NOTES

On a Geometrical Formula Involving Medians and Bimedians

ALESSANDRO FONDA
Università di Trieste
Trieste, Italy
a.fonda@units.it

The following proposition concerning a triangle is well known.

PROPOSITION 1. *For any triangle, the sum of the squares of its three medians is equal to three fourths of the sum of the squares of its sides.*

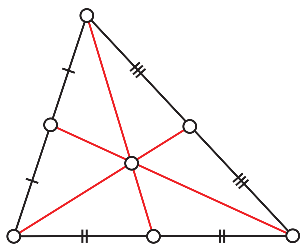

Figure 1 The medians of a triangle

A bit less well known is that there are analogous formulas for the tetrahedron. Indeed, defining a median as the segment joining a vertex to the barycenter of the opposite face (FIGURE 2), we have the following [**1**, page 57].

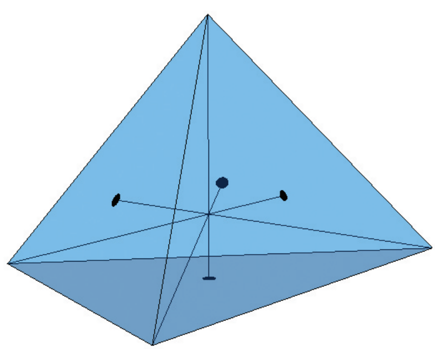

Figure 2 The medians of a tetrahedron

PROPOSITION 2. *For any tetrahedron, the sum of the squares of its four medians is equal to four ninths of the sum of the squares of its edges.*

Math. Mag. **86** (2013) 351–357. doi:10.4169/math.mag.86.5.351. © Mathematical Association of America

A quite different analogue is obtained if we consider bimedians instead of medians. Recalling that a bimedian is the segment joining the midpoints of two opposite edges of the tetrahedron (FIGURE 3), we have the following [**1**, p. 56] [**2**].

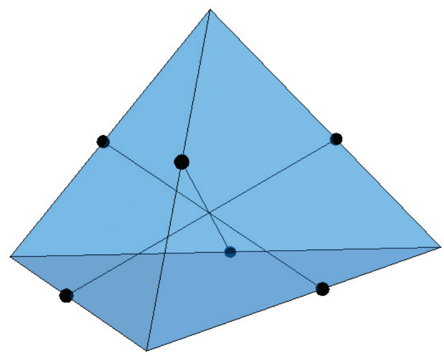

Figure 3 The bimedians of a tetrahedron

PROPOSITION 3. *For any tetrahedron, the sum of the squares of its three bimedians is equal to one fourth of the sum of the squares of its edges.*

In this note we provide a generalization of these three propositions for an arbitrary number of points.

The general formula

To understand our strategy, it will be useful to look at some simple proofs of Propositions 1, 2, and 3. We write the Euclidean distance between any two points A and B as $d(A, B)$. It can be expressed through the norm and the scalar product, as follows:

$$d(A, B) = \|A - B\| = \sqrt{(A - B) \cdot (A - B)}.$$

Using properties of the scalar product (and quite a bit of algebra) we can verify that, given three points A, B, C,

$$\begin{aligned} &d\left(A, \frac{B+C}{2}\right)^2 + d\left(B, \frac{A+C}{2}\right)^2 + d\left(C, \frac{A+B}{2}\right)^2 \\ &\quad = \frac{3}{4}\left[d(A,B)^2 + d(A,C)^2 + d(B,C)^2\right], \end{aligned}$$

thus proving Proposition 1. Similarly, given four points A, B, C, D, we directly verify that

$$\begin{aligned} &d\left(A, \frac{B+C+D}{3}\right)^2 + d\left(B, \frac{A+C+D}{3}\right)^2 \\ &\quad + d\left(C, \frac{A+B+D}{3}\right)^2 + d\left(D, \frac{A+B+C}{3}\right)^2 \\ &= \frac{4}{9}\left[d(A,B)^2 + d(A,C)^2 + d(A,D)^2 + d(B,C)^2 + d(B,D)^2 + d(C,D)^2\right], \end{aligned}$$

proving Proposition 2, and

$$d\left(\frac{A+B}{2}, \frac{C+D}{2}\right)^2 + d\left(\frac{A+C}{2}, \frac{B+D}{2}\right)^2 + d\left(\frac{A+D}{2}, \frac{B+C}{2}\right)^2$$
$$= \frac{1}{4}\Big[d(A, B)^2 + d(A, C)^2 + d(A, D)^2 + d(B, C)^2 + d(B, D)^2 + d(C, D)^2\Big],$$

proving Proposition 3.

