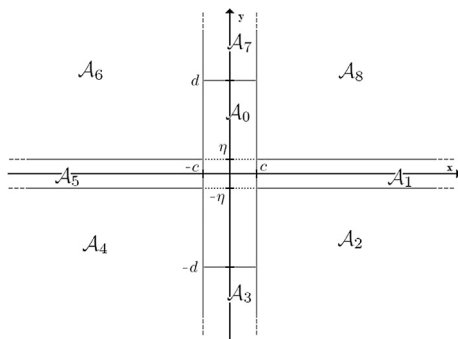


Fig. 2. The considered regions  $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_8$ .

$$\begin{aligned} \mathcal{A}_1 &= [c, +\infty[ \times ]-\eta, \eta], \\ \mathcal{A}_2 &= [c, +\infty[ \times ]-\infty, -\eta], \\ \mathcal{A}_3 &= [-c, c] \times ]-\infty, -d], \\ \mathcal{A}_4 &= ]-\infty, -c] \times ]-\infty, -\eta], \\ \mathcal{A}_5 &= ]-\infty, -c] \times ]-\eta, \eta], \\ \mathcal{A}_6 &= ]-\infty, -c] \times [\eta, +\infty[, \\ \mathcal{A}_7 &= [-c, c] \times [d, +\infty[, \\ \mathcal{A}_8 &= [c, +\infty[ \times [\eta, +\infty[. \end{aligned}$$

The construction in the regions with an even index makes use of some energy estimates. Recalling (7), notice that, for every index  $i$ , it is possible to find two continuous functions  $g_{i,j} : \mathbb{R} \rightarrow \mathbb{R}$ , with  $j = 1, 2$ , such that, for every sufficiently large positive  $R$ ,

$$g_{i,1}(x_i) < g_i^R(t, x_i, \lambda^i) < g_{i,2}(x_i) = -1, \quad \text{when } x_i \leq -c,$$

and

$$1 = g_{i,1}(x_i) < g_i^R(t, x_i, \lambda^i) < g_{i,2}(x_i), \quad \text{when } x_i \geq c.$$

Consider then their primitives

$$G_{i,j}(\xi) = \int_0^\xi g_{i,j}(u) du,$$

and the associated energy functions  $E_{i,j}(\xi, v) = \frac{1}{2}v^2 + G_{i,j}(\xi)$ . It is easy to see that, for every solution  $z = (x, y)$  of  $(P_R)$ , with  $x = (x_1, \dots, x_N)$  and  $y = (y_1, \dots, y_N)$ ,

$$\frac{d}{dt} E_{i,1}(x_i(t), y_i(t)) < 0, \quad \text{if } (x_i(t), y_i(t)) \in \mathcal{A}_4 \cup \mathcal{A}_8,$$

and

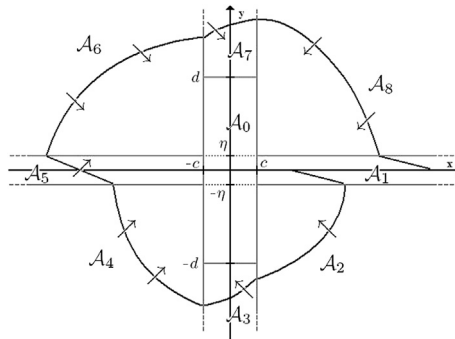


Fig. 3. The construction of the first lap of the admissible spiral.

$$\frac{d}{dt} E_{i,2}(x_i(t), y_i(t)) < 0, \quad \text{if } (x_i(t), y_i(t)) \in \mathcal{A}_2 \cup \mathcal{A}_6.$$

So, the spiral  $\gamma^i$  will be chosen as a level curve of  $E_{i,1}$  in  $\mathcal{A}_4$  and  $\mathcal{A}_8$ , and as a level curve of  $E_{i,2}$  in  $\mathcal{A}_2$  and  $\mathcal{A}_6$ .

