*Advanced Nonlinear Studies* **12** (2012), 395–408

# Periodic Solutions of Pendulum-Like Hamiltonian Systems in the Plane

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*Dedicated to Jean Mawhin, in the occasion of his 70th birthday*

Received 21 December 2011
*Communicated by Fabio Zanolin*

**Abstract**

By the use of a generalized version of the Poincaré–Birkhoff fixed point theorem, we prove the existence of at least two periodic solutions for a class of Hamiltonian systems in the plane, having in mind the forced pendulum equation as a particular case. Our approach is closely related to the one used by Franks in [15], but the proof remains at a more elementary level.

*2010 Mathematics Subject Classification*. 34C25.
*Key words*. Pendulum equation; Poincaré–Birkhoff theorem; nonlinear dynamics.

## 1 Introduction

For many years, the periodically forced pendulum equation

$$\ddot{x} + a \sin x = e(t) \tag{1.1}$$

has been a fruitful source of mathematical ideas. We refer to [24] for a comprehensive review on this subject.

In 1922, Hamel [17] proved that, if $e(t)$ is $T$-periodic and has mean value equal to zero, i.e.,

$$\int_0^T e(t)\,dt = 0\,, \tag{1.2}$$

then equation (1.1) has at least one $T$-periodic solution. This result, forgotten for a long time, was rediscovered independently by Dancer [8] and Willem [36].

In 1984, Mawhin and Willem [26] proved that, if the above assumption (1.2) holds, equation (1.1) has indeed at least two geometrically distinct $T$-periodic solutions (i.e., solutions not differing by a multiple of $2\pi$). Their proof uses variational methods; the first solution is obtained by minimizing the action functional, as already observed in [8, 17, 36]. Clearly, by the periodicity in $x$, this means that there are infinitely many minimum points, differing from each other by a multiple of $2\pi$. So, considering two of them and applying a mountain pass argument, a critical point of saddle type is obtained, which corresponds to a solution which differs from all the previous ones. The result by Mawhin and Willem, as already noticed in [26], continues to hold if the restoring force $a \sin x$ is replaced by a Carathéodory function of the type $g(t, x)$, which is $T$-periodic in $t$ and $2\pi$-periodic in $x$, and satisfies $\int_0^{2\pi} g(t, x)\,dx = 0$.

In 1988, Franks [15] provided a new proof of the existence of at least two $T$-periodic solutions, by the use of a variant of the Poincaré–Birkhoff fixed point theorem on a cylindrical annulus. His idea is to consider the flow generated by the equivalent first order system on the cylinder obtained by identifying $x = 0$ with $x = 2\pi$, observing that, far from the origin, a twist type property holds on the upper and lower sections of the cylinder.

Recently, in [3] and [2], similar results were obtained for the so-called "relativistic pendulum" equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-(\dot{x}/c)^2}} + a\sin x = e(t)\,,$$

by the use of variational methods combined with topological and nonsmooth analysis techniques. Under the same assumption (1.2), the existence of at least two geometrically distinct $T$-periodic solutions was proved in [2]. On the other hand, in [29], the existence of two $T$-periodic solutions was proved for the equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+(\dot{x}/\gamma)^2}} + a\sin x = e(t)\,,$$

involving a "mean curvature" type operator, where, besides (1.2), a boundedness condition on $E(t) = \int_0^t e(\tau)\,d\tau$ was assumed, i.e., that $\|E\|_\infty < \gamma$. In this case, however, the notion of solution has been extended to the class of functions of bounded variation, so that a solution may exhibit some points of discontinuity.

The pendulum equation has been extended in different directions to systems of differential equations of the type

$$\ddot{x} + \nabla V(t, x) = 0\,,$$

with $x(t) \in \mathbb{R}^N$, or to Hamiltonian systems like

$$J\dot{u} = \nabla H(t, u)\,, \tag{1.3}$$

with $u(t) \in \mathbb{R}^{2N}$, where $J$ is the standard $2N \times 2N$ symplectic matrix, namely

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

See, e.g., [5, 6, 7, 11, 12, 18, 22, 27, 30, 32, 33, 34], where multiplicity results for the periodic problem were proved by variational methods. Following Conley and Zehnder [6, 7], these results are somewhat related to a conjecture by Arnold [1], concerning a possible version of the Poincaré–Birkhoff fixed point theorem in higher dimensions. Recently, some of these results have been extended in [4, 25] to systems of relativistic-type pendulums.

Let us also recall that, under the same assumption (1.2), the existence of infinitely many quasi-periodic solutions for the pendulum equation has been proved in [20, 37] by the use of the Kolmogorov-Arnold-Moser theory.

