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A permanence theorem for local dynamical systems

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ABSTRACT

We provide a necessary and sufficient condition for permanence related to a local dynamical system on a suitable topological space. We then present an illustrative application to a Lotka–Volterra predator–prey model with intraspecific competition.

1. Introduction

A fundamental problem in mathematical biology concerns the long-term survival of all species in an ecological context of interacting populations. Although many criteria have been proposed to define the notion of long-term survival, the most suitable from a biological point of view seems to be the one known as *permanence*, cf. [23,27]. As we will make more precise below, it guarantees that the size of each population asymptotically settles above a certain threshold, and all the populations do not grow indefinitely.

This basic question raises also in other contexts, of very different nature, from chemical reaction networks to evolutionary game dynamics. Different names have sometimes been used in these situations (see, e.g. [1,2,7,17,20,21,26,33]).

We now enter a bit more into details. Let $\pi : X \times \mathbb{R} \to X$ be a continuous semi-dynamical system in a suitable locally compact space X. Typically, when dealing with a population model, X is the set $\mathbb{R}^N_+$ (where $\mathbb{R}_+$ denotes the set of nonnegative real numbers), and π is generated by an autonomous ordinary differential equation. Let us denote by S a closed proper subset of X. This is the subset we would like to avoid, when aiming for permanence. In the case described above, S consists of those elements of $\mathbb{R}^N_+$ having at least one coordinate equal to zero.

Definition 1. The system π is said to be *permanent* with respect to S if there exists a compact set $K \subset X \setminus S$ with the following property: for every $x \in X \setminus S$ there exists a $T_x \geq 0$ such that $\pi(x, t) \in K$, for every $t \geq T_x$.

The above definition can be easily adapted to the case of a discrete semi-dynamical system. On the other hand, we will discuss in Section 4 on how to deal with local dynamical systems.

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The notion of permanence is strongly related with the one of *uniform persistence*, that means requiring S to be a uniform repeller, cf. [5,11–13,19,32]. If X is a compact metric space, then permanence is equivalent to uniform persistence with respect to S. Otherwise, in general, permanence is a stronger condition than uniform persistence. In the applications, it is very often *asked as an assumption*, or intrinsically contained in the model, that the number of individuals of the populations remain bounded, motivating this by the fact that our world is indeed bounded (see, e.g. [10,24,25,28,29]). This dissipativity property of the system π permits to recover the desired compactness.

The aim of this paper is to provide a general theorem characterizing permanence, in the same spirit of a theorem provided by the first author in [11] for uniform persistence. A crucial point in [11, Theorem 1] was to assume that S is a compact set. In order to deal with the general situation when S can be non-compact, we will make use here of the Alexandroff compactification of the space X, so to recover the needed compactness.

Let us notice that compactification arguments have already been introduced in this setting, cf. [9,18]. However, it seems that no simple characterization of permanence has been given yet, in the non-compact case.

In recent years, a large number of applications have been provided for uniform persistence theorems in different fields (see [27], and the references therein). In view of the increasing interest in this field, we trust that our main theorem will facilitate further applications to the permanence problem.

The paper is organized as follows. In Section 2 we provide a version of [11, Theorem 1], originally proved for metric spaces, in the case of compact Hausdorff spaces. This will be used in Section 3 to prove our main result, through the above mentioned compactification of the space. In Section 4 we will extend the result to local dynamical systems. In Section 5 we present an illustrative application of our main theorem to a generalized Lotka–Volterra predator–prey model, where an effect of intraspecific competition is introduced. We will show how permanence can be obtained by a small modification of the classical model.

2. Semi-dynamical systems in compact topological spaces

In this section, we consider a continuous semi-dynamical system $\pi: X \times \mathbb{R}_+ \to X$, in the case when X is a compact Hausdorff space.

Theorem 2. *If X is a compact Hausdorff space, $\pi: X \times \mathbb{R}_+ \to X$ is a continuous semi-dynamical system and S is a closed proper subset of X such that $X \setminus S$ is positively invariant, then a sufficient condition for π to be permanent with respect to S is that there exist an open neighborhood U of S and a function $P: X \to \mathbb{R}_+$ such that*

(a_1) *P is continuous;*
(a_2) *$P(x) = 0$ if and only if $x \in S$;*
(a_3) *for every $x \in U \setminus S$ there exists a $t_x > 0$ such that $P(\pi(x, t_x)) > P(x)$.*

Moreover, if X is perfectly normal, the above condition is also necessary.

We recall that a Hausdorff space is *normal* if for any two disjoint closed subsets E and F there is a continuous function $f: X \to [0, 1]$ such that $f^{-1}(0) \supseteq E$ and $f^{-1}(1) \supseteq F$. Normal Hausdorff spaces are also known as T4 spaces.

Furthermore, a Hausdorff space is *perfectly normal* if for any two disjoint closed subsets E and F there is a continuous function $f: X \to [0, 1]$ such that $f^{-1}(0) = E$ and $f^{-1}(1) = F$. Perfectly normal Hausdorff spaces are also known as T6 spaces. For instance, any metric space is perfectly normal, cf. [8, Proposition IX.5.2].

Proof. We first prove the sufficiency, following [11]. For every positive real number p, we consider the sets

$$I(p) := P^{-1}([0, p]) = \{x \in X: P(x) \leq p\}.$$

By (a_1) and (a_2), these are closed neighborhoods of S, and we can fix a sufficiently small $\bar{p} > 0$ so that $I(\bar{p}) \subseteq U$. For every positive real number $q \in (0, \bar{p})$, we define the closed sets

$$V(q) := P^{-1}([q, \bar{p}]) = \{x \in X: q \leq P(x) \leq \bar{p}\}.$$

We are going to prove the following.