The branches of  $\gamma^i$  in  $\mathcal{A}_3$  and  $\mathcal{A}_7$  are built following [11, Lemma 4.2]: by (8), in these regions the superlinearity property of  $g_i$  does not affect the estimate of the radial speed of each component of the solutions, since we have that  $|\rho'_i(t)| \leq c + M$ ; on the other hand, as shown above, the angular speed here is such that  $-\vartheta'_i(t) > 1/2$ . Hence, we have the estimate

$$|\rho'_i(t)| < 2(c + M)(-\vartheta'_i(t)).$$

Therefore, the spiral in  $\mathcal{A}_3$  and  $\mathcal{A}_7$  can be easily defined in polar coordinates  $(\rho, \theta)$  by setting  $d\rho/d\vartheta = -2(c + M)$ .

In the regions  $\mathcal{A}_1$  and  $\mathcal{A}_5$  the curve  $\gamma^i$  can be defined as a straight line, with a sufficiently small negative slope  $-\chi$ . Indeed, taking  $\chi \in ]0, 1/\eta[$ , in  $\mathcal{A}_1$  one has

$$\langle (x'_i(t), y'_i(t)), (-\chi, -1) \rangle = g_i^R(t, x_i(t), y_i(t)) - \chi y_i(t) > 1 - \chi \eta > 0,$$

and a similar computation holds in  $\mathcal{A}_5$ .

We can now choose a starting point  $P = (r_0, 0)$  lying in  $\mathcal{A}_1$  and draw the first lap of the spiral  $\gamma^i$  passing in the eight regions (see Fig. 3). We take  $r_0$  large enough, so that  $\gamma^i$  does not intersect  $\mathcal{A}_0$ . We then continue the construction of  $\gamma^i$  iterating the same procedure, so to obtain an infinite number of laps. Moreover, choosing the slope  $\chi$  of the lines in  $\mathcal{A}_1$  and  $\mathcal{A}_5$  small enough at each lap, we can ensure that the spiral is injective and grows to infinity in norm. In this way, we have finally constructed all the admissible spirals  $\gamma^i$ , with  $i = 1, \dots, N$ .

Let  $\mathcal{R} > \max\{\sqrt{c^2 + d^2}, \bar{R}\}$  be such that the open ball  $B_{\mathcal{R}}$  fully contains all the first  $K_0 + 1$  laps of the spirals  $\gamma^i$ . This radius represents the a priori bound we are looking for. In fact, assume by contradiction that there exists a  $T$ -periodic solution  $z = (x, y)$  to system  $(P_R)$ , with  $R > \mathcal{R}$ , satisfying  $\text{rot}^i(z) \leq K_0$  for every  $i$ , and that there also exist an index  $i_0$  and an instant  $t_1$  for which  $|z_{i_0}(t_1)| \geq \mathcal{R}$ . We know that if such a solution satisfies (9), then  $\text{rot}^{i_0}(z) > K_0$ . Hence, there must exist an instant  $t_2 \in [t_1, t_1 + T]$  at which  $z_{i_0}(t_2) \in \mathcal{A}_0$ . Therefore, the  $i_0$ -th component of  $z$  must go from  $\mathcal{A}_0$  to  $\mathbb{R}^2 \setminus B_{\mathcal{R}}$  in the time-interval  $[t_2, t_1 + T]$ , guided by the spiral  $\gamma^{i_0}$ , thus performing at least  $K_0 + 1$  rotations around the origin in this time interval  $[t_2, t_1 + T]$ . On the other hand, since

$x'_{i_0} = y_{i_0}$ , the  $i_0$ -th component of the solutions can never perform counterclockwise rotations. So, the  $i_0$ -th component has to perform more than  $K_0$  rotations around the origin in the time interval  $[0, T]$ , a contradiction which ends the proof of the lemma.  $\square$

### 2.2. The Poincaré–Birkhoff geometrical setting

To complete the proof of [Theorem 1](#), we will apply a higher dimensional version of the Poincaré–Birkhoff theorem for Poincaré maps of Hamiltonian systems recently obtained in [\[12\]](#). Precisely, let  $0 < R_1 < R_2$  and consider the  $N$ -annulus in  $\mathbb{R}^{2N}$  defined as

$$\Omega = [\bar{B}_{R_2} \setminus B_{R_1}]^N = (\bar{B}_{R_2} \setminus B_{R_1}) \times \cdots \times (\bar{B}_{R_2} \setminus B_{R_1}).$$

As a direct consequence of [\[12, Theorem 8.2\]](#), the following statement holds.