The aim of this paper is to generalize the result obtained by Mawhin and Willem for the pendulum equation to a planar Hamiltonian system of the type (1.3), by the use of a generalized version of the Poincaré–Birkhoff fixed point theorem on a planar annulus (see [9, 31]). We will assume uniqueness and global existence for the solutions of Cauchy problems and, writing $u = (x, y)$, that the Hamiltonian function $H(t, x, y)$ is $2\pi$-periodic in $x$, and satisfies some kind of monotonicity condition in $y$, for $|y|$ large enough. More precisely, we will ask the function $y \mapsto H(t, x, y)$ to be strictly increasing when $y$ is large and positive, and strictly decreasing for $y$ large and negative.

The idea of the proof is to transform the Hamiltonian system into a new one, where $x$ plays the role of an angle and $y$ is related to the radial displacement. For a sufficiently large positive number $\alpha$, we restrict our analysis to the half plane

$$\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 : y > -\alpha\},$$

and, by the transformation

$$\varphi_\alpha(x, y) = \Big( \sqrt{2(y+\alpha)}\cos x\,, -\sqrt{2(y+\alpha)}\sin x\Big),$$

we obtain a new Hamiltonian system to which the Poincaré–Birkhoff theorem can be applied.

Our method applies to the relativistic pendulum equation and to equations involving the curvature operator, as well. It also provides, under suitable assumptions, the existence and multiplicity of the so-called "periodic solutions of the second kind", i.e., those satisfying, for some integer $k$,

$$u(t+T) = u(t) + (2\pi k, 0)\,, \qquad \text{for every } t \in \mathbb{R}\,.$$

Such a problem has already been faced for pendulum-like equations in [23]. We will also consider the existence of "subharmonic solutions of the second kind", defined as above, when $T$ is replaced by an integer multiple of $T$.

# 2 The main results

We consider the planar Hamiltonian system

$$J\dot{u} = \nabla H(t, u) \tag{2.1}$$

where $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is $T$-periodic in the first variable $t$, and such that $\nabla H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, the gradient with respect to the second variable $u$, is a Carathéodory function, locally Lipschitz continuous in $u$. More precisely,

(*i*) $\nabla H(\cdot, u)$ is measurable, for every $u \in \mathbb{R}^2$;

(*ii*) for every compact set $K \subseteq \mathbb{R}^2$, there exists a function $\ell_K \in L^1(0,T)$ such that, for almost every $t \in [0,T]$ and for every $u_1, u_2 \in K$,

$$|\nabla H(t,u_1) - \nabla H(t,u_2)| \leq \ell_K(t)|u_1 - u_2|.$$

We look for $T$-periodic solutions of system (2.1). To this aim, we introduce the following three assumptions. The first one says that $\nabla H(t,u)$ has an at most linear growth in $u$.

**Assumption A1.** *There are a constant $c_1 \geq 0$ and a function $c_2 \in L^1(0,T)$ such that*

$$|\nabla H(t,u)| \leq c_1|u| + c_2(t)\,, \qquad \text{for a.e. } t \in [0,T] \text{ and every } u \in \mathbb{R}^2.$$

In order to state the next two assumptions, it is more convenient to write system (2.1) in its equivalent form

$$\dot{x} = \frac{\partial H}{\partial y}(t,x,y)\,, \qquad \dot{y} = -\frac{\partial H}{\partial x}(t,x,y)\,.$$

We now ask that $H(t,x,y)$ be $2\pi$-periodic in $x$.

**Assumption A2.** *The function $H$ satisfies*

$$H(t,x+2\pi,y) = H(t,x,y)\,, \qquad \text{for a.e. } t \in [0,T] \text{ and every } (x,y) \in \mathbb{R}^2.$$

In the third assumption we impose a sign condition on $\frac{\partial H}{\partial y}(t,x,y)$ when $|y|$ is large enough.

**Assumption A3.** *There is a constant $d > 0$ such that*

$$|y| \geq d \quad \Rightarrow \quad \mathrm{sgn}(y)\,\frac{\partial H}{\partial y}(t,x,y) > 0\,,$$

*for almost every $t \in [0,T]$ and every $x \in [0,2\pi]$.*

We can now state our main result. We say that two solutions are *geometrically distinct* if they do not differ by a multiple of $2\pi$. In the sequel, when quoting multiplicity results, we always refer to geometrically distinct solutions, even if not explicitly stated.

