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Periodic Bouncing Solutions for Nonlinear Impact Oscillators

(To Klaus Schmitt, with esteem)

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Abstract

We prove the existence of a periodic solution to a nonlinear impact oscillator, whose restoring force has an asymptotically linear behavior. To this aim, after regularizing the problem, we use phase-plane analysis, and apply the Poincaré–Bohl fixed point Theorem to the associated Poincaré map, so to find a periodic solution of the regularized problem. Passing to the limit, we eventually find the "bouncing solution" we are looking for.

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1 Introduction and main result

We consider the differential equation

$$x'' + g(t, x) = 0\,, \tag{1.1}$$

where $g : \mathbb{R}\times[0, +\infty[\to \mathbb{R}$ is a continuous function, which is T-periodic in its first variable. We look for T-periodic "bouncing solutions", i.e., nonnegative solutions such that, if $x(t_0) = 0$, for some t_0, then $x'(t_0^-) = -x'(t_0^+)$, where

$$x'(t_0^-) = \lim_{t \to t_0^-} x'(t), \qquad x'(t_0^+) = \lim_{t \to t_0^+} x'(t).$$

Notice that, by the continuity of g, these limits, when they exist, are finite. Let us make more precise this notion of solution, recalling the definition given in [3].

Definition 1.1 A *bouncing solution* to equation (1.1) is a continuous function $x(t)$, defined on some interval $]a,b[$, such that $x(t) \geq 0$ for every $t \in]a,b[$, satisfying the following properties:

i. if $t_0 \in]a,b[$ is such that $x(t_0) > 0$, then $x(t)$ is twice differentiable at $t = t_0$, and $x''(t_0) + g(t_0, x(t_0)) = 0$;

ii. if $t_0 \in]a,b[$ is an isolated zero of $x(t)$, then $x'(t_0^-)$ and $x'(t_0^+)$ exist and $x'(t_0^-) = -x'(t_0^+)$;

iii. if $t_0 \in]a,b[$ is such that $x(t_0) = 0$ and, either $x'(t_0^-)$, or $x'(t_0^+)$, exists and is different from 0, then t_0 is an isolated zero of $x(t)$;

iv. if $x(t) = 0$ for all t in a non-trivial interval $I \subseteq]a,b[$, then $g(t,0) \geq 0$ for every $t \in I$.

A brief comment on the above definition. We can imagine a bouncing solution as describing a particle which, as long as it remains to the right of an obstacle (the origin), it satisfies the differential equation (1.1). If it reaches the obstacle at a nonzero speed, then it bounces elastically, so that its velocity simply changes its sign. On the other hand, if the particle reaches the obstacle with zero speed, then it could remain attached to it for some time, as long as the restoring force g pushes it against it, but, once the restoring force becomes repulsive, the particle has to leave the obstacle again.

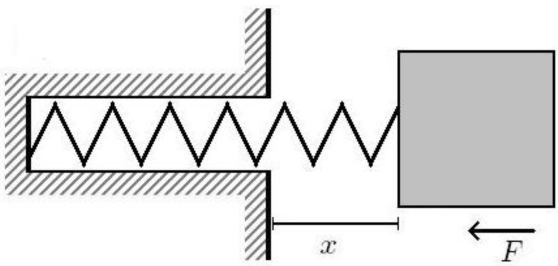

Figure 1: *A model of an impact oscillator.*

If $g(t,x) = \lambda x + e(t)$, for some $\lambda > 0$, where $e(t)$ is a T-periodic forcing term, this is the classical model of a linear "impact oscillator", see Figure 1. In this case, in order to find a T-periodic solution, one has to avoid some "resonance values" of λ (see [15]), which are given by the eigenvalues of the corresponding Dirichlet problem, precisely

$$\lambda \notin \left\{ \left(\frac{N\pi}{T} \right)^2 : N \in \mathbb{N} \right\}. \tag{1.2}$$

Our aim is to consider a nonlinear function $g(t,x)$, which however asymptotically preserves a linear-like behavior.

Here is our main result.