**Theorem 5.** *Assume that every solution  $z$  of  $(P_R)$ , departing from  $z(0) \in \partial\Omega$ , is defined on  $[0, T]$  and satisfies*

$$z_i(t) \neq (0, 0), \text{ for every } t \in [0, T] \text{ and } i = 1, \dots, N.$$

*Assume moreover that there are positive integers  $K_1, \dots, K_N$  such that, for each  $i = 1, \dots, N$*

$$\text{rot}^i(z) < K_i \text{ if } |z_i(0)| = R_1, \text{ and } \text{rot}^i(z) > K_i \text{ if } |z_i(0)| = R_2.$$

*Then, the Hamiltonian system has at least  $N + 1$  distinct  $T$ -periodic solutions  $z$ , with  $z(0) \in \Omega$ , such that  $\text{rot}^i(z) = K_i$ , for every  $i = 1, \dots, N$ .*

We have already proved that, if  $z = (x, y)$  is a solution to  $(P_R)$ , with a nontrivial component  $z_i = (x_i, y_i)$ , then  $z_i(t) \neq (0, 0)$ , for every  $t$ . Moreover, [Lemma 3](#) provides the estimate of the rotation number for the components starting near the origin. We now need to estimate the number of rotations performed by the components which start sufficiently far from the origin.

**Lemma 6.** *Given a positive integer  $K_0$ , let  $\mathcal{R} = \mathcal{R}(K_0)$  be the constant provided by [Lemma 4](#). If  $z = (x, y)$  is any solution to  $(P_R)$ , with  $R > \mathcal{R}$ , such that  $|z_i(0)| = \mathcal{R}$  for some index  $i$ , then  $\text{rot}^i(z) > K_0$ .*

**Proof.** Following the proof of [Lemma 4](#), we only need to consider the case of a solution, starting with  $|z_i(0)| = \mathcal{R}$  for some index  $i$ , such that  $z_i = (x_i, y_i)$  enters the region  $\mathcal{A}_0$  at a certain time  $t_0 \in ]0, T]$ . If we change the time variable  $t$  into  $s = t_0 - t$ , we obtain a solution  $\tilde{z} = (\tilde{x}, \tilde{y})$  to the system

$$\begin{cases} \tilde{x}'_i = \tilde{y}_i \\ -\tilde{y}'_i = g_i^R(t_0 - s, \tilde{x}_1, \dots, \tilde{x}_N) \end{cases} \quad i = 1, \dots, N,$$

such that  $\tilde{z}_i(0) \in \mathcal{A}_0$  and  $|\tilde{z}_i(t_0)| = \mathcal{R}$ . Clearly, the admissible spiral  $\gamma^i$  still guides also the solution  $\tilde{z}$  of this system, forcing the component  $\tilde{z}_i = (\tilde{x}_i, \tilde{y}_i)$  to rotate clockwise at least  $K_0 + 1$  times around the origin in the time interval  $[0, t_0]$ . By the same argument in the proof of [Lemma 4](#), we then have that  $\text{rot}^i(\tilde{z}) > K_0$ . The proof is thus completed, since  $\text{rot}^i(z) = \text{rot}^i(\tilde{z})$ .  $\square$

We are now ready to conclude the proof of [Theorem 1](#). Using [Lemma 3](#), we find the integer  $K$ . Then, once the integers  $K_1, \dots, K_N \geq K$  are given, we can define  $K_0 = \max\{K_1, \dots, K_N\}$  and apply [Lemma 4](#) so to find the constant  $\mathcal{R} = \mathcal{R}(K_0)$ . Choose  $R > \mathcal{R}$  and consider the Poincaré map associated to the Hamiltonian system  $(P_R)$ . By [Lemmas 3 and 6](#), choosing  $R_1 = \delta$  and  $R_2 = \mathcal{R}$ , we can apply [Theorem 5](#) above so to find  $N + 1$  distinct  $T$ -periodic solutions whose  $i$ -th component performs exactly  $K_i$  clockwise rotations around the origin in the time interval  $[0, T]$ . [Lemma 4](#) guarantees us that all these solutions are also solutions to system  $(S)$ , since  $g_i(t, x_i, \lambda^i) = g_i^R(t, x_i, \lambda^i)$  when  $|x_i| < \mathcal{R}$ . The proof of [Theorem 1](#) is thus completed.

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