**Theorem 2.1** *If Assumptions* A1*,* A2*, and* A3 *hold, then system (2.1) has at least two geometrically distinct $T$-periodic solutions.*

As an example, let us consider the forced pendulum equation

$$\ddot{x} + a\sin x = e(t)\,, \tag{2.2}$$

where $e : \mathbb{R} \to \mathbb{R}$ is a $T$-periodic, locally integrable function, satisfying the zero mean condition (1.2). Setting $E(t) = \int_0^t e(\tau)\,d\tau$, we can write the equivalent system

$$\dot{x} = y + E(t)\,, \qquad \dot{y} = -a\sin x\,.$$

We thus have a Hamiltonian system, with

$$H(t,x,y) = \tfrac{1}{2}\,y^2 + E(t)\,y - a\cos x\,,$$

which clearly satisfies Assumptions A1, A2 and A3. So, Theorem 2.1 guarantees the existence of at least two $T$-periodic solutions to equation (2.2), a result first proved by Mawhin and Willem [26].

The same will be true for an equation like

$$\frac{d}{dt}\phi(\dot{x}) + g(t,x) = e(t)\,. \tag{2.3}$$

Here, $\phi : ]-c,c[\, \to \mathbb{R}$, with $c \in\, ]0,+\infty]$, is a diffeomorphism for which

$$\inf\{\phi'(s) : s \in\, ]-c,c\,[\,\} > 0\,. \tag{2.4}$$

Moreover, the function $g(t,x)$ is Carathéodory, $T$-periodic in $t$, locally Lipschitz continuous and $2\pi$-periodic in $x$, and such that

$$\int_0^{2\pi} g(t,x)\,dx = 0\,, \tag{2.5}$$

while the function $e : \mathbb{R} \to \mathbb{R}$ is $T$-periodic, locally integrable, and satisfying (1.2).

Writing the equivalent system

$$\dot{x} = \phi^{-1}(y + E(t))\,, \qquad \dot{y} = -g(t,x)\,, \tag{2.6}$$

condition (2.4) ensures that $\phi^{-1}$ is Lipschitz continuous, and it is easily seen that Assumptions A1, A2 and A3 are satisfied, so that Theorem 2.1 guarantees the existence of at least two $T$-periodic solutions to equation (2.3). We thus recover a recent result by Bereanu and Torres [2]. Notice however that, in [2], the function $g(t,x)$ was not assumed to be locally Lipschitz continuous in $x$. As an example, we could have

$$\phi(s) = \frac{s}{\sqrt{1-(s/c)^2}}\,,$$

leading to the so-called "relativistic pendulum", see [2, 3].

In Section 5 we will see how to generalize this situation so to consider a diffeomorphism $\phi : I_1 \to I_2$, where $I_1$ and $I_2$ are open intervals in $\mathbb{R}$, thus including also the case of the mean curvature operator.

# 3 Proof of Theorem 2.1

Let us first state the version of the Poincaré–Birkhoff theorem [9, 31] which we will use in the proof of our main result (see also [10], and [13] for a review on this theorem).

**Theorem 3.1** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two positive numbers, with $\mathcal{R}_1 < \mathcal{R}_2$, and consider the annulus*

$$\mathcal{A} = \{(u,v) \in \mathbb{R}^2 : \mathcal{R}_1 \le \sqrt{u^2+v^2} \le \mathcal{R}_2\}\,.$$

*Let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving homeomorphism such that $\Psi(0,0) = (0,0)$. On the universal covering space $\{(\varphi,r) : \varphi \in \mathbb{R},\ r > 0\}$, with the standard covering projection $\Pi : (\varphi,r) \mapsto (r\cos\varphi, r\sin\varphi)$, consider a lifting of $\Psi|_{\mathcal{A}}$ of the form*

$$\mathcal{F}(\varphi,r) = (\varphi + \gamma(\varphi,r), \eta(\varphi,r))\,,$$

*where $\gamma(\varphi,r)$ and $\eta(\varphi,r)$ are continuous functions, $2\pi$-periodic in their first variable. Assume the twist condition*

$$\gamma(\varphi,r) > 2\pi\kappa\,, \ \ \textit{if}\ \ r = \mathcal{R}_1\,; \qquad \gamma(\varphi,r) < 2\pi\kappa\,, \ \ \textit{if}\ \ r = \mathcal{R}_2\,,$$

