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# A general method for the existence of periodic solutions of differential systems in the plane

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ABSTRACT

We propose a general method to prove the existence of periodic solutions for planar systems of ordinary differential equations, which can be used in many different circumstances. Applications are given to some nonresonant cases, even for systems with superlinear growth in some direction, or with a singularity. Systems "at resonance" are also considered, provided a Landesman–Lazer type of condition is assumed.



## 1. Introduction

The aim of this paper is to provide a general method for obtaining the existence of periodic solutions for a planar system of the type

$$u' = f(t, u). \tag{1}$$

Here, we assume $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ to be a continuous function, $T$-periodic in its first variable. Notice, however, that most of our results will still hold in the Carathéodory setting.

The first step is to construct an unbounded curve spiralling around the origin, which controls all the solutions of the differential equation, in the sense that they cannot cross it from the inner to the outer part. As a consequence, a solution which grows in norm towards infinity has to perform infinitely many revolutions around the origin.

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Once such a curve has been found, we need to control those solutions which remain sufficiently far from the origin for all the time in the interval $[0, T]$. If, in view of this control, we can deduce that the number of revolutions of those solutions is bounded and cannot be an integer, as a consequence we get the existence of at least one $T$-periodic solution of (1).

Such a procedure was already used in [7], where Fabry and Habets deal with the scalar equation

$$x'' + h(t, x) = 0. \tag{2}$$

They consider a nonresonance situation with respect to the Dancer–Fučík spectrum (cf. [3,12]), when the function $h$ is allowed to have a superlinear growth on one side. As a consequence of our main theorem, we will show how to generalize the existence result by Fabry and Habets to some systems having a superlinear growth in one direction.

We will also illustrate how our main theorem applies to "nonresonance" situations, when the nonlinearity is controlled by some Hamiltonian functions, and in the case of "resonance", when a Landesman–Lazer type condition is assumed.

The above technique can be adapted to the case where the function $f$ in (1) is only defined on an open subset of the type $\mathbb{R} \times \mathcal{A}$, where $\mathcal{A}$ is, e.g., star-shaped in $\mathbb{R}^2$. One can find in [8] an example of application for the scalar second order equation (2), in the case of a function $h$ having a singularity, generalizing an existence result by Del Pino, Manásevich and Montero [4]. In this case, the set $\mathcal{A}$ is an open half-plane. We will show how our technique applies to generalize the existence result in [8], as well.

The proof of our main result is an application of the Poincaré–Bohl Fixed Point Theorem, which we recall here for the reader's convenience.

**Theorem** *(Poincaré–Bohl). Let $\Omega \subset \mathbb{R}^m$ be an open bounded set containing the origin, and $\varphi : \overline{\Omega} \to \mathbb{R}^m$ be a continuous function such that*

$$\varphi(u) \neq \lambda u, \quad \text{for every } u \in \partial\Omega \text{ and } \lambda > 1.$$

*Then, $\varphi$ has a fixed point in $\overline{\Omega}$.*

In order to use this theorem, we will need to approximate the function $f$ with more regular functions for which the Poincaré map is well defined. The Poincaré–Bohl Theorem applies to these maps, thus providing the existence of a $T$-periodic solution for the approximating equations. The solution to our system is then obtained by a limit procedure.

A few words about the notations. We denote by $\langle\cdot,\cdot\rangle$ the Euclidean scalar product in $\mathbb{R}^2$, and by $|\cdot|$ the corresponding norm. As usual, the open ball, centered at the origin, with radius $R > 0$ is $B_R = \{v \in \mathbb{R}^2\colon |v| < R\}$, and by $S^1$ we denote the set $\{v \in \mathbb{R}^2\colon |v| = 1\}$. The cone determined by a set $\mathcal{A} \subseteq S^1$ is defined as

$$\Theta(\mathcal{A}) = \{v \in \mathbb{R}^2\colon v = \rho e^{i\theta},\ \rho \geqslant 0,\ e^{i\theta} \in \mathcal{A}\}.$$