Claim. *Given $q \in (0, \bar{p}]$, there exists $T \geq 0$ with the property that, taken any $x \in V(q)$, there is a $t_x \in (0, T]$ such that $P(\pi(x, t_x)) \notin I(\bar{p})$.*

By assumption (a_3), for every $y \in I(\bar{p}) \setminus S$ there exist $T_y > 0$ and $\varepsilon_y > 0$ such that $P(\pi(y, T_y)) \geq P(y) + 2\varepsilon_y$. By the continuity of P, there exists an open neighborhood B_y of y such that, for all $z \in B_y$,

$$P(\pi(z, T_y)) \geq P(z) + \varepsilon_y. \tag{1}$$

The open sets B_y, as y varies in $V(q)$, cover $V(q)$. Since $V(q)$ is compact, there exists a finite subcover $B_{y_1}, \ldots, B_{y_k}$.

Setting $\varepsilon = \min_{i=1,\ldots,k} \varepsilon_{y_i}$, by (1), for every $z \in V(q)$ there exists an index $i \in \{1, \ldots, k\}$ such that

$$P(\pi(z, T_{y_i})) \geq P(z) + \varepsilon.$$

Defining $\hat{T} = \max_{i=1,\dots,k} T_{y_i}$, it follows that for all $z \in V(q)$ there exists a $T_z \in [0, \hat{T}]$ such that

$$P(\pi(z, T_z)) \geq P(z) + \varepsilon. \tag{2}$$

Let n be the integer satisfying

$$q + (n-1)\varepsilon \leq \bar{p} < q + n\varepsilon,$$

and set $T = n\hat{T}$. We affirm that this choice of T satisfies the claim.

Suppose by contradiction that there exists $x \in V(q)$ such that $\pi(x, t) \in I(\bar{p})$ for all $t \in [0, T]$. Since $x \in V(q)$, by (2) there exists $t_1 = T_x \in [0, \hat{T}]$ such that

$$P(\pi(x, t_1)) \geq P(x) + \varepsilon \geq q + \varepsilon.$$

If $n = 1$, then $P(\pi(x, t_1)) > \bar{p}$ and we get the contradiction. If $n > 1$, since we are assuming $\pi(x, t) \in I(\bar{p})$ for all $t \in [0, T]$, we get $\pi(x, t_1) \in V(q)$ and hence by (1) we can define $t_2 = T_{\pi(x,t_1)} \in [0, \hat{T}]$, which is such that

$$P(\pi(x, t_1 + t_2)) \geq P(\pi(x, t_1)) + \varepsilon \geq q + 2\varepsilon.$$

If $n = 2$, then $P(\pi(x, t_1 + t_2)) > \bar{p}$ and we get the contradiction, since $t_1 + t_2 \leq 2\hat{T} = T$. Else, if $n > 2$, we can repeat the same argument and define $t_3, \dots, t_n$. We have

$$P(\pi(x, t_1 + t_2 + \dots + t_n)) \geq q + n\varepsilon > \bar{p}.$$

Since $t_1 + t_2 + \dots + t_n \leq n\hat{T} = T$, this contradicts the fact that $\pi(x, t) \in I(\bar{p})$ for all $t \in [0, T]$. Thus the claim is proved.

The claim has some important consequences. The first fact is that for every $x \in I(\bar{p}) \setminus S$ there exists a $t_x \geq 0$ such that $\pi(x, t_x) \notin I(\bar{p})$. This can be seen by fixing any $q < P(x)$. Now let us consider a point x outside $I(\bar{p})$. Its orbit either always stays out of $I(\bar{p})$ or, fixed any $\bar{q} \in (0, \bar{p}]$, it enters $V(\bar{q})$. By the claim, it follows that there is a $T > 0$ such that, whenever an orbit enters $V(\bar{q})$, the orbit must go out of $I(\bar{p})$ within a time at most equal to T. From these two facts it follows straightforwardly that the compact set

$$K = \{\pi(x, t) : P(x) \geq \bar{p},\ t \in [0, T]\} = \pi\left(\overline{X \setminus I(\bar{p})} \times [0, T]\right)$$

verifies the definition of permanence with respect to S.

Let us now prove the necessity, in the case when X is perfectly normal. We consider the disjoint closed subsets S and K, where $K \subseteq X \setminus S$ is the compact set provided by the definition of permanence. Since X is perfectly normal, there exists a function $P: X \to [0, 1]$ such that $P^{-1}(0) = S$ and $P^{-1}(1) = K$. Clearly P satisfies (a_1) and (a_2). Let $U = X \setminus K$, and take a certain point $x \in U \setminus S$, so that $P(x) < 1$. By permanence, there is $t_x > 0$ such that $\pi(x, t_x) \in K$. This implies that $P(\pi(x, t_x)) = 1 > P(x)$, and hence also (a_3) is satisfied. □

Notice that, in the above statement, U has to be a proper subset of X. Indeed, since X is compact, if it were $U = X$, then P would have a maximum point in X, and (a_3) would not be possible.

As a consequence of Theorem 2, we have the following.