*for some $\kappa \in \mathbb{Z}$. Then, $\Psi$ has two fixed points $(u_1, v_1)$, $(u_2, v_2)$ in the interior of $\mathcal{A}$, such that*

$$\gamma(\Pi^{-1}(u_1, v_1)) = \gamma(\Pi^{-1}(u_2, v_2)) = 2\pi\kappa\,.$$

We now start the proof of Theorem 2.1. For any $u_0 \in \mathbb{R}^2$, let us denote by $u(t; t_0, u_0)$ the solution of the Cauchy problem

$$\begin{cases} J\dot{u} = \nabla H(t, u) \\ u(t_0) = u_0\,, \end{cases} \tag{3.1}$$

which is unique and globally defined, by the assumptions made on $H$. Writing $u_0 = (x_0, y_0)$, and correspondingly

$$u(t; t_0, u_0) = (x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))\,,$$

we see that the periodicity Assumption A2 implies that

$$x(t; t_0, x_0 + 2\pi, y_0) = x(t; t_0, x_0, y_0) + 2\pi\,,$$
$$y(t; t_0, x_0 + 2\pi, y_0) = y(t; t_0, x_0, y_0)\,.$$

Moreover, as a consequence of the linear growth Assumption A1 and the periodicity Assumption A2, we have that

$$\left|\frac{\partial H}{\partial x}(t, x, y)\right| \leq c_1|y| + \tilde{c}_2(t)\,, \tag{3.2}$$

with $\tilde{c}_2(t) = 2\pi c_1 + c_2(t)$, for almost every $t \in \mathbb{R}$ and every $(x, y) \in \mathbb{R}^2$. The following proposition is a rather classical application of the Gronwall inequality, once (3.2) is known.

**Proposition 3.1** *One has*

$$\lim_{y_0 \to +\infty} y(t; t_0, x_0, y_0) = +\infty\,, \qquad \lim_{y_0 \to -\infty} y(t; t_0, x_0, y_0) = -\infty\,,$$

*uniformly for every $t, t_0 \in [0, T]$ and $x_0 \in [0, 2\pi]$.*

Let $d > 0$ be the constant introduced in Assumption A3. By Proposition 3.1 there is a $D \geq d$ such that

$$\begin{aligned} y_0 \leq -D \ &\Rightarrow\ y(t; 0, x_0, y_0) \leq -d\,, \\ y_0 \geq D \ &\Rightarrow\ y(t; 0, x_0, y_0) \geq d\,, \end{aligned} \tag{3.3}$$

for every $t \in [0, T]$ and $x_0 \in [0, 2\pi]$. Moreover, there is a $\beta > D$ such that

$$y_0 \geq -D \ \Rightarrow\ y(t; 0, x_0, y_0) \geq -\beta\,, \quad \text{for every } t \in [0, T] \text{ and } x_0 \in [0, 2\pi]\,. \tag{3.4}$$

We will now transform system (2.1) into a new system, for which it will be possible to apply the Poincaré–Birkhoff fixed point theorem. Let us fix an $\alpha > \beta$. Set

$$\Omega_\alpha = \{(x, y) \in \mathbb{R}^2 : y > -\alpha\}\,,$$

and let $\varphi_\alpha : \Omega_\alpha \to \mathbb{R}^2 \setminus \{(0, 0)\}$ be defined as

$$\varphi_\alpha(x, y) = \left(\sqrt{2(y + \alpha)}\cos x\,, -\sqrt{2(y + \alpha)}\sin x\right).$$

Notice that the determinant of the Jacobian matrix $\varphi'_\alpha(x,y)$ is equal to 1, for every $(x,y) \in \Omega_\alpha$. Setting

$$\widehat{\Omega}_\alpha = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2\pi,\ y > -\alpha\},$$

we thus have that the function

$$\widehat{\varphi}_\alpha : \widehat{\Omega}_\alpha \to \mathbb{R}^2 \setminus \{(\xi,\eta) \in \mathbb{R}^2 : \xi \geq 0,\ \eta = 0\},$$

with $\widehat{\varphi}_\alpha(u) = \varphi_\alpha(u)$, is a symplectic diffeomorphism, i.e.,

$$\widehat{\varphi}_\alpha'(u)^{\mathcal{T}} J\, \widehat{\varphi}_\alpha'(u) = J\,, \tag{3.5}$$

for every $u \in \widehat{\Omega}_\alpha$, cf. [16]. (Here $\mathbb{M}^{\mathcal{T}}$ denotes the transposed of a matrix $\mathbb{M}$.) Then, using (3.5), one easily sees that the function $\widehat{\varphi}_\alpha$ transforms the Hamiltonian system (2.1) on $\widehat{\Omega}_\alpha$ into the Hamiltonian system

$$J\dot{v} = \nabla L_\alpha(t,v)\,, \tag{3.6}$$

on $\mathbb{R}^2 \setminus \{(\xi,\eta) \in \mathbb{R}^2 : \xi \geq 0,\ \eta = 0\}$, where

$$L_\alpha(t,v) = H(t, \widehat{\varphi}_\alpha^{-1}(v))\,.$$

(As usual, we denote by $\nabla L_\alpha$ the gradient with respect to the second variable.)