(It will be sometimes convenient to use the complex notation for the points in $\mathbb{R}^2$.) If, in particular, the set $\mathcal{A}$ is an arc determined by two angles $\theta_1 < \theta_2$, we will simply write

$$\Theta(\theta_1, \theta_2) = \{v \in \mathbb{R}^2\colon v = \rho e^{i\theta},\ \rho \geqslant 0,\ \theta \in [\theta_1, \theta_2]\}.$$

The closed segment joining two points $v_1$ and $v_2$ is denoted by $[v_1, v_2]$. Finally, we use the standard notation

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

## 2. Main results

We start by defining what we will call a *regular spiral* in the plane. Roughly speaking, it is a piecewise continuously differentiable injective curve which rotates infinitely many times around the origin, and grows in norm to infinity.

**Definition 2.1.** A *clockwise rotating regular spiral* is a continuous and injective curve

$$\gamma : [0, +\infty[ \to \mathbb{R}^2,$$

satisfying the following properties:

1. there exists an unlimited strictly increasing sequence

$$0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_k < \sigma_{k+1} < \cdots$$

such that the restriction of $\gamma$ to every closed interval $[\sigma_k, \sigma_{k+1}]$ is continuously differentiable, and such that

$$\langle J\dot{\gamma}(s), \gamma(s)\rangle > 0, \quad \text{for every } s \in [\sigma_k, \sigma_{k+1}]; \tag{3}$$

2. the curve grows in norm to infinity:

$$\lim_{s\to+\infty} \left|\gamma(s)\right| = +\infty; \tag{4}$$

3. the curve rotates clockwise infinitely many times:

$$\int_0^{+\infty} \frac{\langle J\dot{\gamma}(s), \gamma(s)\rangle}{|\gamma(s)|^2}\, ds = +\infty. \tag{5}$$

A similar definition can be given for a counter-clockwise rotating regular spiral, by changing the inequality in (3), and requiring the integral in (5) to be equal to $-\infty$.

In the following, we will only concentrate on clockwise rotating regular spirals. However, all our results have their analogues in the counter-clockwise case. For simplicity, we will assume that such a curve is parametrized in clockwise polar coordinates, so that $\gamma(s) = |\gamma(s)|(\cos s, -\sin s)$, and, in particular, for any nonnegative integer $n$, the point $\gamma(2\pi n)$ lies on the positive $x$-axis. Being $\gamma$ injective, we will have

$$\left|\gamma(s)\right| < \left|\gamma(s + 2\pi)\right|, \quad \text{for every } s > 0. \tag{6}$$

It is convenient to define, for every $n \in \mathbb{N}$, the set $\Omega_n$: it is the open region delimited by the Jordan curve $\Gamma_n$ obtained by gluing together the piece of curve $\gamma$ going from $\gamma(2\pi n)$ to $\gamma(2\pi(n+1))$, and the segment joining the two endpoints:

$$\Gamma_n = \bigl\{\gamma(s):\ s \in \bigl[2\pi n, 2\pi(n+1)\bigr]\bigr\} \cup \bigl[\gamma(2\pi n), \gamma\bigl(2\pi(n+1)\bigr)\bigr].$$

(See Fig. 1.)

We consider now the differential equation (1), for which we are going to select a particular kind of clockwise rotating regular spiral.