Theorem 1.1 *Let* $g : \mathbb{R} \times [0,+\infty[\to \mathbb{R}$ *be a continuous function, which is* T*-periodic in the first variable, and such that*

$$\mu_1 \leq \liminf_{x\to+\infty} \frac{g(t,x)}{x} \leq \limsup_{x\to+\infty} \frac{g(t,x)}{x} \leq \mu_2 \,, \tag{1.3}$$

uniformly in $t \in [0,T]$*, where* μ_1 *and* μ_2 *satisfy*

$$\left(\frac{N\pi}{T}\right)^2 < \mu_1 \leq \mu_2 < \left(\frac{(N+1)\pi}{T}\right)^2 \,, \tag{1.4}$$

for a suitable nonnegative integer N*. Then, there exists at least one* T*-periodic bouncing solution to equation (1.1).*

The above theorem generalizes a result by Bonheure and Fabry [3, Theorem 1], where $g(t,x)-\lambda x$ was assumed to be bounded, for some $\lambda > 0$ satisfying (1.2). Its proof will be carried out in Section 2. The idea is to approximate equation (1.1) by regular differential equations, without bouncing, and then obtain the solution we are looking for by a limit procedure. This device, suggested in [14], has already been used, for example, in [3, 16, 19]. In our case, in order to get the T-periodic solutions of the approximating equations, we use some phase-plane analysis methods developed in our previous papers [10, 11].

In recent years, different problems related to linear or nonlinear impact oscillators have been studied by many authors, using topological and variational methods. There is a vast literature on this subject, due to its great interest in physics and engineering (see, e.g., [1, 2], and the references therein). Let us just quote a few papers which perhaps are more related to our approach. The problem of the approximation of solutions was considered in the eighties in [5, 6, 8, 9]. In 1992, Lazer and McKenna [14] introduced the periodic problem with friction, opening the road to the analysis of many possible situations, like in [3, 12, 16, 18, 19, 21]. The existence of invariant tori was studied in [15, 17, 22]. Concerning other types of dynamics and the possibility of chaotic behavior of the solutions, see, e.g., [4, 7, 13, 20].

2 Proof of Theorem 1.1

The proof is divided in two steps. In the first one we find a candidate for the T-periodic solution, following a procedure similar to the one in [3] (see also [14]): we introduce a sequence of equations which approximates (1.1) and, once we have found a T-periodic solution for each approximating equation, we pass to the limit. In the second step we verify that this limit function satisfies the conditions defining a bouncing solution.

1st *step: find a candidate* $\bar{x}$. Let $C > 0$ be such that

$$|g(t,x)| \leq \frac{C}{2}(x+1) \,, \qquad \text{for every } t \in [0,T] \text{ and } x \in [0,+\infty[\;. \tag{2.5}$$

Fix $\delta \in]0, \frac{1}{2}[$ and let $(g_n)_n$ be a sequence of continuous functions, which are T-periodic in the first variable and Lipschitz continuous in the second one, converging uniformly to g. Define, for every positive integer n,

$$h_n(t,x) = \begin{cases} g_n(t,x) & \text{if } x \geq \frac{1}{n} \\ nx(g_n(t,x)+\delta) - \delta & \text{if } 0 < x < \frac{1}{n} \\ nx - \delta & \text{if } x \leq 0\,. \end{cases} \tag{2.6}$$

We can assume without loss of generality that all h_n verify (2.5), i.e.,

$$|h_n(t,x)| \leq \frac{C}{2}(x+1)\,, \qquad \text{for every } t \in [0,T] \text{ and } x \in [0,+\infty[\ , \tag{2.7}$$

and that (1.3) holds for g_n instead of g, uniformly in n, slightly modifying, if necessary, the constants μ_1 and μ_2, without affecting (1.4).

Consider the equation

$$x'' + h_n(t,x) = 0\,. \tag{2.8}$$

We will prove the existence of a T-periodic solution to this equation, for every sufficiently large integer n, using [10, Theorem 2.3]. In order to have the necessary estimates on the solutions, we will briefly recall the main points of the proof given there.