In the last formula, we could have considered six bimedians, each one starting from one edge and joining it to the opposite edge. We only wrote three of them because the other ones coincide with these, two by two. To avoid possible misunderstandings in the sequel, it will be preferable to deal with means, rather than sums. In general, given a finite set of real numbers $\mathcal{S} = \{x_1, \dots, x_N\}$, we will use the notation

$$\text{Mean } \mathcal{S} = \frac{x_1 + \cdots + x_N}{N}.$$

Our aim is to find a generalization of the three formulas above to any set of n points. The idea is, first, to fix integers j and k satisfying $j \geq 1$, $k \geq 1$, and $j + k \leq n$. Then, from among the n points, take two distinct subsets, one consisting of j points and one consisting of k points. Compute the barycenter of each subset, and compute the squared distance between these two barycenters. Average over all possible choices of the two subsets. Our claim is that there is a constant α with the following property:

The mean of the squares of the distances between the couples of barycenters thus obtained is equal to the constant α multiplied by the mean of the squares of all segments joining the n points.

And this turns out to be true, as the following theorem states.

THEOREM 1. *Let n, j and k be three integers such that $j \geq 1$, $k \geq 1$, $j + k \leq n$, and set*

$$\alpha_{j,k} = \frac{j+k}{2jk}.$$

Then, for any given n points $A_1, A_2, \dots, A_n$,

$$\text{Mean}\left\{\begin{array}{c} d\left(\dfrac{A_{i_1} + \cdots + A_{i_j}}{j}, \dfrac{A_{i_{j+1}} + \cdots + A_{i_{j+k}}}{k}\right)^2 : \\ 1 \leq i_1 < \cdots < i_j \leq n, 1 \leq i_{j+1} < \cdots < i_{j+k} \leq n \\ \{i_1, \dots, i_j\} \cap \{i_{j+1}, \dots, i_{j+k}\} = \phi \end{array}\right\}$$
$$= \alpha_{j,k} \text{ Mean}\{d(A_p, A_q)^2 : 1 \leq p < q \leq n\}.$$

Concerning the above formula, we notice that the arithmetic mean on the left-hand side involves $\binom{n}{j} \cdot \binom{n-j}{k}$ terms, while on the right-hand side we have the *mean square distance* of the points $A_1, \dots, A_n$, which involves $\binom{n}{2}$ numbers. Remarkably, the constant α does not depend on n, but only on j and k.

Before carrying out the proof of the theorem, let us consider some particular situations.

The case $n = 3$, $j = 1$, $k = 2$ yields Proposition 1. On the other hand, taking $n = 4$, if $j = 1$ and $k = 3$ we get Proposition 2, while if $j = 2$ and $k = 2$ we obtain Proposition 3. In these cases, we have

$$\alpha_{1,2} = \frac{3}{4}, \quad \alpha_{1,3} = \frac{2}{3}, \quad \alpha_{2,2} = \frac{1}{2}.$$

More generally, let us define a *median*, in the general n points case, as the segment joining one of the points to the barycenter of the remaining $n-1$ points. We then easily deduce the following generalization of Propositions 1 and 2.

COROLLARY 1. *Given n points, the sum of the squares of the n medians is equal to $n/(n-1)^2$ times the sum of the squares of all segments joining the n points.*

Let us now define a *bimedian*, in the general n points case, as the segment joining the midpoint of one segment to the barycenter of the remaining $n-2$ points. Then, the following generalization of Proposition 3 holds.

COROLLARY 2. *Given n points, the sum of the squares of the $n(n-1)/2$ bimedians is equal to $n/(4n-8)$ times the sum of the squares of all segments joining the n points.*

Notice that, in the case $n = 4$, the six bimedians considered in the above corollary coincide two by two. This explains why, in this case, we now have *one half* of the sum of the squares of its edges, instead of *one fourth*, as stated in Proposition 3.

Let us mention that the results stated here hold in any real or complex inner product space.

Proof of the formula

We now go for the proof of Theorem 1. For simplicity, the proof will be carried out in the real case, but only minor modifications are needed if we consider a complex inner product space. We intend to prove that

$$\sum_{\{i_1,\ldots,i_{j+k}\}} \left\| \frac{A_{i_1} + \cdots + A_{i_j}}{j} - \frac{A_{i_{j+1}} + \cdots + A_{i_{j+k}}}{k} \right\|^2 = \nu_{n,j,k} \sum_{1\le p<q\le n} \|A_p - A_q\|^2,$$

where

$$\nu_{n,j,k} = \frac{\binom{n}{j}\binom{n-j}{k}}{\binom{n}{2}}\, \alpha_{j,k},$$

the first sum being taken on all sequences of distinct indices $\{i_1, \ldots, i_{j+k}\}$ in $\{1, \ldots, n\}$ such that $1 \le i_1 < \cdots < i_j \le n$ and $1 \le i_{j+1} < \cdots < i_{j+k} \le n$.