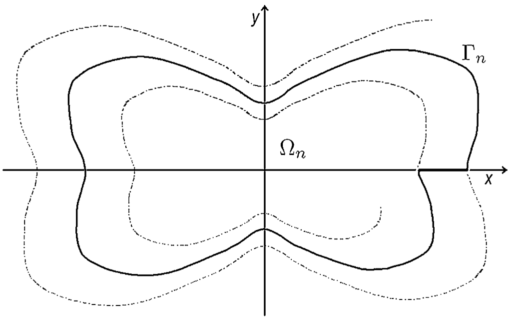


**Fig. 1.** The set $\Omega_n$.

**Definition 2.2.** A clockwise rotating regular spiral $\gamma$ is said to be *admissible for system* (1) if, when restricted to any subinterval $[\sigma_k, \sigma_{k+1}]$, it satisfies

$$\big\langle J\dot{\gamma}(s), f\big(t, \gamma(s)\big)\big\rangle < 0, \quad \text{for every } t \in [0, T] \text{ and } s \in [\sigma_k, \sigma_{k+1}]. \tag{7}$$

(The sequence $\{\sigma_k\}_k$ is the one introduced in Definition 2.1.)

Hence, roughly speaking, if $\gamma$ is an admissible clockwise rotating regular spiral, and if a solution of (1) ever reaches $\gamma$, then, at the crossing point, the solution will have to cross $\gamma$ from its outer part towards its inner part. The idea of controlling the solutions by the use of some guiding curves has been already used by many authors: see, e.g. [13,14].

We now state our general result.

**Theorem 2.3.** *Let the following assumptions hold*:

(H1) *there exists a clockwise rotating regular spiral $\gamma$, which is admissible for* (1);
(H2) *there exists $R > 0$ such that, for any solution $u : [0, T] \to \mathbb{R}^2$ of* (1) *satisfying*

$$\big|u(t)\big| \geqslant R, \quad \textit{for every } t \in [0, T],$$

*one has that, either $|u(T)| < |u(0)|$, or*

$$\int_0^T \frac{\langle Ju'(t), u(t)\rangle}{|u(t)|^2}\, dt \notin 2\pi\mathbb{N};$$

(H3) *there exist $C > 0$ and $\theta_1 < \theta_2$ such that*

$$\big\langle Jf(t, v), v\big\rangle \leqslant C\big(|v|^2 + 1\big), \quad \textit{for every } t \in [0, T] \textit{ and } v \in \Theta(\theta_1, \theta_2).$$

*Then, Eq.* (1) *has a $T$-periodic solution.*

Before starting the proof, let us spend a few words to explain the meaning of the above assumptions. Writing the solution $u(t)$ in polar coordinates

$$u(t) = \rho(t)\big(\cos\big(\vartheta(t)\big), \sin\big(\vartheta(t)\big)\big), \tag{8}$$

it is easily seen that

$$-\vartheta'(t) = \frac{\langle Ju'(t), u(t)\rangle}{|u(t)|^2} = \frac{\langle Jf(t, u(t)), u(t)\rangle}{|u(t)|^2}.$$

So, condition (H2) says that, for every large amplitude solution, either $\rho(T) < \rho(0)$, or

$$\vartheta(T) \neq \vartheta(0) - 2\pi k, \quad k = 0, 1, 2, 3, \dots. \tag{9}$$

A similar assumption can be found, e.g., in [15, Theorem 3].

Condition (H3) is needed in order to avoid that solutions clockwise rotate too rapidly around the origin. Indeed, it implies that

$$\vartheta(t) \in [\theta_1, \theta_2] \pmod{2\pi} \quad \Rightarrow \quad -\vartheta'(t) \leqslant C\left(1 + \frac{1}{\rho^2(t)}\right).$$

It could be intuitively thought of as a kind of angular speed controller.

**Proof.** We assume $R > 1$ such that $\overline{\Omega}_0 \subseteq B_R$. (Recall that $\Omega_0$ is the open and bounded set delimited by $\Gamma_0$.) Let $m_1$ be a positive integer such that $\overline{B}_R \subseteq \Omega_{m_1}$, and let $\bar{n}$ be an integer such that

$$\bar{n} > \frac{(C+1)T}{\theta_2 - \theta_1}. \tag{10}$$

We can find an $R_1 > R$ such that $\overline{\Omega}_{m_1+\bar{n}+1} \subseteq B_{R_1}$. In the same way we can find an integer $m_2 > m_1 + \bar{n} + 1$ such that $\overline{B}_{R_1} \subseteq \Omega_{m_2}$, and $R_2 > R_1$ such that $\overline{\Omega}_{m_2+\bar{n}+1} \subseteq B_{R_2}$.