The previous equation, written in the phase-plane setting, becomes the first order system

$$\begin{cases} x' = y \\ y' = -h_n(t,x)\,, \end{cases} \tag{2.9}$$

which we can rewrite as $u' = f_n(t,u)$, being $u = (x,y)$ and $f_n(t,u) = (y, -h_n(t,x))$. It is easily seen that, for a suitable $D > 0$,

$$h_n(t,x)x + y^2 \geq D(x^2+y^2) = D\|u\|^2\,, \tag{2.10}$$

when u is large enough in norm, thus giving a bound from below to the angular velocity of those solutions $(x(t), y(t))$ to (2.9) which remain large enough in norm, for every $t \in [0,T]$. Moreover, by (1.3) and (2.6), the angular speed of those solutions is controlled by the angular speed of the solutions of the equations

$$x'' + \mu_i x^+ - nx^- = 0\,, \qquad i = 1,2\,,$$

in the phase-plane, whose periods satisfy

$$\lim_{n\to\infty}\left(\frac{\pi}{\sqrt{\mu_i}} + \frac{\pi}{\sqrt{n}}\right) = \frac{\pi}{\sqrt{\mu_i}}\,.$$

Hence, by (1.4), any solution to (2.9) which remains large enough in norm makes more than N, and less than $N+1$ clockwise rotations around the origin, in the time T.

For every sufficiently large n, by [10, Proposition 2.5], it is possible to construct an admissible spiral γ_n (cf. [10, Definition 2.2]), so that [10, Theorem 2.3] provides a T-periodic solution to equation (2.9). We recall that, roughly speaking, an admissible spiral is a curve in the plane, rotating clockwise around the origin, whose amplitude goes to infinity together with the number of rotations, which controls the solutions of the differential equation, in the sense that, if a solution intersects the curve at a certain point, then it has to cross it from the outer to the inner part of it.

Unfortunately, this achievement is not sufficient for us here, since we need more precise estimates on these solutions, independently of n. Indeed, we need a uniform bound, leading, by a compactness argument, to the convergence of these solutions to a limit function, which will be the candidate for being a bouncing solution. Hence, we now briefly describe how to modify the argument in the proof of [10, Theorem 2.3] in order to overcome this difficulty. In particular, we need to construct a more suitable admissible spiral γ_n.

Set $\Pi^+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ and $\Pi^- = \{(x, y) \in \mathbb{R}^2 : x \leq 0\}$. By (2.7), we can find two positive constants C_1 and C_2 such that, when $x \geq 0$,

$$\begin{aligned} \langle f_n(t, u) \,, u\rangle &= xy - y\, h_n(t, x) \\ &\leq |y| \left(x + \frac{C}{2}(x + 1) \right) \\ &\leq C_1(x^2 + y^2) + C_2 \,, \end{aligned} \tag{2.11}$$

for every n. In this way, we have an estimate of the radial growth of a solution to (2.9), which, together with the behavior of the angular velocity in (2.10), gives us a control on the direction of the vector field associated to (2.9). This permits us to build, as in [10, Proposition 2.5], a branch of the spiral γ_n in Π^+ starting from a point $P_0 = (0, y_0)$, with $y_0 > 0$ (see also [11, Lemma 4.2]). The choice of y_0 will be clarified later. This curve will rotate clockwise in Π^+ intersecting the y-axis in $P_1 = (0, -y_1)$ with $y_1 > y_0 > 0$. Recalling that the values C_1 and C_2 in (2.11) could be found independently of n, we can choose all γ_n to coincide in this region, starting from P_0, with final point P_1.

Now we explain how to construct γ_n in Π^-, for a fixed n. In this region all the solutions to (2.8) are such that

$$x'' + nx - \delta = 0 \,. \tag{2.12}$$

If we extend this equation to the whole real line, we find that the orbits of this equation in the phase-plane are ellipses determined by a non-negative parameter c satisfying

$$y^2 + nx^2 - 2\delta x = c^2 \,. \tag{2.13}$$

We will identify c^2 as the energy of the orbit. Notice that all these ellipses intersect the y-axis at the same points $(0, \pm c)$, independently of n. On the other hand, the intersections with the x-axis are

$$x_1(n) = \frac{\delta - \sqrt{\delta^2 + c^2 n}}{n} \,, \qquad x_2(n) = \frac{\delta + \sqrt{\delta^2 + c^2 n}}{n} \,. \tag{2.14}$$

We can see that the sequence $(x_1(n))_n$ is strictly increasing and converges to 0. We define the part of the spiral γ_n in Π^-, starting from the point $P_1 = (0, -y_1)$, as the curve, parametrized by the polar angle, with linearly increasing energy (see Figure 2), so that all the solutions to (2.9) which intersect γ_n will necessarily enter inside it. Precisely, let, for $\theta \in [0, \pi]$,

$$\gamma_n(\theta) = -|\gamma_n(\theta)|(\sin\theta, \cos\theta) = (\xi_n(\theta), \upsilon_n(\theta))$$

where

$$\upsilon_n(\theta)^2 + n\,\xi_n(\theta)^2 - 2\delta\,\xi_n(\theta) = \left(y_1 + \frac{\theta}{\pi} \right)^2 \,.$$

The final point of γ_n in Π^- is $P_2 = (0, y_1 + 1)$, which is independent of n. Now the construction continues, iterating this procedure. It is important to notice that all the intersection points with the vertical axis are independent of n.