Corollary 3. *If X is a compact perfectly normal space, $\pi: X \times \mathbb{R}_+ \to X$ is a continuous semi-dynamical system and S is a closed proper subset of X such that $X \setminus S$ is positively invariant, then a necessary and sufficient condition for π to be permanent with respect to S is that there exist an open neighborhood U of S and a function $P: U \to \mathbb{R}_+$ such that conditions (a_1), (a_2) and (a_3) hold.*

Proof. Let $K_0 = X \setminus U$. Then, there exists a continuous function $f: X \to [0, 1]$ such that $f^{-1}(0) = S$ and $f^{-1}(1) = K_0$. Set

$$K_1 = \left\{x \in X : f(x) \geq \frac{1}{2}\right\},$$

and define $P_1: X \to \mathbb{R}_+$ as follows:

$$P_1(x) = \begin{cases} P(x) & \text{if } x \in X \setminus K_1, \\ 1 - 2(1 - f(x))(1 - P(x)) & \text{if } x \in K_1 \setminus K_0, \\ 1 & \text{if } x \in K_0. \end{cases}$$

Then, the conclusion follows applying Theorem 2, with P_1 instead of P. □

Remark 4. It is easy to verify that analogues of Theorem 2 and Corollary 3 hold also for discrete semi-dynamical systems. For briefness, we do not enter into details.

3. Dynamical systems in locally compact spaces

In this section, we begin by considering a continuous dynamical system $\pi: X \times \mathbb{R} \to X$, in the case when X a is locally compact Hausdorff space.

We construct the *Alexandroff compactification* $\widetilde{X}$ of X introducing an abstract point ∞, and setting $\widetilde{X} = X \cup \{\infty\}$. If $x = \infty$, a base of neighborhoods is given by the complements of the compact subsets of X, while, if $x \in X$, any base of neighborhoods in X is also a base in $\widetilde{X}$. It is well known that $\widetilde{X}$, provided with this topology, is a compact Hausdorff space, cf. [8]. The inclusion map $c: X \hookrightarrow \widetilde{X}$ is an embedding (i.e. a homeomorphism between X and $c(X)$), the so-called *Alexandroff extension*.

Remark 5. If, for instance, $X = \mathbb{R}^N$, then its Alexandroff compactification is homeomorphic to the sphere S^N and a possible choice of the inclusion map c is given by the inverse of the stereographic projection.

Given a function $F: X \to \mathbb{R}$, we set

$$\liminf_{x\to\infty} F(x) = \liminf_{y\to\infty} F(c^{-1}(y)),$$

where the inferior limit on the right hand side is taken on the Alexandroff compactification $\widetilde{X}$. The same notation is adopted also for the limit, when it exists. When ∞ is an isolated point of $\widetilde{X}$, i.e. when X is compact, usually the notion of limit is not defined. However, since any extension of the function F to ∞ would be continuous, for practical convenience we agree that, in this case, any statement concerning the limit (or lim inf) is to be considered automatically true.

Here is the main result of this section.

Theorem 6. *If X is a locally compact Hausdorff space, $\pi: X \times \mathbb{R} \to X$ is a continuous dynamical system and S is a closed proper subset of X such that $X \setminus S$ is positively invariant, then a sufficient condition for π to be permanent with respect to S is that there exist an open neighborhood U of S such that $X \setminus U$ is compact, and a function $P: X \to \mathbb{R}_+$ such that*

(A_1) *P is continuous and $\lim_{x\to\infty} P(x) = 0$;*
(A_2) *$P(x) = 0$ if and only if $x \in S$;*
(A_3) *for every $x \in U \setminus S$ there exists a $t_x > 0$ such that $P(\pi(x, t_x)) > P(x)$.*

Moreover, if $\widetilde{X}$ is perfectly normal, the above condition is also necessary.

Proof. If X is compact, then the theorem reduces to Theorem 2, in view of the adopted agreement on the limits at the isolated point ∞. Hence, we hereafter consider the case of a non-compact space.

We want to extend π to a dynamical system $\tilde{\pi}$ on $\widetilde{X}$. Thus we define

$$\tilde{\pi}(y, t) = \begin{cases} c(\pi(c^{-1}(y), t)) & \text{if } y \in c(X), \\ \infty & \text{if } y = \infty. \end{cases}$$

All the conditions for $\tilde{\pi}$ being a dynamical system are trivially satisfied except for the continuity of $\tilde{\pi}$ at (∞, t_0), for any $t_0 \in \mathbb{R}$. To prove this, we have to show that, for every compact set K in X, there is a further compact set K' in X and a $\delta > 0$ such that, if $x \in \widetilde{X} \setminus K'$ and $|t - t_0| < \delta$, then $\pi(x, t) \notin K$. This is true since, once K has been chosen, it is sufficient to take $\delta = 1$ and

$$K' = \pi(K \times [-t_0 - 1, -t_0 + 1]).$$

Indeed, if $|t - t_0| < 1$ and $\pi(x, t) \in K$, then,

$$x = \pi(\pi(x, t), -t) \in \pi(K \times [-t_0 - 1, -t_0 + 1]) = K'.$$

Define the set $\widetilde{S} = c(S) \cup \{\infty\}$. Hence, π is permanent with respect to S if and only if $\tilde{\pi}$ is permanent with respect to $\widetilde{S}$. The proof is then easily completed applying Theorem 2 to $\tilde{\pi}$, after defining $\widetilde{U} = c(U) \cup \{\infty\}$, and $\widetilde{P}: \widetilde{X} \to \mathbb{R}_+$ as

$$\widetilde{P}(y) = \begin{cases} P(c^{-1}(y)) & \text{if } y \in c(X), \\ 0 & \text{if } y = \infty. \end{cases} \quad \square$$

In order to ensure that $\widetilde{X}$ is perfectly normal, a possibility is to assume that the locally compact space X is second-countable (indeed, for locally compact spaces, X is second-countable if and only if $\widetilde{X}$ is metrizable, see [8, Theorem XI.8.6]). More generally, we have the following.