Consider a sequence $(f_n)_n$ of locally Lipschitz continuous functions converging to $f$ uniformly on $[0, T] \times \overline{B}_{R_2}$. By (7), as long as $\gamma(s)$ belongs to $\overline{B}_{R_2}$, then, for $n$ large enough,

$$\big\langle J\dot{\gamma}(s), f_n\big(t, \gamma(s)\big)\big\rangle < 0, \quad \text{for every } t \in [0, T]. \tag{11}$$

Moreover, by (H3), for $n$ sufficiently large,

$$\frac{\langle Jf_n(t, v), v\rangle}{|v|^2} \leqslant C + 1, \quad \text{for every } t \in [0, T] \text{ and } v \in \Theta(\theta_1, \theta_2) \cap (\overline{B}_{R_2} \setminus B_R). \tag{12}$$

The solutions to the Cauchy problems associated to

$$u' = f_n(t, u) \tag{13}$$

are unique and, if $u_n$ is a solution satisfying $|u_n(0)| \leqslant R_1$, then, for sufficiently large $n$,

$$\big|u_n(t)\big| < R_2, \quad \text{for every } t \in [0, T].$$

Indeed, assuming by contradiction that $\max\{|u_n(t)|\colon t \in [0, T]\} \geqslant R_2$, there would be $t_1, t_2$ in $[0, T]$, with $t_1 < t_2$, such that

$$\big|u_n(t_1)\big| = R_1, \qquad \big|u_n(t_2)\big| = R_2,$$

and

$$R_1 < \left|u_n(t)\right| < R_2, \quad \text{for every } t \in \left]t_1, t_2\right[.$$

Then, for $t$ varying from $t_1$ to $t_2$, by (11) the solution would be driven by the curve $\gamma$ to make at least $\bar{n}+1$ clockwise revolutions around the origin, thus crossing at least $\bar{n}$ times the cone $\Theta(\theta_1, \theta_2)$, in the clockwise sense. Writing the solution in polar coordinates (8), from (12) we have that, if $\theta_1 \leqslant \vartheta_n(t) \leqslant \theta_2$, then

$$-\vartheta_n'(t) = \frac{\langle Jf_n(t, u_n(t)), u_n(t)\rangle}{|u_n(t)|^2} \leqslant C+1.$$

So, the time to cross the cone $\Theta(\theta_1, \theta_2)$ in the clockwise sense is at least $(\theta_2 - \theta_1)/(C+1)$, and then, by (10), the time to cross it $\bar{n}$ times should be greater than $T$. Hence, $t_2 - t_1 > T$, which is impossible.

The Poincaré map associated to (13) is then well defined on $\overline{B}_{R_1}$. Let us now see that the Poincaré–Bohl Theorem can be applied, taking as $\Omega$ the set $B_{R_1}$.

Assume by contradiction that, for every $n$, there exists $u_n^0 \in \partial B_{R_1}$ and a constant $\lambda_n > 1$ such that the solution $u_n(t)$ of (13) with $u_n(0) = u_n^0$ satisfies $u_n(T) = \lambda_n u_n^0$. We claim that, for $n$ large enough, it has to be

$$R < \left|u_n(t)\right| < R_2, \quad \text{for every } t \in [0, T]. \tag{14}$$

Indeed, we already proved above that $\max\{|u_n(t)|\colon\ t \in [0, T]\} < R_2$. Assume by contradiction that $\min\{|u_n(t)|\colon\ t \in [0, T]\} \leqslant R$. Then, since $|u_n(T)| > R_1$, there would be $\hat{t}_1, \hat{t}_2$ in $[0, T]$, with $\hat{t}_1 < \hat{t}_2$, such that

$$\left|u_n(\hat{t}_1)\right| = R, \qquad \left|u_n(\hat{t}_2)\right| = R_1,$$

and

$$R < \left|u_n(t)\right| < R_1, \quad \text{for every } t \in \left]\hat{t}_1, \hat{t}_2\right[.$$