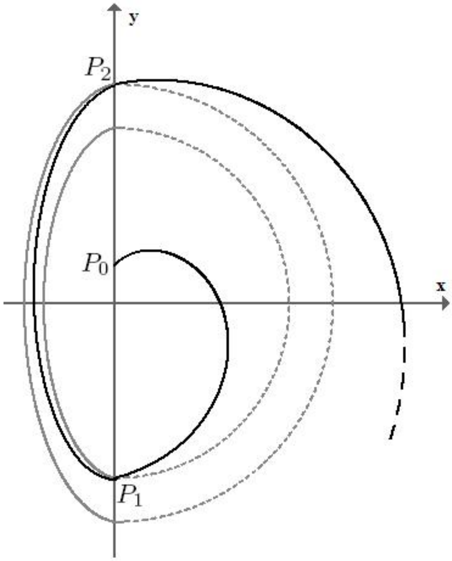

Figure 2: *It is shown how to construct the first lap of the spiral γ_n, in black. In Π^+ two half-balls are drawn, in dotted grey, and in Π^- two branches of two orbits of equation (2.12), in grey. These branches locate two particular examples of the set E_r^n.*

Define $E_r^n = (\Pi^- \cap \Omega_r^n) \cup (\Pi^+ \cap B_r)$, where B_r is the open ball of radius r centered in the origin and Ω_r^n is the interior region delimited by the orbit with energy r^2 of equation (2.12) (see Figure 2). By the above arguments, it is possible to find a positive integer n_0 and a constant $R_1 > 0$ such that, for every $n \geq n_0$, all the solutions to (2.9) which remain outside $E_{R_1}^n$ for all times in $[0, T]$ necessarily make more than N, and less than $N + 1$ clockwise rotations around the origin, in the time T.

We now fix $y_0 = R_1$, so that the construction of the spiral γ_n is made starting from $P_0 = (0, R_1)$. After $N + 1$ laps around the origin, the spiral γ_n intersects the y-axis in a certain point $(0, R_2)$, and after $2N + 2$ laps in $(0, R_3)$. We have $R_1 < R_2 < R_3$, and these constants are independent of n.

It is now possible to apply the Poincaré–Bohl Theorem to the Poincaré map associated to (2.9), restricted to the closure of the set $E_{R_2}^n$. This map takes its values in $E_{R_3}^n$, since a solution starting from a point in the closure of $E_{R_2}^n$ would have to perform at least $N + 1$ rotations around the origin to exit from $E_{R_3}^n$, thus needing a time larger than T to do this. In order to verify the hypothesis of the Poincaré–Bohl Theorem we take $Q \in \partial E_{R_2}^n$ and distinguish two cases: a solution $u_n(t)$ to (2.9), starting from Q, enters $E_{R_1}^n$, for some $t \in [0, T]$, or not. In the first case, once entered $E_{R_1}^n$, the solution cannot exit from $E_{R_2}^n$ in the time T (since it would have to perform $N+1$ rotations around the origin). In the other case, we know that the solution cannot perform an integer number of rotations around the origin, in the time T, for every sufficiently large n. In any case, $u_n(T) \neq \lambda Q$ for every $\lambda \geq 1$, so that the Poincaré–Bohl Theorem can be applied (cf. [10]).

So, for every $n \geq n_0$, there exists a fixed point of the Poincaré map in the closure of $E_{R_2}^n$, giving us a T-periodic solution u_n to (2.9). Moreover, $u_n(t) \in E_{R_3}^n$ for every $t \in [0, T]$. Set $\Sigma = E_{R_3}^{n_0}$. Since

R_1, R_2 and R_3 do not depend on n, and being

$$E_r^n \supset E_r^{n+1} \qquad \text{for every } n \in \mathbb{N} \text{ and } r > 0\,,$$

we have that

$$u_n(t) = \big(x_n(t), x_n'(t)\big) \in \Sigma\,, \text{ for every } t \in [0,T] \text{ and every } n \geq n_0\,. \tag{2.15}$$

We have thus obtained the needed estimates we were looking for.