We now need the periodicity Assumption A2 in order to extend the new Hamiltonian function $L_\alpha$ to $\mathbb{R}^2 \setminus \{(0,0)\}$, preserving the regularity assumptions. We will still denote by $L_\alpha(t,v)$ such an extension. Notice that, if $u = (x,y)$ refers to the original system (2.1), then, for the new system (3.6), $x$ plays the role of the clockwise angular displacement.

In order to apply Theorem 3.1, we need to modify (3.6). Consider a $C^2$-function $\chi : [0,+\infty[\, \to [0,+\infty[$ , such that

$$\chi(r) = \begin{cases} 1 & \text{if } r \geq \sqrt{2(\alpha-\beta)}\,, \\[2ex] 0 & \text{if } r \leq \dfrac{1}{2}\sqrt{2(\alpha-\beta)}\,. \end{cases}$$

Let $\widetilde{L}_\alpha : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\widetilde{L}_\alpha(t,v) = \begin{cases} \chi(|v|) L_\alpha(t,v) & \text{if } v \neq 0\,, \\[2ex] 0 & \text{if } v = 0\,. \end{cases}$$

This function is $T$-periodic in the first variable $t$, and $\nabla\widetilde{L}_\alpha : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$, the gradient with respect to the second variable $v$, is a Carathéodory function, locally Lipschitz continuous in $v$, and has an at most linear growth in $v$. Since $\widetilde{L}_\alpha$ and $L_\alpha$ coincide on the set $\{v \in \mathbb{R}^2 : |v| \geq \sqrt{2(\alpha-\beta)}\}$, we have that the Hamiltonian system

$$J\dot{v} = \nabla\widetilde{L}_\alpha(t,v) \tag{3.7}$$

still describes the original system (2.1), as long as $u = (x,y)$ satisfies $y \geq -\beta$. We will denote by $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ the Poincaré map associated to this new system.

Let us define the annulus

$$\begin{aligned}\mathcal{A} &= \varphi_\alpha(\overline{\Omega}_{-D} \setminus \Omega_D) \\ &= \overline{B}\Big((0,0),\ \sqrt{2(D+\alpha)}\Big) \setminus B\Big((0,0),\ \sqrt{2(-D+\alpha)}\Big).\end{aligned}$$

Notice that $L_\alpha$ and $\widetilde{L}_\alpha$ coincide on $\mathcal{A}$. We want to apply Theorem 3.1 to the map $\mathcal{P}$ over the annulus $\mathcal{A}$. By Liouville Theorem we know that $\mathcal{P}$ is an area-preserving homeomorphism, and since it coincides with the identity on the disk $B\Big((0,0), \frac{1}{2}\sqrt{2(\alpha-\beta)}\Big)$, we have that $\mathcal{P}(0,0) = (0,0)$.

We now check the twist condition. If $|v_0| = \sqrt{2(-D+\alpha)}$, let $x_0 \in [0, 2\pi[$ be such that

$$\varphi_\alpha^{-1}(v_0) = \{(x_0 + 2k\pi, -D) : k \in \mathbb{Z}\}.$$

Then, by (3.3) and (3.4),

$$-\beta \le y(t; 0, x_0, -D) \le -d, \qquad \text{for every } t \in [0,T], \tag{3.8}$$

and, by Assumption A3, $x(\cdot\,; 0, x_0, -D)$ is strictly decreasing on $[0,T]$, so we conclude that

$$x(T; 0, x_0, -D) < x_0.$$

Similarly, if $|v_0| = \sqrt{2(D+\alpha)}$, taking $x_0 \in [0, 2\pi[$ for which

$$\varphi_\alpha^{-1}(v_0) = \{(x_0 + 2k\pi, D) : k \in \mathbb{Z}\},$$

we have that $y(t; 0, x_0, D) \ge d$, for every $t \in [0,T]$. Hence, by Assumption A3, $x(\cdot\,; 0, x_0, D)$ is strictly increasing on $[0,T]$, so that

$$x(T; 0, x_0, D) > x_0.$$

Since, as long as $y \ge -\beta$, we know that $x$ gives the clockwise angular displacement for system (3.7), the twist condition is verified, and Theorem 3.1 applies, with $\kappa = 0$, providing the existence of two fixed points $v_0^{(1)}, v_0^{(2)}$ of $\mathcal{P}$ in $\mathcal{A}$, such that, taking $u_0^{(j)} = (x_0^{(j)}, y_0^{(j)}) \in \varphi_\alpha^{-1}(v_0^{(j)})$, one has

$$x(T; 0, x_0^{(j)}, y_0^{(j)}) = x_0^{(j)},$$

for $j = 1, 2$. Therefore, $u_0^{(1)}$ and $u_0^{(2)}$ are the starting points of the two $T$-periodic solutions of system (2.1) we are looking for. The proof is thus completed.