Then, for $t$ varying from $\hat{t}_1$ to $\hat{t}_2$, by (11) the solution would be driven by the curve $\gamma$ to make at least $\bar{n}+1$ clockwise revolutions around the origin, thus crossing at least $\bar{n}$ times the cone $\Theta(\theta_1, \theta_2)$, in the clockwise sense. Arguing as above, we see that $\hat{t}_2 - \hat{t}_1 > T$, which is impossible.

By (14), necessarily it has to be

$$1 < \lambda_n < \frac{R_2}{R_1},$$

so, up to subsequences, we can assume that:

$$\lambda_n \to \bar{\lambda} \in \left[1, \frac{R_2}{R_1}\right] \quad \text{and} \quad u_n^0 \to \bar{u} \in \partial B_{R_1}.$$

Moreover, since $(f_n)_n$ converges to $f$ uniformly in $[0, T] \times \overline{B}_{R_2}$, there is a constant $M > 0$ such that

$$\left|f_n(t, u)\right| \leqslant M, \quad \text{for every } n \in \mathbb{N},\ t \in [0, T] \text{ and } u \in \overline{B}_{R_2}.$$

Then, $(u_n)_n$ is bounded in $C^1([0,T])$ and, by the Ascoli–Arzelà Theorem, there is a continuous function $u:[0,T]\to\mathbb{R}^2$ such that, up to a subsequence, $u_n \to u$ uniformly. Passing to the limit in

$$u_n(t) = u_n^0 + \int_0^t f_n\big(\tau, u_n(\tau)\big)\,d\tau,$$

we obtain

$$u(t) = \bar{u} + \int_0^t f\big(\tau, u(\tau)\big)\,d\tau,$$

so that $u$ is a solution to Eq. (1) with initial value $u(0) = \bar{u} \in \partial B_{R_1}$. By the above estimates,

$$R \leqslant \big|u(t)\big| \leqslant R_2, \quad \text{for every } t \in [0,T], \tag{15}$$

and $u(T) = \bar{\lambda} u(0)$. Hence, $|u(T)| \geqslant |u(0)|$ and, using polar coordinates as in (8), there is an integer $k$ such that

$$\vartheta(T) = \vartheta(0) - 2\pi k.$$

As a consequence of (H2), by (9) it has to be $k \leqslant -1$. Let $\bar{m} \in \mathbb{Z}$ be such that

$$\big|\gamma\big(-\vartheta(0) + 2\pi(\bar{m}-1)\big)\big| < \big|u(0)\big| \leqslant \big|\gamma\big(-\vartheta(0) + 2\pi\bar{m}\big)\big|.$$

(Recall that $\gamma$ is parametrized in clockwise polar coordinates.) Then, by the admissibility of the curve $\gamma$ and (15), since $B_R$ contains $\overline{\Omega}_0$, it has to be

$$\big|u(t)\big| < \big|\gamma\big(-\vartheta(t) + 2\pi\bar{m}\big)\big|, \quad \text{for every } t \in\, ]0,T].$$

So, using (6),

$$\begin{aligned}\big|u(T)\big| &< \big|\gamma\big(-\vartheta(T) + 2\pi\bar{m}\big)\big| = \big|\gamma\big(-\vartheta(0) + 2\pi(\bar{m}+k)\big)\big| \\ &\leqslant \big|\gamma\big(-\vartheta(0) + 2\pi(\bar{m}-1)\big)\big| < \big|u(0)\big|,\end{aligned}$$

and we get a contradiction with the fact that $|u(T)| \geqslant |u(0)|$.