Proposition 7. *Let X be a σ-locally compact perfectly normal Hausdorff space. Then its Alexandroff compactification $\widetilde{X}$ is perfectly normal.*

We recall that a locally compact Hausdorff space is σ-locally compact if it is the union of countably many compact sets. The proof of Proposition 7 is provided, for the reader's convenience, in the Appendix.

As a straightforward consequence of Corollary 3, we can state its analogue, in this situation.

Corollary 8. *If X is a σ-locally compact perfectly normal Hausdorff space, $\pi: X \times \mathbb{R} \to X$ is a continuous dynamical system and S is a closed proper subset of X such that $X \setminus S$ is positively invariant, then a necessary and sufficient condition for π to be permanent with respect to S is that there exist an open neighborhood U of S such that $X \setminus U$ is compact, and a function $P: U \to \mathbb{R}_+$ such that conditions (A_1), (A_2) and (A_3) hold.*

Analogues of Theorem 6 and Corollary 8 can be stated for discrete dynamical systems, as well. Instead, for continuous or discrete semi-dynamical systems, the extended system $\tilde{\pi}$ can be discontinuous at ∞ and our compactification method no longer applies. We do not know whether our results still hold in such situations.

4. Permanence for local dynamical systems

In this section we extend our results to local dynamical systems [3,15]. We recall that a local dynamical system on a topological space X consists of a continuous map $\pi: \mathcal{D}_\pi \to X$, where $\mathcal{D}_\pi$ is an open subset of $X \times \mathbb{R}$, with the property that, for every $x \in X$, there exist $\alpha_x < 0 < \omega_x$ (with possibly $\alpha_x = -\infty$ or $\omega_x = +\infty$) such that $(x, t) \in \mathcal{D}_\pi$ if and only if $t \in (\alpha_x, \omega_x)$. Moreover it is required that, for every $x \in X$, one has $\pi(x, 0) = x$, and

$$\pi(\pi(x, t), s) = \pi(x, t + s),$$

whenever the left-hand side is defined. The definition of permanence can be easily adapted in the following way.

Definition 9. The local dynamical system π is said to be *permanent* with respect to S if there exists a compact set $K \subset X \setminus S$ with the following property: for every $x \in X \setminus S$ there exists a $T_x \in [0, \omega_x)$ such that $\pi(x, t) \in K$, for every $t \in [T_x, \omega_x)$.

We emphasize that, although permanence implies global existence in the future, this property is not a priori known and therefore the generalization to local dynamical systems is useful also under this framework, in addition to the fact that the global existence in the past is no longer required.

We want to show that, under little requirements on the space, it is possible to restrict the problem to the case of (global) dynamical systems for which we can apply our main result.

We say that two local dynamical systems $\pi: \mathcal{D}_\pi (\subseteq X \times \mathbb{R}) \to X$ and $\rho: \mathcal{D}_\rho (\subseteq Y \times \mathbb{R}) \to Y$ are isomorphic (in the sense of Gottschalk and Hedlund, [14,30,31]) if there exists a homeomorphism $h: X \to Y$ and a continuous map $\phi: \mathcal{D}_\pi \to \mathbb{R}$ such that

- $\phi(x, 0) = 0$, for every $x \in X$,
- $\phi(x, \cdot): (\alpha_x, \omega_x) \to (\alpha_{h(x)}, \omega_{h(x)})$ is a homeomorphism for every $x \in X$,
- $h(\pi(x, t)) = \rho(h(x), \phi(x, t))$, for every $(x, t) \in \mathcal{D}_\pi$.

In [6], Carlson proved the following generalization of Vinograd's Theorem.

Theorem 10. *Let $\pi: \mathcal{D}_\pi \to X$ be a local dynamical system on a topological space X. If $X \times \mathbb{R}$ is a normal Hausdorff space, then there exists a (global) dynamical system $\rho: X \times \mathbb{R} \to X$ which is isomorphic to π.*

We recall that the space $X \times \mathbb{R}$ is normal if, for instance, X is metrizable. Let us state the main result of this section.

Theorem 11. *If X is a locally compact, second-countable Hausdorff space, $\pi: \mathcal{D}_\pi (\subseteq X \times \mathbb{R}) \to X$ is a continuous local dynamical system and S is a closed proper subset of X such that $X \setminus S$ is positively invariant, then a necessary and sufficient condition for π to be permanent with respect to S is that there exist an open neighborhood U of S such that $X \setminus U$ is compact, and a function $P: X \to \mathbb{R}_+$ satisfying* (A$_1$), (A$_2$) *and* (A$_3$).

Proof. If X is locally compact and second-countable, then it is metrizable (more precisely, locally compact Hausdorff spaces are regular [8, Theorem XI.6.4] and second-countable regular spaces are metrizable by Urysohn metrizability Theorem [8, Theorem IX.9.2]). Then, by Theorem 10, there exists an isomorphic dynamical system $\rho: X \times \mathbb{R} \to X$. Both permanence and the hypotheses of the corollary are preserved by isomorphisms, so the result follows applying Theorem 6 to the system ρ. □

Notice that, as for Corollaries 3 and 8, the function P could be assumed to be defined only on U.

