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ORIGINAL ARTICLE

Periodic motions in a gravitational central field with a rotating external force

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Abstract We consider a Kepler problem, with an additional rotating external force, and study the existence of periodic solutions when a small perturbative term is introduced. Surprisingly enough, we always get at least one of such solutions. Moreover, if a nonresonance assumption is added, then the existence of a second solution is also proved.

Keywords Periodic solutions · Two-body Kepler problem · Perturbative methods

Mathematics Subject Classification 34C25

1 Introduction and main result

In this paper we consider the existence of periodic solutions of a second order differential equation in the plane of the type

$$\ddot{x} + \gamma \frac{x}{|x|^3} - \alpha \frac{e^{i\omega t}}{|x|^\beta} = \varepsilon F(t, x, \dot{x}; \varepsilon). \tag{1}$$

Hence, $x(t)$ is in $\mathbb{R}^2$, which will be identified, to simplify the notation, with the complex plane $\mathbb{C}$. All constants γ, α, ω and ε are assumed to be positive, and we take $\beta \in [0, 2]$. In particular,

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ω is the frequency of the periodic rotating external force, which has period $T = 2\pi/\omega$. Moreover, ε is assumed to be a small parameter, and the function $F : \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$ is continuous, T-periodic in its first variable, and locally Lipschitz continuous in its second and third variables.

Notice that, when $\alpha = 0$ and $\varepsilon = 0$, Eq. (1) is the classical two-body Kepler problem, for which it is well-known that, for any given period, there is a continuum of periodic orbits. When $\alpha \neq 0$, the additional term $\alpha e^{i\omega t}/|x|^\beta$ introduces a rotating component which naturally leads to the study of T-periodic solutions. To our knowledge, such a model has not yet been considered in the literature.

Equation (1) belongs to the wide family of systems with singularities, that have been studied without interruption since the very foundation of Celestial Mechanics, by using a variety of mathematical techniques including analytical, geometrical, topological and variational methods. Classically, the involved models are autonomous (i.e., not explicitly depending on time), but more recently there has been an increasing interest in the study of non-autonomous systems. An additional parametric or external forcing term may model the influence of anisotropies, or external periodic gravitational fields. A comprehensive revision of the state-of-the-art is virtually impossible, therefore we only quote some examples of the related literature. The use of topological methods in the study of singular systems goes back to Poincaré (1892). More recently, variational methods have been successfully employed in, e.g., Ambrosetti and Coti Zelati (1989), Bahri and Rabinowitz (1989), Bertotti (1991), Capozzi et al. (1988), Chen (2010), Degiovanni et al. (1987), Gordon (1975), Serra and Terracini (1994). See also the book (Ambrosetti and Coti Zelati 1993), and the references therein.

Similar problems have already been considered by the authors of this paper. Radially symmetric systems are studied in Fonda and Toader (2008, 2012), Fonda and Ureña (2011) by using topological methods. In this case, the analysis is simplified by the fact that the radial coordinate is ruled by a scalar equation. In Franco and Torres (2008), Torres (2004) (see also Franco and Webb 2006), fixed-point theorems on compression-expansion of conical sections are applied, leading to periodic solutions that do not rotate around the origin.

The strategy proposed in this paper is different from the previous ones and falls into the family of perturbative methods. Generally speaking, Perturbation Theory aims to find solutions close to known solutions of the unperturbed system. For the importance of Perturbation Theory in natural sciences in general and Celestial Mechanics in particular, one can read the excellent historical reviews (Ito and Tanikawa 2007; Paul 2007). The perturbed two-body problem is a classical topic (see for instance Moser and Zehnder 2005; Siegel and Moser 1971). Recent works where perturbative arguments are applied to singular systems are (Amster et al. 2011; Margheri et al. 2012). Concerning the Kepler problem, there seems to be a still growing interest, as shown by the recent papers (Cabral and Vidal 2000; Cordani 2000; Diacu et al. 2005; Escalona-Buendía and Pérez-Chavela 2008; Gutzwiller 1973; Vidal 2001), just to quote a few.

In the case under consideration, the aim is to identify periodic solutions close to circular ones. As we will see, the inclusion of the additional forcing term $\alpha e^{i\omega t}/|x|^\beta$ fixes the period and, if $\varepsilon = 0$, only two circular solutions survive. The objective is then to find conditions to locally continue such solutions for $\varepsilon \neq 0$ small enough.

Our main result is as follows.