So, up to a subsequence, for every $u_n^0 \in \partial B_{R_1}$, the solution $u_n$ of (13) with $u_n(0) = u_n^0$ is such that $u_n(T) \neq \lambda u_n^0$, for every $\lambda > 1$. We can then apply the Poincaré–Bohl Theorem to find a $T$-periodic solution $v_n(t)$ of (13) starting from a point $v_n^0 \in \overline{B}_{R_1}$. Using the Ascoli–Arzelà Theorem again, we find that, up to a subsequence, $(v_n)_n$ converges to a $T$-periodic solution of Eq. (1). $\square$

**Remark 2.4.** Condition (H3) has been used to forbid the large amplitude solutions to rotate too rapidly. One could imagine many different situations, where (H3) is replaced by some other type of control of the angular speed of the solutions.

The existence of an admissible regular spiral is guaranteed, e.g., if the large amplitude solutions rotate clockwise not too slowly, and have a controlled radial velocity, as the following proposition proves.

**Proposition 2.5.** *Let the following two assumptions hold*:

(H4) *there exist $R > 0$ and $\eta > 0$ such that*

$$|v| \geqslant R \quad \Rightarrow \quad \langle Jf(t,v), v\rangle \geqslant \eta |v|^2, \quad \textit{for every } t \in [0,T];$$

(H5) *there exists a continuous function $\chi : [0, +\infty[ \to ]0, +\infty[$ such that*

$$\langle f(t,v), v\rangle \leqslant \chi(|v|), \quad \textit{for every } t \in [0,T] \textit{ and } v \in \mathbb{R}^2,$$

*and*

$$\int_0^{+\infty} \frac{r\,dr}{\chi(r)} = +\infty.$$

*Then,* (H1) *is satisfied.*

**Proof.** We define the curve $\gamma : [0, +\infty[ \to \mathbb{R}^2$ as

$$\gamma(s) = r(s)(\cos s, -\sin s),$$

where $r(s)$ is the solution of the Cauchy problem

$$\dot{r} = \frac{2}{\eta}\frac{\chi(r)}{r}, \qquad r(0) = R.$$

Since this curve is smooth, the sequence $(\sigma_k)_k$, in this case, is arbitrary. Clearly, (3) and (5) hold, since $s$ is the angle in clockwise polar coordinates. We see that $r(s)$ is strictly increasing, and remains bounded for $s$ bounded. Moreover, $r(s) \to +\infty$ for $s \to +\infty$, so that condition (4) is satisfied, as well. Hence $\gamma$ is a clockwise rotating regular spiral. In order to show that it is admissible for (1), we compute

$$\langle J\dot{\gamma}(s), f(t, \gamma(s))\rangle = \frac{\dot{r}(s)}{r(s)}\langle J\gamma(s), f(t,\gamma(s))\rangle + \langle \gamma(s), f(t, \gamma(s))\rangle.$$

Using the assumptions, we have that

$$\langle J\dot{\gamma}(s), f(t,\gamma(s))\rangle \leqslant -\eta \dot{r}(s) r(s) + \chi(r(s)) < 0,$$

thus completing the proof. □

**Remark 2.6.** If the function $f$ has an at most linear growth, i.e., there exists $C > 0$ such that

$$|f(t,v)| \leqslant C(|v| + 1), \quad \text{for every } t \in [0,T] \text{ and } v \in \mathbb{R}^2,$$

then (H3) follows from the Cauchy–Schwarz inequality, and (H5) holds, with $\chi(r) = Cr(r+1)$. In a preliminary version of our paper, we had assumed the linear growth condition instead of (H5). In that case, we constructed the admissible curve $\gamma$ as a logarithmic spiral. Condition (H5) was suggested to us by Christian Fabry, whose contribution we acknowledge here.

As a straightforward consequence, we have the following.

**Corollary 2.7.** *If* (H2), (H4) *hold, and $f$ has an at most linear growth, then Eq.* (1) *has a $T$-periodic solution.*

In the applications, however, we will not necessarily need that the function $f$ has an at most linear growth. Indeed, the construction of the admissible regular spiral can sometimes be made directly.