# 4 Periodic solutions of the second kind

In this section we study the existence of the so-called "periodic solutions of the second kind", those satisfying, for some integer $k$,

$$u(t+T) = u(t) + (2\pi k, 0), \qquad \text{for every } t \in \mathbb{R}. \tag{4.1}$$

These are sometimes called "running solutions". Clearly, if $k = 0$, we recover the usual periodicity. Notice that (4.1) is equivalent to ask that the function

$$\tilde{u}_1(t) = u(t) - \Big(\frac{2\pi k}{T}t, 0\Big)$$

be $T$-periodic. Let us introduce a new assumption.

**Assumption A4.** *There are two integers $k_1$, $k_2$, with $k_1 \leq k_2$ and two constants $d_1 < d_2$, such that*

$$y \leq d_1 \quad \Rightarrow \quad \frac{\partial H}{\partial y}(t, x, y) < \frac{2\pi k_1}{T}\,,$$
$$y \geq d_2 \quad \Rightarrow \quad \frac{\partial H}{\partial y}(t, x, y) > \frac{2\pi k_2}{T}\,,$$

*for almost every $t \in [0, T]$ and every $x \in [0, 2\pi]$.*

The following theorem generalizes Theorem 2.1.

**Theorem 4.1** *If Assumptions* A1, A2, *and* A4 *hold, then, for every integer $k \in \{k_1, k_1 + 1, \ldots, k_2\}$, system (2.1) has at least two geometrically distinct periodic solutions of the second kind, satisfying (4.1).*

The proof is a straightforward modification of the proof of Theorem 2.1. The only difference is that Theorem 3.1 applies, this time, with $\kappa = k-1$, with $k = k_1, k_1+1, \ldots, k_2$, so that Assumption A4 provides the necessary twist property.

For a second order differential equation like (2.3), we have the following.

**Corollary 4.1** *Taking $\phi : ]-c, c[ \to \mathbb{R}$ as in Section 2, with $c \in ]0, +\infty]$, for any integer $k$ in the interval $]-\frac{T}{2\pi}c, \frac{T}{2\pi}c[$, equation (2.3) has at least two geometrically distinct periodic solutions of the second kind, satisfying*

$$x(t + T) = x(t) + 2\pi k\,, \quad \textit{for every } t \in \mathbb{R}\,.$$

*Proof.* It is sufficient to observe that, in this case,

$$\lim_{y\to-\infty} \frac{\partial H}{\partial y}(t, x, y) = \lim_{y\to-\infty} \phi^{-1}(y + E(t)) = -c\,,$$
$$\lim_{y\to+\infty} \frac{\partial H}{\partial y}(t, x, y) = \lim_{y\to+\infty} \phi^{-1}(y + E(t)) = c\,,$$

uniformly for every $t \in [0, T]$ and every $x \in [0, 2\pi]$. If $c$ is finite, we take $k_2$ to be the largest integer less than $\frac{T}{2\pi}c$; otherwise, if $c = +\infty$, $k_2$ can be an arbitrary positive integer. Setting $k_1 = -k_2$, we see that Assumption A4 is verified. Theorem 4.1 then applies. ■

We could also look for "subharmonic solutions of the second kind", those satisfying, for some integer $k$, and for some positive integer $m$,

$$u(t + mT) = u(t) + (2\pi k, 0)\,, \quad \text{for every } t \in \mathbb{R}\,. \tag{4.2}$$

To this aim, we introduce the following.

**Assumption A5.** *For some positive integer $m$, there are two integers $k_1$, $k_2$, with $k_1 \leq k_2$ and two constants $d_1 < d_2$, such that*

$$y \leq d_1 \quad \Rightarrow \quad \frac{\partial H}{\partial y}(t, x, y) < \frac{2\pi k_1}{mT}\,,$$
$$y \geq d_2 \quad \Rightarrow \quad \frac{\partial H}{\partial y}(t, x, y) > \frac{2\pi k_2}{mT}\,,$$

*for almost every $t \in [0, T]$ and every $x \in [0, 2\pi]$.*

Let us now state the corresponding existence result.