Theorem 1 *For any choice of the positive constants γ, α, ω and of $\beta \in [0, 2]$, there is an $\varepsilon_0 > 0$ such that, if $\varepsilon \in [0, \varepsilon_0]$, then Eq. (1) has at least one T-periodic solution. Moreover, there are at least two T-periodic solutions if the rotating periodic force is such that*

$$\beta \in [0, 2[\,, \quad \text{or} \quad \beta = 2 \text{ and } \alpha < \gamma\,,$$

and

$$\frac{\omega^{2(2-\beta)}}{\alpha^3} \neq \frac{(a(k)+1)^{\beta+1}}{\gamma^{\beta+1}\, a(k)^3}, \quad \textit{for every integer } k \geq 2, \tag{2}$$

where $a(k)$ is defined by

$$a(k) = \begin{cases} \dfrac{(1-\beta)k^2 - 3 + \sqrt{(\beta-3)^2 k^4 + 2(5\beta-7)k^2 + 9}}{2(2-\beta)} & \textit{if } \beta \in [0, 2[, \\ \dfrac{k^2(k^2-1)}{k^2+3} & \textit{if } \beta = 2. \end{cases} \tag{3}$$

Condition (2) is surely satisfied if

$$\gamma \left(\frac{\omega^{2(2-\beta)}}{\alpha^3} \right)^{\frac{1}{\beta+1}} > \frac{19}{12}. \tag{4}$$

The proof of Theorem 1 relies on the spectral analysis of the linearization around equilibria of the system written in a synodic framework. This will be done in the next section. Assumption (2) is introduced in order to avoid some kind of resonance. It is remarkable that it is only needed to guarantee the existence of a second T-periodic solution, while no such condition is necessary for the existence of the first one.

At a first sight, it could seem strange that the perturbing function F is allowed to contain also friction terms. In the case when $\alpha = 0$, this would not be possible, since a friction term would lead to dissipation of energy, thus eventually making the particle fall into the singularity. On the contrary, we will show that, when $\alpha \neq 0$, the rotating term $\alpha e^{i\omega t}/|x|^\beta$ forces the particle to maintain an almost constant rotation speed, and to remain at an almost constant distance from the singularity. More information on the role of dissipative effects in Celestial Mechanics can be found, e.g., in Celletti et al. (2001), Margheri et al. (2012), and the references therein.

2 Proof of theorem 1

After the time rescaling $s = \omega t$, setting

$$z(s) = \left(\frac{\omega^2}{\alpha}\right)^{\frac{1}{\beta+1}} x\left(\frac{s}{\omega}\right),$$

and defining $G : \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ as

$$G(s, z, \zeta; \varepsilon) = \left(\frac{1}{\alpha\omega^{2\beta}}\right)^{\frac{1}{\beta+1}} F\left(\frac{s}{\omega}, \left(\frac{\alpha}{\omega^2}\right)^{\frac{1}{\beta+1}} z,\ \omega\left(\frac{\alpha}{\omega^2}\right)^{\frac{1}{\beta+1}} \zeta\,;\, \varepsilon\right),$$

we get the equivalent equation

$$\ddot{z} + c\frac{z}{|z|^3} = \frac{e^{is}}{|z|^\beta} + \varepsilon G(s, z, \dot{z}\,;\varepsilon), \tag{5}$$

with

$$c = \gamma \left(\frac{\omega^{2(2-\beta)}}{\alpha^3} \right)^{\frac{1}{\beta+1}}. \tag{6}$$

(Notice that, here and below, we maintain the notation $\dot{z}$, $\ddot{z}$ to denote the derivatives with respect to s.) Passing to polar coordinates

$$z(s) = \rho(s)e^{i\theta(s)}\,,$$

Eq. (5) becomes

$$\left(\ddot{\rho} - \rho\,\dot{\theta}^2 + \frac{c}{\rho^2}\right) + i\,(2\dot{\rho}\,\dot{\theta} + \rho\,\ddot{\theta}) = \left(\frac{e^{is}}{\rho^\beta} + \varepsilon G(s, \rho e^{i\theta}, (\dot{\rho} + i\rho\dot{\theta})e^{i\theta}; \varepsilon)\right) e^{-i\theta}\,,$$