We now want to find some more explicit conditions for permanence, in the case when X is a subset of $\mathbb{R}^N$, provided with the Euclidean distance, and the continuous dynamical system π is generated by an autonomous ordinary differential equation, for which X is positively invariant. The derivative along the orbits of a function $P: X \to \mathbb{R}$, when it exists, is defined, as usual, by

$$\dot{P}(x) = \frac{d}{dt} P(\pi(x, t))\bigg|_{t=0}.$$

Given a point $x \in X$, we recall that the ω-limit set $\omega(x)$ of x is the set of all $z \in X$ such that there exists a sequence $(t_n)_n$ in $\mathbb{R}_+$, with $t_n \to \omega_x$, for which $\pi(x, t_n) \to z$. For a subset M of X, we write

$$\Omega(M) = \bigcup_{x \in M} \omega(x).$$

We propose the following version of [22, Theorem 2.5] (see also [17]).

Corollary 12. *Let X be a subset of $\mathbb{R}^N$, $\pi : \mathcal{D}_\pi (\subseteq X \times \mathbb{R}^N) \to X$ a local dynamical system generated by an autonomous ordinary differential equation, and S a proper subset of X such that $X \setminus S$ is open in $\mathbb{R}^N$ and positively invariant. Assume that there exist a function $P \in \mathcal{C}(X, \mathbb{R}_+) \cap \mathcal{C}^1(X \setminus S, \mathbb{R}_+)$, a lower semicontinuous function $\psi: X \to \mathbb{R}$, bounded below, and a constant $\alpha \in [0, 1]$ such that*

(i) $\lim_{x\to\infty} P(x) = 0$ *and* $\liminf_{x\to\infty} \psi(x) > 0$;
(ii) $P(x) = 0$ *if and only if* $x \in S$;
(iii) $\dot{P}(x) \geq [P(x)]^\alpha \psi(x)$ *for all* $x \in X \setminus S$;
(iv) *for every* $z \in S$, *we have* $\sup_{0\leq T<\omega_z} \int_0^T \psi(\pi(z,s))\,ds > 0$.

Then, the local dynamical system π *is permanent with respect to* S.

Assume now the following dissipativity condition on S*: there exists a compact set* $\mathcal{K}$ *in* X *such that, for every* $x \in S$*, one has that* $\omega_x = +\infty$*, and there is a* $\tau_x \geq 0$ *for which* $\pi(x,t) \in \mathcal{K}$*, for every* $t \geq \tau_x$*. The same conclusion then holds true if (iv) is replaced by*

(v) *for every* $z \in \overline{\Omega(S)}$, *we have* $\sup_{T\geq 0} \int_0^T \psi(\pi(z,s))\,ds > 0$.

Proof. We start proving the first statement. By assumption, P satisfies (A_1) and (A_2). Let us prove that (A_3) also holds, for some open neighborhood U of S such that $X \setminus U$ is compact.
By (iv), for every $z \in S$ there is a $t_z \in (0, \omega_z)$ such that $\int_0^{t_z} \psi(\pi(z,s))\,ds > 0$. By the lower semicontinuity of ψ, there exists an open neighborhood B_z of z such that, for every $x \in B_z$, we have $\int_0^{t_z} \psi(\pi(x,s))\,ds > 0$. Hence, by (iii), if $x \in B_z \setminus S$,

$$0 < \int_0^{t_z} \frac{\dot{P}(\pi(x,s))}{[P(\pi(x,s))]^\alpha}\,ds = \begin{cases} \dfrac{1}{1-\alpha}[P(\pi(x,t_z))^{1-\alpha} - P(x)^{1-\alpha}] & \text{if } \alpha < 1, \\ \log \dfrac{P(\pi(x,t_z))}{P(x)} & \text{if } \alpha = 1, \end{cases}$$

and thus $P(\pi(x,t_z)) > P(x)$. Moreover, since $\liminf_{x\to\infty} \psi(x) > 0$, there is an open neighborhood U_∞ of ∞ where ψ is positive, so that, by (iii), we have that $\dot{P}(x) > 0$ for every $x \in (X \cap U_\infty) \setminus S$, and then $P(\pi(x,t)) > P(x)$ for any sufficiently small $t > 0$. Taking $U = \bigcup_{z\in S} B_z \cup U_\infty$, we have that $X \setminus U$ is compact and P satisfies (A_3), so the proof follows from Theorem 11.

Assume now the dissipativity condition on S, and that (v) holds. By Fatou's Lemma, for every fixed $T \geq 0$ the function $z \mapsto \int_0^T \psi(\pi(z,s))\,ds$ is lower semicontinuous. So, also the function $z \mapsto \sup_{T\geq 0} \int_0^T \psi(\pi(z,s))\,ds$ is lower semicontinuous. The set $\overline{\Omega(S)}$ is compact, being contained in $\mathcal{K}$, so by (v) there exists a $\delta > 0$ and an open neighborhood W of $\overline{\Omega(S)}$ such that, for every $w \in W$,

$$\sup_{T\geq 0} \int_0^T \psi(\pi(w,s))\,ds \geq \delta. \tag{3}$$

Fix now $z \in S$. Since the positive semiorbit of z is bounded and W is a neighborhood of $\omega(z)$, there exists $t_z \geq 0$ such that $\pi(z,t) \in W$ for all $t \geq t_z$, cf. [16, Theorem I.8.1]. So, by repeated use of (3), we get

$$\sup_{T\geq 0} \int_0^T \psi(\pi(z,s))\,ds = +\infty.$$

We have thus shown that (iv) is satisfied, and the proof is completed. □

5. An application to a Lotka–Volterra model

In this section we present a simple application, without any special biological disclosure, with the only purpose of illustrating our main theorem.