We have found a sequence $(x_n)_n$ of T-periodic C^1-functions which are uniformly bounded, together with their derivatives. By Ascoli–Arzelà Theorem we can then find a T-periodic continuous function $\bar{x}$ such that, up to a subsequence, $x_n \to \bar{x}$ uniformly in $[0,T]$. This function $\bar{x}$ is the candidate for being the bouncing solution.

2nd step: prove that $\bar{x}$ is a bouncing solution. First of all, let us check that $\bar{x}(t)$ is non negative. Since $\big(x_n(t), x_n'(t)\big) \in E^n_{R_3}$, for every sufficiently large n, using (2.14) we see that

$$x_n(t) \geq \frac{\delta - \sqrt{\delta^2 + R_3^2\, n}}{n}\,, \qquad \text{for every } t \in [0,T]\,.$$

Hence, passing to the limit, we have that $\bar{x}(t) \geq 0$ for every $t \in [0,T]$.

Now we verify the four properties that characterize a bouncing solution.

`First property.` Suppose that there exists a t_0 such that $\bar{x}(t_0) > 0$. Therefore, there exist $\varepsilon > 0$ and a positive integer m such that, for every $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$,

$$\bar{x}(t) > \frac{1}{m} \quad \text{and} \quad x_n(t) > \frac{1}{m}\,, \qquad \text{for every } n \geq m\,.$$

Hence, $h_n(t, x_n(t)) = g_n(t, x_n(t))$ in $[t_0 - \varepsilon, t_0 + \varepsilon]$, for every $n \geq m$, thus converging uniformly to $g(t, \bar{x}(t))$. Using (2.15), a standard compactness argument shows that the sequence $(x_n)_n$ C^2-converges to $\bar{x}$ on $[t_0 - \varepsilon, t_0 + \varepsilon]$, and $\bar{x}$ solves the differential equation $x'' + g(t,x) = 0$ in that interval.

`Second property.` Let now t_0 be an isolated zero of $\bar{x}$. Then, there exists $\alpha > 0$ such that $0 < \bar{x}(t) \leq 1$, for every $t \in [t_0 - \alpha, t_0 + \alpha] \setminus \{t_0\}$ and, by (2.5), $|\bar{x}''(t)| = |g(t, \bar{x}(t))| \leq C$. We claim that the limit $\lim_{t \to t_0^-} \bar{x}'(t)$ exists and is finite. On the contrary, there would exist a constant $\chi > 0$ and two sequences $(a_k)_k$ and $(b_k)_k$, such that $a_k, b_k \in]t_0 - \frac{1}{k}, t_0[$ and $|\bar{x}'(a_k) - \bar{x}'(b_k)| \geq \chi$, for every k. By Lagrange Theorem, for some ξ_k between a_k and b_k,

$$C \geq |\bar{x}''(\xi_k)| = \frac{|\bar{x}'(a_k) - \bar{x}'(b_k)|}{|a_k - b_k|} \geq k\chi\,,$$

which gives a contradiction when k is large enough. Similarly, $\lim_{t \to t_0^+} \bar{x}'(t)$ exists and is finite, too.

We now multiply the equation $x_n'' + h_n(t, x_n) = 0$ by x_n' and integrate in $[t_0 - \varepsilon, t_0 + \varepsilon]$, taking $\varepsilon < \alpha$, thus obtaining

$$0 = \frac{1}{2} x_n'(t_0 + \varepsilon)^2 - \frac{1}{2} x_n'(t_0 - \varepsilon)^2 + \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} h_n(t, x_n(t))\, x_n'(t)\, dt\,.$$

Since

$$\lim_{n\to\infty} h_n(t, x_n(t))x_n'(t) = g(t, \bar{x}(t))\bar{x}'(t)\,,$$

pointwise in $[t_0 - \varepsilon, t_0 + \varepsilon] \setminus \{t_0\}$, using Lebesgue dominated convergence Theorem we have

$$0 = \frac{1}{2}\bar{x}'(t_0 + \varepsilon)^2 - \frac{1}{2}\bar{x}'(t_0 - \varepsilon)^2 + \int_{t_0-\varepsilon}^{t_0+\varepsilon} g(t, \bar{x}(t))\,\bar{x}'(t)\,dt\,.$$

Passing to the limit as $\varepsilon \to 0$, we see that $\bar{x}'(t_0^+)^2 = \bar{x}'(t_0^-)^2$. Clearly, the only reasonable conclusion is that $\bar{x}'(t_0^+) = -\bar{x}'(t_0^-)$.