which is equivalent to the system

$$\begin{cases} \ddot{\rho} - \rho\,\dot{\theta}^2 + \dfrac{c}{\rho^2} = \dfrac{\cos(s-\theta)}{\rho^\beta} + \varepsilon \widetilde{G}_1(s, \rho, \theta, \dot{\rho}, \dot{\theta}; \varepsilon) \\[2ex] 2\dot{\rho}\,\dot{\theta} + \rho\,\ddot{\theta} = \dfrac{\sin(s-\theta)}{\rho^\beta} + \varepsilon \widetilde{G}_2(s, \rho, \theta, \dot{\rho}, \dot{\theta}; \varepsilon)\,, \end{cases} \tag{7}$$

where

$$\widetilde{G}_1(s, \rho, \theta, \dot{\rho}, \dot{\theta}; \varepsilon) = \Re\Big(G(s, \rho e^{i\theta}, (\dot{\rho} + i\rho\dot{\theta})e^{i\theta}; \varepsilon)\, e^{-i\theta}\Big)\,,$$
$$\widetilde{G}_2(s, \rho, \theta, \dot{\rho}, \dot{\theta}; \varepsilon) = \Im\Big(G(s, \rho e^{i\theta}, (\dot{\rho} + i\rho\dot{\theta})e^{i\theta}; \varepsilon)\, e^{-i\theta}\Big).$$

(For any complex number w, we denote by $\Re(w)$ and $\Im(w)$ its real and imaginary parts, respectively.) This change of variables is justified, since the solutions we are looking for never attain the singularity. Setting $\eta = \dot{\rho}$, $\varphi = s - \theta$, $v = \dot{\varphi}$ and defining

$$\Gamma_1(s, \rho, \eta, \varphi, v; \varepsilon) = \widetilde{G}_1(s, \rho, s - \varphi, \eta, 1 - v; \varepsilon)\,,$$
$$\Gamma_2(s, \rho, \eta, \varphi, v; \varepsilon) = \widetilde{G}_2(s, \rho, s - \varphi, \eta, 1 - v; \varepsilon)\,,$$

we transform (7) into the first order system

$$\begin{cases} \dot{\rho} = \eta \\ \dot{\eta} = \rho\,(1-v)^2 - \dfrac{c}{\rho^2} + \dfrac{\cos\varphi}{\rho^\beta} + \varepsilon\Gamma_1(s, \rho, \eta, \varphi, v; \varepsilon) \\ \dot{\varphi} = v \\ \dot{v} = \dfrac{1}{\rho}\Big(2\eta(1-v) - \dfrac{\sin\varphi}{\rho^\beta} - \varepsilon\Gamma_2(s, \rho, \eta, \varphi, v; \varepsilon)\Big). \end{cases} \tag{8}$$

Let us first consider the unperturbed system, taking $\varepsilon = 0$. This system always has the equilibrium

$$\mathcal{E}_1 = (\rho_1, 0, \pi, 0)\,,$$

where ρ_1 satisfies

$$\rho_1^3\left(1 - \frac{1}{\rho_1^{\beta+1}}\right) = c. \tag{9}$$

Notice that $\rho_1 > 1$. In this case, the orbit of the solution is tuned in such a way that the rotating force is always directed in the same direction as the gravitational force. For instance, in the particular case $\beta = 2$, this equilibrium corresponds to the solution of (1), with $\varepsilon = 0$, given by

$$x_1(t) = -\sqrt[3]{\frac{\gamma + \alpha}{\omega^2}}\, e^{i\omega t}.$$

Moreover, if $\beta \in [0, 2[$, or $\beta = 2$ and $\alpha < \gamma$, then there is a second equilibrium

$$\mathcal{E}_2 = (\rho_2, 0, 0, 0)\,,$$

where ρ_2 satisfies

$$\rho_2^3 \left(1 + \frac{1}{\rho_2^{\beta+1}}\right) = c. \tag{10}$$

In this case, the rotating force is always directed in the opposite direction with respect to the gravitational force. For instance, if $\beta = 2$, this second equilibrium corresponds to the solution of (1), with $\varepsilon = 0$, given by

$$x_2(t) = \sqrt[3]{\frac{\gamma - \alpha}{\omega^2}}\, e^{i\omega t}.$$

Let us first consider the linearization of system (8) with $\varepsilon = 0$ at the equilibrium point $\mathcal{E}_1$, i.e., writing $u = (\rho, \eta, \varphi, v)$ and using (9),

$$\dot{u} = \mathbb{A}_1(u - \mathcal{E}_1)\,,$$

with

$$\mathbb{A}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 - \dfrac{2-\beta}{\rho_1^{\beta+1}} & 0 & 0 & -2\rho_1 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{2}{\rho_1} & \dfrac{1}{\rho_1^{\beta+1}} & 0 \end{pmatrix}.$$