**Theorem 4.2** *If Assumptions* A1, A2, *and* A5 *hold, then, for every nonzero integer* $k \in \{k_1, k_1 + 1, \dots, k_2\}$*, with* $|k|$ *and* $m$ *relatively prime, system (2.1) has at least two geometrically distinct subharmonic solutions of the second kind, satisfying (4.2), for which there is no other* $\ell \in \{1, 2, \dots, m-1\}$ *such that, for some integer* $j$*,*

$$u(t + \ell T) = u(t) + (2\pi j, 0)\,, \quad \textit{for every } t \in \mathbb{R}\,. \tag{4.3}$$

*Proof.* Let $k$ and $m$ be as in the statement. By Theorem 4.1, with $T$ replaced by $mT$, we find two solutions satisfying (4.2). Let $u$ be one of them. Assume by contradiction that (4.3) holds, for some integers $\ell \in \{1, 2, \dots, m-1\}$ and $j$. Let $p$ and $n$ be positive integers such that $p\ell = nm$. By (4.2), it has to be

$$u(t + nmT) = u(t) + (2\pi kn\,, 0), \quad \text{for every } t \in \mathbb{R}\,,$$

and, by (4.3),

$$u(t + p\ell T) = u(t) + (2\pi jp\,, 0), \quad \text{for every } t \in \mathbb{R}\,.$$

Hence, $nk = pj$, so that

$$\frac{k}{m} = \frac{j}{\ell}\,,$$

a contradiction, since $|k|$ and $m$ are relatively prime, and $1 \le \ell \le m-1$. ∎

# 5 Further results

In this section we discuss on possible generalizations of our assumptions and on different applications of our results.

As an alternative to Assumption A3, let us introduce the following.

**Assumption A3′.** *There exists a* $\zeta \in L^1(0, T)$ *such that*

$$\operatorname{sgn}(y)\,\frac{\partial H}{\partial y}(t, x, y) \ge \zeta(t)\,,$$

*for almost every* $t \in [0, T]$ *and every* $(x, y) \in \mathbb{R}^2$*, and*

$$\int_0^T \limsup_{y \to -\infty} \Big( \sup_{x \in [0, 2\pi]} \frac{\partial H}{\partial y}(t, x, y) \Big)\, dt < 0 < \int_0^T \liminf_{y \to +\infty} \Big( \inf_{x \in [0, 2\pi]} \frac{\partial H}{\partial y}(t, x, y) \Big)\, dt.$$

This is a Landesman-Lazer type of condition. Notice, however, that such kind of condition usually regards the inferior and superior limits as $x$ goes to $\pm\infty$, when dealing with second order scalar equations. Here, since the nonlinearity is periodic in $x$, and we deal with a more general Hamiltonian system, we propose the above version, with the inferior and superior limits as $y$ goes to $\pm\infty$.

We can then state the following variant of Theorem 2.1.

**Theorem 5.1** *If Assumptions* A1*,* A2*, and* A3′ *hold, then system (2.1) has at least two geometrically distinct $T$-periodic solutions.*

*Proof.* Following [14, Lemma 1], if Assumption A3′ holds, there are a constant $d \geq 1$ and two functions $\psi_1, \psi_2 \in L^1(0, T)$ such that

$$\frac{\partial H}{\partial y}(t, x, y) \leq \psi_1(t), \text{ for a.e. } t \in [0, T], \text{ every } x \in [0, 2\pi], \text{ and every } y \leq -d\,,$$
$$\frac{\partial H}{\partial y}(t, x, y) \geq \psi_2(t), \text{ for a.e. } t \in [0, T], \text{ every } x \in [0, 2\pi], \text{ and every } y \geq d\,,$$

and

$$\int_0^T \psi_1(t)\, dt < 0 < \int_0^T \psi_2(t)\, dt\,.$$

We now follow the lines of the proof of Theorem 2.1, until we obtain (3.8), when $y_0 = -D$. At this point, writing for simplicity $x(t)$, $y(t)$ instead of $x(t; 0, x_0, -D)$, $y(t; 0, x_0, -D)$, respectively, we have that

$$x(T; 0, x_0, -D) - x_0 = \int_0^T \frac{\partial H}{\partial y}(t, x(t), y(t))\, dt \leq \int_0^T \psi_1(t)\, dt < 0\,,$$

so that $x(T; 0, x_0, -D) < x_0$. Similarly, when $y_0 = D$, we get $x(T; 0, x_0, D) > x_0$, and the proof is completed. ■

Let us now focus our attention on Assumption A1: It has been used in order that the Poincaré map be well defined, and to obtain Proposition 3.1. To this aim, it would be enough to assume global existence for the solutions of the Cauchy problems (3.1), cf. [19]. Or else, we could try to follow the proof of Theorem 2.1 anyway, being careful that the Poincaré map be well defined on the annulus we are interested in, and obtaining the desired estimates anyway.