Let us recall the classical Lotka–Volterra predator–prey model (cf. [4])

$$\begin{cases} \dot{x} = x(a - by), \\ \dot{y} = y(-c + dx). \end{cases} \tag{4}$$

Here all constants a, b, c, d are positive. Usually this system is studied on $\mathbb{R}^2_+$, interpreting x as the population of preys and y as that of predators. Permanence is considered with respect to $(\{0\} \times \mathbb{R}_+) \cup (\mathbb{R}_+ \times \{0\})$, corresponding, in some sense, to the survival in the long-term of both species. System (4), however, is not permanent since, besides an equilibrium point $(\bar{x}, \bar{y})$, with

$$\bar{x} = \frac{c}{d}, \qquad \bar{y} = \frac{a}{b},$$

its orbits in $\mathbb{R}^2_+ \setminus S$ are all periodic (rotating counter-clockwise around $(\bar{x}, \bar{y})$), they are arbitrarily near S, and become arbitrarily large.

As a modification of (4), we consider the system

$$\begin{cases} \dot{x} = x(a - by + f_1(x,y)), \\ \dot{y} = y(-c + dx + f_2(x,y)), \end{cases} \tag{5}$$

where the real functions f_1, f_2 are assumed to be locally Lipschitz continuous. It will be useful to introduce the function f, defined as

$$f(x,y) = (f_1(x,y), f_2(x,y)).$$

We observe that the perturbation f can compromise the local existence for the solutions of (5) on the boundary of $\mathbb{R}^2_+$. To overcome this issue, we consider the system as defined on $X = \mathbb{R}^2$. In this way we obtain a local dynamical system $\pi\colon \mathcal{D}_\pi (\subseteq \mathbb{R}^2 \times \mathbb{R}) \to \mathbb{R}^2$. Equivalently, permanence will be studied with respect to the set

$$S = \mathbb{R}^2 \setminus (0, +\infty)^2.$$

We remark that a locally Lipschitz continuous function $\hat{f}\colon \mathbb{R}^2_+ \to \mathbb{R}^2$ can always be extended to a locally Lipschitz continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, for instance setting $f(x,y) = \hat{f}(\hat{x},\hat{y})$, where $(\hat{x},\hat{y})$ is the point of $\mathbb{R}^2_+$ with minimum distance from $(x,y) \in \mathbb{R}^2$. Provided that $\mathbb{R}^2 \setminus S$ is positively invariant, the choice of the prolongation does not affect permanence.

Corollary 13. *Assume that, for every $x > 0$ and $y > 0$, either $f(x,y) = 0$, or*

$$\langle f(x,y), (d(x-\bar{x}), b(y-\bar{y}))\rangle < 0. \tag{6}$$

If, moreover, there exist a point (x_0, y_0) in ∂S and a neighborhood U_0 of (x_0, y_0) such that $f(x,y) \neq 0$ for every $(x,y) \in U_0 \setminus S$, then system (5) is permanent.

We notice that condition (6) also reads as

$$(dx - c)f_1(x,y) + (by - a)f_2(x,y) < 0,$$

and it says that the field $f(x,y)$ points inward with respect to the orbits of (4).

Proof. In $\mathbb{R}^2 \setminus S$, as long as $f(x,y) = 0$, the orbits follow the periodic orbits of the classical system (4), and if $f(x,y) \neq 0$, then they cross those periodic orbits from the outer to the inner regions. Let U_1 be the set of all points in $\mathbb{R}^2 \setminus S$ whose orbits in system (4) cross U_0. Clearly, $U = U_1 \cup S$ is an open neighborhood of S, and its complement $\mathbb{R}^2 \setminus U$ is a compact set.

Let, for $x > 0$ and $y > 0$,

$$V(x,y) = d(\bar{x}\ln x - x) + b(\bar{y}\ln y - y),$$

and define the continuous function $P\colon \mathbb{R}^2 \to \mathbb{R}_+$ as

$$P(x,y) = \begin{cases} e^{V(x,y)} & \text{if } (x,y) \in \mathbb{R}^2 \setminus S, \\ 0 & \text{if } (x,y) \in S. \end{cases}$$

The level sets of this function are precisely the orbits of system (4) and, by the above reasoning, for every $(x,y) \in U$, there is a time $t_{(x,y)} > 0$ for which $P(\pi((x,y), t_{(x,y)})) > P(x,y)$. So, all the assumptions of Theorem 11 are satisfied, and the proof is completed. $\square$

We propose two examples where the above corollary applies.

Example 14. We introduce in the Lotka–Volterra system (4) a negative intraspecific effect for the preys, which becomes effective only when their number crosses a certain threshold. We are thus considering the system

$$\begin{cases} \dot{x} = x(A(x) - by), \\ \dot{y} = y(-c + dx), \end{cases} \tag{7}$$

where the continuous function $A\colon \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$A(x) = \begin{cases} a & \text{if } x \in [0, \alpha], \\ a + g_1(x) & \text{if } x > \alpha. \end{cases}$$

All the constants a, b, c, d and α are assumed to be positive, with $d\alpha \geq c$ (i.e. $\alpha \geq \bar{x}$), and $g_1\colon (\alpha, +\infty) \to \mathbb{R}$ is a negative function. Then, system (7) satisfies the hypotheses of Corollary 13, with

$$f_1(x,y) = \begin{cases} 0 & \text{if } x \in [0, \alpha], \\ g_1(x) & \text{if } x > \alpha, \end{cases}$$

and $f_2(x,y)$ identically zero. Hence, system (7) is permanent.