Let us evaluate the eigenvalues of $\mathbb{A}_1$, so to see whether they can or cannot be of the type $\lambda = ik$, with $k \in \mathbb{Z}$. We have

$$\det(\mathbb{A}_1 - \lambda I) = \lambda^4 + \left(1 - \frac{\beta - 1}{\rho_1^{\beta+1}}\right)\lambda^2 + \frac{1}{\rho_1^{\beta+1}}\left(3 - \frac{2-\beta}{\rho_1^{\beta+1}}\right).$$

Setting $r_1 = 1/\rho_1^{\beta+1}$ and $\lambda^2 = \xi$, we have to solve the second degree equation

$$\xi^2 + (1 - (\beta - 1)r_1)\xi + r_1(3 - (2 - \beta)r_1) = 0\,, \tag{11}$$

whose discriminant is

$$\Delta = (1 - (\beta - 1)r_1)^2 - 4r_1(3 - (2 - \beta)r_1).$$

If $\Delta < 0$, the solutions of (11) are nonreal, hence the eigenvalues of $\mathbb{A}_1$ are complex with nonzero real part. Otherwise, if $\Delta \geq 0$, the two solutions ξ_1, ξ_2 of (11) are real, and, since

$$\xi_1\xi_2 = r_1(3 - (2 - \beta)r_1) > 0\,,$$

(recall that $\rho_1 > 1$, hence $r_1 < 1$), they have the same sign. Being

$$\xi_1 + \xi_2 = (\beta - 1)r_1 - 1 \in]-2, 0[\,,$$

the only possibility of resonance is that $\xi_i = -1$, for some $i = 1, 2$. But, if this would happen, substituting in (11) would imply that $\beta < 2$ and $r_1 = (2 + \beta)/(2 - \beta)$, which is impossible, since $r_1 < 1$. We thus conclude that the matrix $\mathbb{A}_1$ can never have eigenvalues of the type $\lambda = ik$, with $k \in \mathbb{Z}$.

Let us now assume that $\beta \in [0, 2[$, or $\beta = 2$ and $\alpha < \gamma$, and consider the linearization of system (8) with $\varepsilon = 0$ at the equilibrium point $\mathcal{E}_2$, i.e., using (10),

$$\dot{u} = \mathbb{A}_2(u - \mathcal{E}_2)\,,$$

with

$$\mathbb{A}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 + \dfrac{2-\beta}{\rho_2^{\beta+1}} & 0 & 0 & -2\rho_2 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{2}{\rho_2} & -\dfrac{1}{\rho_2^{\beta+1}} & 0 \end{pmatrix}.$$

Let us evaluate the eigenvalues of $\mathbb{A}_2$. We have

$$\det(\mathbb{A}_2 - \lambda I) = \lambda^4 + \left(1 + \frac{\beta - 1}{\rho_2^{\beta+1}}\right)\lambda^2 - \frac{1}{\rho_2^{\beta+1}}\left(3 + \frac{2-\beta}{\rho_2^{\beta+1}}\right).$$

Setting $r_2 = 1/\rho_2^{\beta+1}$ and $\lambda^2 = \xi$, we have to solve the second degree equation

$$\xi^2 + (1 + (\beta - 1)r_2)\xi - r_2(3 + (2 - \beta)r_2) = 0\,,$$

whose discriminant is

$$\Delta = (1 + (\beta - 1)r_2)^2 + 4r_2(3 + (2 - \beta)r_2) > 0.$$

Hence its two solutions ξ_1, ξ_2 are real, and, since $\xi_1\xi_2 < 0$, one of them, say ξ_1, is negative, with

$$\xi_1 = -\tfrac{1}{2}(1 + (\beta - 1)r_2 + \sqrt{\Delta}).$$

A direct computation shows that $\xi_1 < -1$. Moreover, for every $k \in \mathbb{Z}$ with $|k| \geq 2$, we have

$$\xi_1 = -k^2 \quad \Longleftrightarrow \quad (2 - \beta)r_2^2 + ((\beta - 1)k^2 + 3)r_2 + k^2(1 - k^2) = 0.$$