To show how this can be done, consider again the equation

$$\frac{d}{dt}\phi(\dot{x}) + g(t, x) = e(t)\,, \tag{5.1}$$

and assume now that $\phi : ]-c, c[ \to ]-\gamma, \gamma[$ is a diffeomorphism satisfying $\phi(0) = 0$, with $c, \gamma \in ]0, +\infty]$, and such that $\phi^{-1}$ is locally Lipschitz continuous. As before, the function $g(t, x)$ is assumed to be Carathéodory, $T$-periodic in $t$, locally Lipschitz continuous and $2\pi$-periodic in $x$, and such that (2.5) holds. The function $e : \mathbb{R} \to \mathbb{R}$ is $T$-periodic, locally integrable, and satisfying (1.2), so that we can write the equation in its equivalent form (2.6).

By the above assumptions, there is an $h \in L^1(0, T)$ such that

$$|g(t, x)| \leq h(t)\,, \qquad \text{for a.e. } t \in [0, T] \text{ and every } x \in \mathbb{R}\,.$$

Set, for $\varepsilon > 0$ sufficiently small,

$$\tilde{d} = \|E\|_\infty + \varepsilon\,, \quad \tilde{D} = \tilde{d} + \|h\|_1\,, \quad \tilde{\beta} = \tilde{D} + \|h\|_1\,.$$

Then, for any solution of the corresponding Cauchy problem, since

$$|\dot{y}(t; 0, x_0, y_0)| \leq h(t)\,, \quad \text{for a.e. } t \in [0, T]\,,$$

we have that

$$y_0 \notin ]-\tilde{D}, \tilde{D}[ \quad \Rightarrow \quad y(t;0,x_0,y_0) \notin ]-\tilde{d}, \tilde{d}\,[\,,$$

and

$$y_0 \in [-\tilde{D}, \tilde{D}] \quad \Rightarrow \quad y(t;0,x_0,y_0) \in [-\tilde{\beta}, \tilde{\beta}]\,,$$

for every $t \in [0,T]$ and $x_0 \in [0,2\pi]$. Following the lines of the proof of Theorem 2.1, we can thus conclude that, if

$$2(\|E\|_\infty + \|h\|_1) < \gamma\,,$$

then equation (5.1) has at least two geometrically distinct $T$-periodic solutions. Moreover, in the spirit of Theorem 4.1, for every integer $k$ such that

$$|k| < \frac{T}{2\pi}\,\phi^{-1}(\gamma - 2\|E\|_\infty - 2\|h\|_1)\,,$$

equation (5.1) has at least two geometrically distinct periodic solutions of the second kind, satisfying

$$x(t+T) = x(t) + 2\pi k\,, \qquad \text{for every } t \in \mathbb{R}\,.$$

Notice that, since adding a constant to $E(t)$ does not affect the original differential equation, we can always replace $\|E\|_\infty$ by $\frac{1}{2}(\max E - \min E)$.

As an example, we could take

$$\phi(s) = \frac{s}{\sqrt{1+(s/\gamma)^2}}\,, \tag{5.2}$$

(here $c = +\infty$), leading to the mean curvature operator. We then get the following immediate consequence.

**Corollary 5.1** *Assume that $e(t)$ satisfies (1.2) and let $a > 0$ and $\gamma > 0$ be such that $2(\|E\|_\infty + aT) < \gamma$. Then, the equation*

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+(\dot{x}/\gamma)^2}} + a\sin x = e(t)$$

*has at least two geometrically distinct $T$-periodic solutions. Moreover, for every integer $k$ such that*

$$|k| < \frac{T}{2\pi}\,\phi^{-1}(\gamma - 2\|E\|_\infty - 2aT)\,,$$

*where $\phi$ is given by (5.2), there are at least two geometrically distinct periodic solutions of the second kind, satisfying*

$$x(t+T) = x(t) + 2\pi k\,, \qquad \textit{for every } t \in \mathbb{R}\,.$$

The problem of subharmonic solutions of the second kind can be treated similarly, in the spirit of Theorem 4.2. For briefness, we prefer not entering into details.

This type of equations, with the mean curvature operator, has been studied using a variational method by Obersnel and Omari [29], who proved that, under the weaker condition $\|E\|_\infty < \gamma$, there are at least two geometrically distinct $T$-periodic "bounded variation" solutions. Here, however, we are always dealing with regular solutions. See also [35], where a different topological approach has been used. Notice that, as proved in [29, Proposition 1.1], the zero mean condition (1.2) is not sufficient to ensure the existence of $T$-periodic solutions, so that some additional assumption is indeed needed.

More general situations can be considered, where $\phi$ is a diffeomorphism between any two open intervals $I_1$ and $I_2$. These intervals may be bounded or not on one or both sides. We avoid the details, for briefness.

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