Notice that, in the above example, we are "penalizing" the preys, when they become too numerous, so to have permanence. This could at first seem counter-intuitive.

Example 15. In this second example we consider a different perturbation of the Lotka–Volterra system (4), introducing a positive term that affects predators when their population is small. The system we consider is

$$\begin{cases} \dot{x} = x(a - by), \\ \dot{y} = y(C(y) + dx), \end{cases} \tag{8}$$

where the continuous function $C: \mathbb{R}_+ \to \mathbb{R}$ is as follows:

$$C(y) = \begin{cases} -c + g_2(y) & \text{if } y \in [0, \beta), \\ -c & \text{if } y \geq \beta. \end{cases}$$

All the constants a, b, c, d and β are assumed to be positive, with $b\beta \leq a$ (i.e. $\beta \leq \bar{y}$), and $g_2: [0, \beta) \to \mathbb{R}$ is a positive function. Also system (8) satisfies the hypotheses of Corollary 13, and hence is permanent.

So, in the second example above, we are "encouraging" a bit the predators, when rare, and this provides permanence.

Remark 16. As already observed in Remark 5, the compactification needed in the proof of Theorem 6 can be carried out, in this case, by the use of the stereographic projection. Let us describe how this type of compactification transforms system (4). We have two fixed points, 0 and ∞, and two heteroclinic orbits connecting them, coming from the two semiaxes $\{x = 0, y \geq 0\}$ and $\{y = 0, x \geq 0\}$. There is a third fixed point in the interior region, and all other solutions are periodic, rotating around this point. The situation is then surprisingly similar to the one encountered when studying in the phase plane the behavior of an oscillating pendulum.

To conclude, as a variant of Corollary 13, we can easily prove the following.

Corollary 17. *Assume that, for every $x > 0$ and $y > 0$, either $f(x, y) = 0$, or*

$$\langle f(x, y), (d(x - \bar{x}), b(y - \bar{y}))\rangle < 0.$$

If, moreover, there is a constant $R > 0$ such that

$$[x \geq R \text{ and } y \geq R] \Rightarrow f(x, y) \neq 0,$$

then system (5) is permanent.

Appendix. Proof of Proposition 7

In this section, we provide a proof of Proposition 7 which, we recall, states that, if X is a σ-locally compact perfectly normal Hausdorff space, then its Alexandroff compactification $\widetilde{X}$ is perfectly normal.

By assumption, there exists a countable family of compact subsets K_n, with $n \in \mathbb{N}$, such that

$$K_n \cap (\overline{X \setminus K_{n+1}}) = \emptyset, \qquad \bigcup_{n \in \mathbb{N}} K_n = X.$$

Since X is perfectly normal, for all $n \in \mathbb{N}$ there exists a continuous function $l_n: X \to [0, 1]$ such that $l_n^{-1}(0) = \overline{X \setminus K_{n+1}}$ and $l_n^{-1}(1) = K_n$. We define the continuous function $l: X \to (0, 2]$ as

$$l(x) = \sum_{k=0}^{\infty} \frac{l_k(x)}{2^k},$$

and consider the function $\tilde{l}: \widetilde{X} \to [0, 2]$ defined as

$$\tilde{l}(y) = \begin{cases} l(c^{-1}(y)) & \text{if } y \in c(X), \\ 0 & \text{if } y = \infty. \end{cases}$$

Notice that, if $x \in K_{n+1} \setminus K_n$, for some n, then $l(x) = 2^{-n}(l_n(x) + 1)$, whence

$$\lim_{x \to \infty} l(x) = 0.$$

Therefore, $\tilde{l}$ is continuous on the whole $\widetilde{X}$.

Remark 18. If $X = \mathbb{R}^N$, we can take the closed balls $K_n = \overline{B}(0, n)$, and set

$$l_n(x) = \begin{cases} 1 & \text{if } \|x\| \leq n, \\ 2^{n+1-\|x\|} - 1 & \text{if } n \leq \|x\| \leq n + 1, \\ 0 & \text{if } \|x\| \geq n + 1, \end{cases}$$

so that $l(x) = 2^{1-\|x\|}$.

A topological space is perfectly normal if and only if every closed set is a zero set, cf. [8, Section VII.4]. Let $\widetilde{E} \subseteq \widetilde{X}$ be a closed set. Define $E = c^{-1}(\widetilde{E})$; since the function c is an embedding, E is closed in X. We need to consider two cases: either $\widetilde{E} = c(E)$, or $\widetilde{E} = c(E) \cup \{\infty\}$.

If $\infty \notin \widetilde{E}$, then there exists $\bar{n}$ such that $E \subseteq K_{\bar{n}}$. We define the closed set $F = \overline{X \setminus K_{\bar{n}+1}}$. Since X is a perfectly normal space, there exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}(0) = E$ and $f^{-1}(1) = F$. Setting

$$\tilde{f}(y) = \begin{cases} f(c^{-1}(y)) & \text{if } y \in c(X), \\ 1 & \text{if } y = \infty, \end{cases}$$

we have found a continuous function $\tilde{f} : \widetilde{X} \to \mathbb{R}$ such that $\tilde{f}^{-1}(0) = \tilde{E}$.