`Third property.` Let t_0 be such that $\bar{x}(t_0) = 0$, and $\bar{x}'(t_0^-) = -\eta < 0$. There is an $\alpha > 0$ such that $\bar{x}(t) < 1$ for every $t \in]t_0 - \alpha, t_0 + \alpha[$, and

$$-\frac{3}{2}\eta < \bar{x}'(t) < -\frac{1}{2}\eta\,, \qquad \text{for every } t \in]t_0 - \alpha, t_0[\,. \tag{2.16}$$

In particular, $\bar{x}(t) > 0$ in an interval $]t_0 - \alpha, t_0[$. Set

$$\bar{\tau} = \min\left\{\frac{\eta}{24C}\,, \frac{\alpha}{9}\right\}\,, \tag{2.17}$$

where C is the constant introduced in (2.5). We will prove that $\bar{x}(t)$ has no zeros in $]t_0, t_0 + 8\bar{\tau}[$. Let us fix $\tau \in]0, \bar{\tau}[$. By (2.16),

$$0 < \bar{x}(t_0 - \tau) < \frac{3}{2}\eta\tau\,, \qquad -\frac{3}{2}\eta < \bar{x}'(t_0 - \tau) < -\frac{1}{2}\eta\,.$$

In a neighborhood of $t_0 - \tau$ we have that $(x_n)_n$ C^2-converges to $\bar{x}$ so that, for every n large enough,

$$0 < x_n(t_0 - \tau) < \frac{3}{2}\eta\tau\,, \qquad -\frac{3}{2}\eta < \bar{x}_n'(t_0 - \tau) < -\frac{1}{2}\eta\,.$$

Without loss of generality, we can assume that $x_n < 1$ in $]t_0 - \alpha, t_0 + \alpha[$ so that, by (2.7), as long as x_n remains positive, its second derivatives are bounded:

$$|x_n''(t)| \le C\,, \qquad \text{for every } t \in]t_0 - \alpha, t_0 + \alpha[\text{ such that } x_n(t) \ge 0\,. \tag{2.18}$$

Let $p_1 : \mathbb{R} \to \mathbb{R}$ be the parabola characterized by

$$p_1(t_0 - \tau) = \frac{3}{2}\eta\tau\,, \quad p_1'(t_0 - \tau) = -\frac{1}{2}\eta\,, \quad p_1'' \equiv C\,.$$

The function $p_1(t)$ vanishes at two points, the first of which we denote by t_1. It is easy to see that $t_1 \le t_0 + 5\tau$. By (2.18), we have that $x_n(t) < p_1(t)$ and $x_n'(t) < p_1'(t)$ for all those $t \in [t_0 - \tau, t_1]$ having the property that $x_n(s) \ge 0$ for every $s \in [t_0 - \tau, t\,]$. So, x_n must vanish in $]t_0 - \tau, t_1[$, giving the existence of a $t_1^n \in]t_0 - \tau, t_1[$ such that

$$x_n(t_1^n) = 0 \qquad \text{and} \qquad x_n(t) > 0\,, \text{ for every } t \in [t_0 - \tau, t_1^n[\,.$$

Being $t_1^n \le t_0 + 5\tau$, and $\tau < \bar{\tau}$, by (2.17) we see that

$$x_n'(t_1^n) < p_1'(t_1^n) < p_1'(t_1) < -\eta/4\,.$$

So, in a right neighborhood of t_1^n, the solution is negative and satisfies the differential equation $x_n'' + nx_n - \delta = 0$. Therefore, there exists a $t_2^n < t_1^n + \pi/\sqrt{n}$ such that $x_n(t_2^n) = 0$ and, by the symmetry of the equation, $x_n'(t_2^n) = -x_n'(t_1^n) > \eta/4$. We can suppose that $t_2^n < t_1^n + \tau \le t_0 + 6\tau$, choosing n large enough.