This second degree equation has two real solutions, only one of which is positive. If we denote it by $r_2(k)$, it is explicitly given by

$$r_2(k) = \begin{cases} \dfrac{(1-\beta)k^2 - 3 + \sqrt{(\beta-3)^2k^4 + 2(5\beta - 7)k^2 + 9}}{2(2-\beta)} & \text{if } \beta \in [0, 2[\,, \\ \dfrac{k^2(k^2-1)}{k^2+3} & \text{if } \beta = 2. \end{cases} \tag{12}$$

By (10), since $r_2 = 1/\rho_2^{\beta+1}$, we have the corresponding values of c, given by

$$c(k) = \frac{r_2(k) + 1}{r_2(k)^{\frac{3}{\beta+1}}}. \tag{13}$$

Notice that $r_2(-k) = r_2(k)$, so that $c(-k) = c(k)$, as well. It is possible to prove that $r_2(k)$ is strictly increasing with $k \geq 2$, and that $\lim_{k\to+\infty} r_2(k) = +\infty$. So, $c(k)$ is strictly decreasing, and

$$\lim_{k\to+\infty} c(k) = \begin{cases} 0 & \text{if } \beta \in [0, 2[\,, \\ 1 & \text{if } \beta = 2. \end{cases}$$

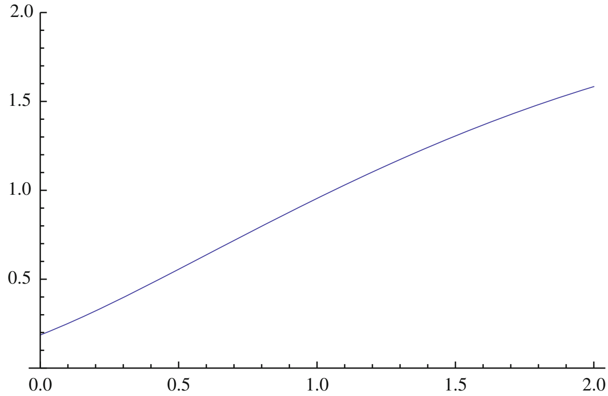

Fig. 1 The function $f(\beta)$

In particular, for every $k \geq 2$,

$$c(k) \leq c(2) = \begin{cases} f(\beta) & \text{if } \beta \in [0, 2[\,, \\ \dfrac{19}{12} & \text{if } \beta = 2\,, \end{cases}$$

with

$$f(\beta) = \frac{5 - 6\beta + \sqrt{16\beta^2 - 56\beta + 97}}{\left(1 - 4\beta + \sqrt{16\beta^2 - 56\beta + 97}\right)^{\frac{3}{\beta+1}}} (4 - 2\beta)^{\frac{2-\beta}{\beta+1}}.$$

One can further prove that $f(\beta)$ is strictly increasing in $\beta \in [0, 2[$, and that $\lim_{\beta \to 2^-} f(\beta) = \frac{19}{12}$ (see Fig. 1). We thus have that $c(k) \leq \frac{19}{12}$, for every integer k, with $|k| \geq 2$.

Now, we can conclude the proof. If $\varepsilon = 0$, the linearized system at the equilibrium point $\mathcal{E}_1$ is always non resonant, i.e., the eigenvalues of $\mathbb{A}_1$ are never of the type ik, with $k \in \mathbb{Z}$. Therefore, since, for $j = 1, 2$, the functions $\Gamma_j(s, \rho, \eta, \varphi, v; \varepsilon)$ are locally Lipschitz continuous in (ρ, η, φ, v), a classical perturbation theorem applies [see, e.g., (Coddington and Levinson, 1955, Chapter 14, Theorem 1.1)], providing the existence of a 2π-periodic solution of (8), for ε small enough. Correspondingly, by the changes of variables made above, we have a T-periodic solution of (1).

On the other hand, if $\beta \in [0, 2[$, or $\beta = 2$ and $\alpha < \gamma$, there is a second equilibrium point $\mathcal{E}_2$ for $\varepsilon = 0$. The analysis of the linearized system at this equilibrium point shows the existence of some resonant values $c(k)$ for the constant c, given by (13), with $r_2(k)$ defined by (12). Since c is defined as in (6), condition (2) (where $a(k) = r_2(k)$) is needed in order to avoid these resonant values. Hence, if (2) holds, the above mentioned perturbation theorem applies, and one concludes analogously.

Since $c(k) \leq \frac{19}{12}$ for every $\beta \in [0, 2]$ and every $k \in \mathbb{Z}$ with $|k| \geq 2$, if (4) is assumed, then (2) surely holds. The proof is thus concluded.

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