Assume now $\infty \in \widetilde{E}$. Since X is perfectly normal and E is closed, there is a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = E$. We define $\tilde{f} : \widetilde{X} \to \mathbb{R}$ as

$$\tilde{f}(y) = \begin{cases} f(c^{-1}(y))\tilde{l}(y) & \text{if } y \in c(X), \\ 0 & \text{if } y = \infty, \end{cases}$$

and we see that $\tilde{f}$ is continuous and $\tilde{f}^{-1}(0) = \widetilde{E}$. The proof is thus completed.

Remark 19. Notice that, in Corollary 8, the assumption of X being σ-locally compact and perfectly normal is weaker than assuming X to be locally compact and second countable. Indeed, if X is locally compact and second countable, then on one hand it is separable, hence σ-compact; on the other hand, $\widetilde{X}$ is metrizable, so X is metrizable, as well.

References

[1] D.F. Anderson, A proof of the global attractor conjecture in the single linkage class case, SIAM J. Appl. Math. 71 (2011) 1487–1508.
[2] D. Angeli, P. De Leenheer, E.D. Sontag, Persistence results for chemical reaction networks with time-dependent kinetics and no global conservation laws, SIAM J. Appl. Math. 71 (2011) 128–146.
[3] N.P. Bhatia, O. Hájek, Local Semi-Dynamical Systems, Springer, Berlin, 1969.
[4] M. Braun, Differential Equations and their Applications: An Introduction to Applied Mathematics, Springer, Berlin, 1993.
[5] G.J. Butler, H.I. Freedman, P. Waltman, Uniformly persistent systems, Proc. Amer. Math. Soc. 96 (1986) 425–430.
[6] D.H. Carlson, A generalization of Vinograd's theorem for dynamical systems, J. Differential Equations 11 (1972) 193–201.
[7] G. Craciun, F. Nazarov, C. Pantea, Persistence and permanence of mass-action and power-law dynamical systems, SIAM J. Appl. Math. 71 (2013) 305–329.
[8] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.
[9] M.L. Fernandes, Uniform repellers for processes with applications to periodic differential systems, J. Differential Equations 86 (1990) 141–157.
[10] M.L. Fernandes, F. Zanolin, Remarks on strongly flow-invariant sets, J. Math. Anal. Appl. 128 (1987) 176–188.
[11] A. Fonda, Uniformly persistent semidynamical systems, Proc. Amer. Math. Soc. 104 (1988) 111–116.
[12] B.M. Garay, Uniform persistence and chain recurrence, J. Math. Anal. Appl. 139 (1989) 372–381.
[13] T.C. Gard, Uniform persistence in multispecies population models, Math. Biosci. 85 (1987) 93–104.
[14] W.H. Gottschalk, G.A. Hedlund, Topological Dynamics, Amer. Math. Soc., Providence, 1955.
[15] O. Hájek, Structure of dynamical systems, Comment. Math. Univ. Carolin. 6 (1965) 53–72.
[16] J.K. Hale, Ordinary Differential Equations, Krieger, Malabar, 1969.
[17] J. Hofbauer, A general cooperation theorem for hypercycles, Monatsh. Math. 91 (1981) 233–240.
[18] J. Hofbauer, On the occurrence of limit cycles in the Volterra–Lotka equation, Nonlinear Anal. 5 (1981) 1003–1007.
[19] J. Hofbauer, A unified approach to persistence, Acta Appl. Math. 14 (1989) 11–22.
[20] J. Hofbauer, P. Schuster, K. Sigmund, Competition and cooperation in catalytic selfreplication, J. Math. Biol. 11 (1981) 155–168.
[21] J. Hofbauer, K. Sigmund, Evolutionary Games and Population Dynamics, Cambridge Univ. Press, Cambridge, 1998.
[22] V. Hutson, A theorem on average Liapunov functions, Monatsh. Math. 98 (1984) 267–273.
[23] V. Hutson, K. Schmitt, Permanence and the dynamics of biological systems, Math. Biosci. 111 (1992) 1–71.
[24] Y. Kang, P. Chesson, Relative nonlinearity and permanence, Theor. Popul. Biol. 78 (2010) 26–35.
[25] A. Margheri, C. Rebelo, N. Bacaër, Persistence in seasonally forced epidemiological models, J. Math. Biol. 64 (2012) 933–949.
[26] P. Schuster, K. Sigmund, R. Wolff, Dynamical systems under constant organization, III: cooperative and competitive behavior of hypercycles, J. Differential Equations 32 (1979) 357–368.
[27] H.L. Smith, H.R. Thieme, Dynamical Systems and Population Persistence, Amer. Math. Soc., Providence, 2010.
[28] H.R. Thieme, Persistence under relaxed point-dissipativity (with application to an endemic model), SIAM J. Math. Anal. 24 (1993) 407–435.
[29] H.R. Thieme, Uniform persistence and permanence for non-autonomous semiflows in population biology, Math. Biosci. 166 (2000) 173–201.
[30] T. Ura, Isomorphism and local characterization of local dynamical systems, Funkcial. Ekvac. 12 (1969) 99–122.
[31] T. Ura, Local isomorphisms and local parallelizability in dynamical systems theory, Math. Syst. Theory 3 (1969) 1–16.
[32] P. Waltman, A brief survey of persistence in dynamical systems, in: S. Busenberg, M. Martelli (Eds.), Delay Differential Equations and Dynamical Systems, 1991, pp. 31–40.
[33] F. Zanolin, Permanence and positive periodic solutions for Kolmogorov competing species systems, Results Math. 21 (1992) 224–250.