Define p_2^n as the parabola such that

$$p_2^n(t_2^n) = 0\,, \quad (p_2^n)'(t_2^n) = \eta/4\,, \quad (p_2^n)'' \equiv -C\,,$$

and let $t_3^n = t_2^n + \eta/2C$ be its second zero. By (2.17) and the previous construction, the following inequalities hold:

$$t_0 - \tau < t_2^n < t_0 + 6\tau < t_0 + 9\tau < \min\{t_3^n, t_0 + \alpha\}\,.$$

In a right neighborhood of t_2^n, the solution x_n is positive. More precisely, by (2.18), $x_n \ge p_2^n$ in the interval $]t_2^n, \min\{t_3^n, t_0 + \alpha\}[$.

Being $p_2^n(t) = -\frac{C}{2}(t - t_2^n)(t - t_3^n)$, we have that $p_2^n(t_0 + 7\tau) \ge C\tau^2$ and $p_2^n(t_0 + 8\tau) \ge C\tau^2$ and, since p_2^n is concave, the same inequality holds for every $t \in [t_0 + 7\tau, t_0 + 8\tau]$, so that

$$x_n(t) \ge C\tau^2, \qquad \text{for every } t \in [t_0 + 7\tau, t_0 + 8\tau]\,.$$

Notice that both this interval and the value $C\tau^2$ do not depend on n.

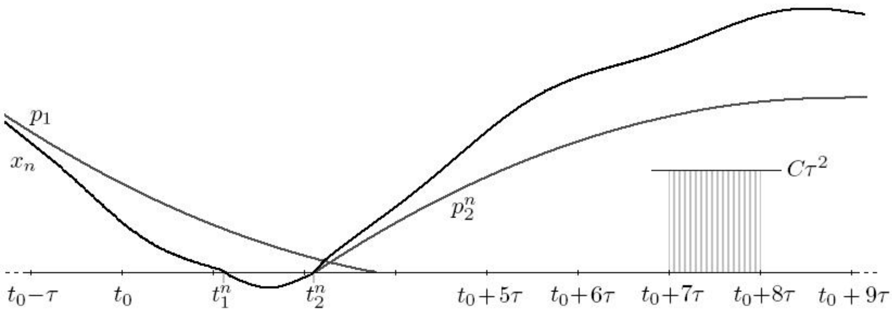

Figure 3: *The figure shows how the two parabolas p_1 and p_2^n control the solution x_n. In the interval* $[t_0 + 7\tau, t_0 + 8\tau]$, *one has that $x_n(t)$ is greater than $C\tau^2$.*

Now we can conclude. Suppose by contradiction that there is a $\bar{t} \in]t_0, t_0 + 8\bar{\tau}[$ such that $\bar{x}(\bar{t}) = 0$. Let $\tau \in]0, \bar{\tau}[$ verify

$$t_0 + 7\tau < \bar{t} < t_0 + 8\tau\,.$$

Then, as shown above, $x_n(\bar{t}) \ge C\tau^2 > 0$ for every n large enough, and we have a contradiction with the fact that $\lim_n x_n(\bar{t}) = \bar{x}(\bar{t}) = 0$.

`Fourth property.` Suppose now that $\bar{x}(t) = 0$ for every t in a non-trivial interval I and assume by contradiction that there exists a $t_0 \in I$ such that $g(t_0, 0) < 0$. Then, there exist $\beta \in]0, \delta[$, a non-trivial interval $J \subset I$ containing t_0, a constant $\varepsilon > 0$ and a positive integer m, with $1/m < \varepsilon$, such that

$$g_n(t, x) < -\beta\,, \qquad \text{for every } t \in J,\ x \in [0, \varepsilon] \text{ and } n > m\,,$$

and

$$x_n(t) \leq \varepsilon \qquad \text{for every } t \in J \text{ and } n > m\,.$$

It is easy to see that, $h_n(t,x) < -\beta$ for every $t \in J$, $x \in [0,\varepsilon]$ and $n > m$. We then have that $x_n''(t) > \beta > 0$, for every $t \in J$ and $n > m$, contradicting the fact that $\lim_n x_n(t) = 0$ for every $t \in J$.

The four properties of a bouncing solution are satisfied by $\bar{x}$, and the proof is thus completed.

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