It is easy to see that

$$\sum_{1\le p<q\le n} \|A_p - A_q\|^2 = (n-1)\sum_{p=1}^{n} \|A_p\|^2 - 2\sum_{1\le p<q\le n} A_p \cdot A_q.$$

Let us now concentrate on the left-hand side of the identity. The sum appearing there, for symmetry reasons, will be developed as

$$\sum_{\{i_1,\ldots,i_{j+k}\}} \left\| \frac{A_{i_1} + \cdots + A_{i_j}}{j} - \frac{A_{i_{j+1}} + \cdots + A_{i_{j+k}}}{k} \right\|^2$$

$$= \beta_{n,j,k} \sum_{p=1}^{n} \|A_p\|^2 + \gamma_{n,j,k} \sum_{1\le p<q\le n} A_p \cdot A_q,$$

where $\beta_{n,j,k}$ and $\gamma_{n,j,k}$ are constants, to be determined. In order to find the first one, let us compute, for instance, the coefficient of $\|A_1\|^2$. If A_1 belongs to the first group, then $\|A_1\|^2$ will have a factor $1/j^2$, and this may happen $\binom{n-1}{j-1} \cdot \binom{n-j}{k}$ times. On the other hand, if A_1 belongs to the second group, then $\|A_1\|^2$ will have a factor $1/k^2$, and this may happen $\binom{n-1}{k-1} \cdot \binom{n-k}{j}$ times. Then, summing the two and simplifying,

$$\begin{aligned}\beta_{n,j,k} &= \binom{n-1}{j-1}\binom{n-j}{k}\frac{1}{j^2} + \binom{n-1}{k-1}\binom{n-k}{j}\frac{1}{k^2} \\ &= \frac{(n-1)!}{j!\,k!\,(n-j-k)!}\,\frac{j+k}{jk} \\ &= (n-1)\nu_{n,j,k}.\end{aligned}$$

In order to find the value of $\gamma_{n,j,k}$, let us now compute, for instance, the coefficient of $A_1 \cdot A_2$. We distinguish four cases.

I. Assume $j \geq 2$ and $k \geq 2$. If A_1 and A_2 both belong to the first group, then $A_1 \cdot A_2$ will have a factor $2/j^2$, and this may happen $\binom{n-2}{j-2} \cdot \binom{n-j}{k}$ times. If A_1 belongs to the first group and A_2 to the second one, then $A_1 \cdot A_2$ will have a factor $-2/(jk)$, and this may happen $\binom{n-2}{j-1} \cdot \binom{n-j-1}{k-1}$ times. The same if A_1 belongs to the second group and A_2 to the first one. If A_1 and A_2 both belong to the second group, then $A_1 \cdot A_2$ will have a factor $2/k^2$, and this may happen $\binom{n-2}{k-2} \cdot \binom{n-k}{j}$ times. Summing up and simplifying,

$$\begin{aligned}\gamma_{n,j,k} &= \binom{n-2}{j-2}\binom{n-j}{k}\frac{2}{j^2} + 2\binom{n-2}{j-1}\binom{n-j-1}{k-1}\frac{-2}{jk} + \binom{n-2}{k-2}\binom{n-k}{j}\frac{2}{k^2} \\ &= -2\,\frac{(n-2)!}{j!\,k!\,(n-j-k)!}\,\frac{j+k}{jk} \\ &= -2\,\nu_{n,j,k}.\end{aligned}$$

II. Assume $j = 1$ and $k \geq 2$. If A_1 belongs to the first group and A_2 to the second one, then $A_1 \cdot A_2$ will have a factor $-2/k$, and this may happen $\binom{n-2}{k-1}$ times. The same if A_1 belongs to the second group and A_2 to the first one. If A_1 and A_2 both belong to the second group, then $A_1 \cdot A_2$ will have a factor $2/k^2$, and this may happen $\binom{n-2}{k-2} \cdot (n-k)$ times. Summing up,

$$\begin{aligned}\gamma_{n,1,k} &= 2\binom{n-2}{k-1}\frac{-2}{k} + \binom{n-2}{k-2}(n-k)\frac{2}{k^2} \\ &= -2\,\frac{(n-2)!}{k!\,(n-1-k)!}\,\frac{1+k}{k} \\ &= -2\,\nu_{n,1,k}.\end{aligned}$$

III. Assume $k = 1$ and $j \geq 2$. As in case II, we find

$$\gamma_{n,j,1} = -2\,\nu_{n,j,1}.$$

IV. Finally, assume $j = k = 1$. If A_1 belongs to the first group and A_2 to the second one, then $A_1 \cdot A_2$ will have a factor -2, and this may happen only once. The same if A_1 belongs to the second group and A_2 to the first one. Then,

$$\gamma_{n,1,1} = 2(-2) = -4 = -2\,\nu_{n,1,1}.$$

So, in all four cases, we have that $\gamma_{n,j,k} = -2\,\nu_{n,j,k}$. In conclusion, we see that

$$\sum_{\{i_1,\ldots,i_{j+k}\}} \left\| \frac{A_{i_1} + \cdots + A_{i_j}}{j} - \frac{A_{i_{j+1}} + \cdots + A_{i_{j+k}}}{k} \right\|^2$$

$$= (n-1)\nu_{n,j,k} \sum_{p=1}^{n} \|A_p\|^2 - 2\,\nu_{n,j,k} \sum_{1\le p<q\le n} A_p \cdot A_q$$

$$= \nu_{n,j,k} \left[(n-1) \sum_{p=1}^{n} \|A_p\|^2 - 2 \sum_{1\le p<q\le n} A_p \cdot A_q \right]$$

$$= \nu_{n,j,k} \sum_{1\le p<q\le n} \|A_p - A_q\|^2,$$

and the proof is completed.

A further example

We have seen how our formula applies to triangles and tetrahedra. We now illustrate another particular example, by considering a regular octahedron. The results in this section are developed as an illustration of our main formula. For comparison, they can also be obtained by more elementary methods.

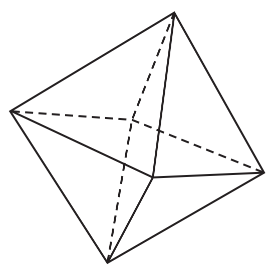

Figure 4 A regular octahedron

In this case we have six points $A_1, \ldots, A_6$, so $n = 6$. For simplicity, we consider only the cases when $j + k = 6$, and we write the formulas for $j = 1, 2, 3$. Let us denote by ℓ the length of the edges of the octahedron. First of all, we notice that the mean square distance of the vertices is

$$\frac{1}{15} \sum_{1\le p<q\le 6} \|A_p - A_q\|^2 = \frac{12\,\ell^2 + 3(\ell\sqrt{2})^2}{15} = \frac{6}{5}\,\ell^2.$$

To fix the ideas, assume that A_1, A_2, A_3 determine a face of the octahedron (i.e., an equilateral triangle), and let A_4, A_5, and A_6 be opposite to A_1, A_2, and A_3, respectively.

The case $j = 1$, $k = 5$. Let us denote by m the length of the six medians. Since $\alpha_{1,5} = \frac{3}{5}$, our formula gives

$$\frac{6\,m^2}{6} = \frac{3}{5} \cdot \frac{6}{5}\,\ell^2,$$

from which we derive

$$m = \frac{3}{5}\sqrt{2}\,\ell.$$

The case $j = 2, k = 4$. We are dealing with bimedians, i.e., the segments joining the midpoint of one edge to the barycenter of the remaining four points. Among these fifteen bimedians, twelve of them have a positive length, which we denote by b, while the remaining three have a zero length, being reduced to the center of the octahedron. Since $\alpha_{2,4} = \frac{3}{8}$, our formula says that

$$\frac{12\,b^2 + 3 \cdot 0^2}{15} = \frac{3}{8} \cdot \frac{6}{5}\,\ell^2,$$

whence

$$b = \frac{3}{4}\,\ell.$$

The case $j = 3, k = 3$. Here we take two groups of three vertices, and we want to compute the distances between their barycenters. However, we have to deal with two possible situations.

As we said above, the points A_1, A_2, A_3 determine one face, and the remaining three points A_4, A_5, A_6 determine the opposite face. There are eight such situations (where pairs of faces are involved), which coincide two by two. Let us denote by d the distance between these two faces.

On the other hand, the points A_1, A_2 and A_4, for instance, do not determine a face, but a right triangle, as well as the complementary points A_3, A_5, A_6. There are twelve such pairs of triangles, even if they coincide two by two. Let us denote by δ the distance between the barycenters of these two triangles. Using the symmetries of the regular octahedron, it is easy to see that

$$\delta = \frac{\ell\sqrt{2}}{3}.$$

Since $\alpha_{3,3} = \frac{1}{3}$, our formula tells us that

$$\frac{8\,d^2 + 12\,\delta^2}{20} = \frac{1}{3} \cdot \frac{6}{5}\,\ell^2,$$

and we deduce that

$$d = \sqrt{\frac{2}{3}}\,\ell.$$

Acknowledgment Thanks to my son Marcello for the drawings of the tetrahedra.

REFERENCES

1. N. A. Court, *Modern Pure Solid Geometry,* Macmillan, New York, 1935.
2. G. Y. Sosnow, A geometry problem, *Amer. Math. Monthly* **25** (1918) 122. http://dx.doi.org/10.2307/2971976

Summary We propose a general formula involving n points in an Euclidean space, which generalizes, on one hand, a well-known formula for the medians of a triangle and, on the other hand, two other formulas involving either the medians or the bimedians of a tetrahedron.