We will now introduce a further condition which, together with (H4), guarantees that (H2) holds. This condition consists in a control of the angular velocity of the solutions of the differential equation (1).

**Proposition 2.8.** *Let* (H4) *and the following assumption hold*:

(H6) *there exist some values $w_1, \dots, w_m \in S^1$ and two positive functions*

$$\psi_1, \psi_2 : S^1 \setminus \{w_1, \dots, w_m\} \to \, ]0, +\infty],$$

*not identically equal to $+\infty$, with the following properties*:

(i) *in each open arc of the domain these functions are either continuous with all values in $\mathbb{R}$, or identically equal to $+\infty$*;

(ii) *one has*

$$\psi_1(w) \leqslant \liminf_{\lambda \to +\infty} \left\langle \frac{Jf(t, \lambda w)}{\lambda}, w \right\rangle \leqslant \limsup_{\lambda \to +\infty} \left\langle \frac{Jf(t, \lambda w)}{\lambda}, w \right\rangle \leqslant \psi_2(w), \tag{16}$$

*uniformly for $t \in [0, T]$ and $w$ in any compact subset of $S^1 \setminus \{w_1, \dots, w_n\}$*;

(iii) *moreover,*

$$\left[ \int_0^{2\pi} \frac{d\theta}{\psi_2(e^{i\theta})}, \int_0^{2\pi} \frac{d\theta}{\psi_1(e^{i\theta})} \right] \cap \left\{ \frac{T}{N} \colon N \in \mathbb{N}_0 \right\} = \varnothing, \tag{17}$$

*where $\mathbb{N}_0$ denotes the set of positive integers.*

*Then, both* (H2) *and* (H3) *are satisfied.*

Notice that, in (17), we use the convention that $\frac{1}{+\infty} = 0$, and we implicitly assume that the integrals have finite values.

**Proof.** Since $\psi_2$ is not identically equal to $+\infty$, it is bounded at least on one arc, and from the last inequality in (16) we deduce that (H3) holds. We now want to estimate the time needed by a solution of (1) to make a revolution around the origin, in order to verify (H2). Set

$$\tau_1 = \int_0^{2\pi} \frac{d\theta}{\psi_1(e^{i\theta})}, \qquad \tau_2 = \int_0^{2\pi} \frac{d\theta}{\psi_2(e^{i\theta})}.$$

By (17), there exists a small enough $\varepsilon > 0$ such that

$$[\tau_2 - \varepsilon, \tau_1 + \varepsilon] \cap \left\{ \frac{T}{N} \colon N \in \mathbb{N}_0 \right\} = \varnothing. \tag{18}$$

Writing a solution of (1) in polar coordinates (8), from (H4) we have that there is an $\hat{R}_1 > 0$ such that, if $|u(t)| \geqslant \hat{R}_1$ for every $t \in [0, T]$, then

$$-\vartheta'(t) = \frac{\langle Jf(t, u(t)), u(t)\rangle}{|u(t)|^2} \geqslant \eta > 0,$$

for every $t \in [0, T]$. So, we can find a large enough compact subset $\mathcal{K} \subseteq S^1 \setminus \{w_1, \dots, w_m\}$, which is a union of closed arcs, such that, if $|u(t)| \geqslant \hat{R}_1$ for every $t \in [0, T]$, then $u(t)$ takes a time less than $\varepsilon$ to cross $\Theta(S^1 \setminus \mathcal{K})$.

Let $K = \{\theta \in [0, 2\pi]:\ e^{i\theta} \in \mathcal{K}\}$. We can enlarge $\mathcal{K}$, if necessary, so that

$$\int_K \frac{d\theta}{\psi_2(e^{i\theta})} \geqslant \tau_2 - \frac{\varepsilon}{2}.$$

Notice that, since $\psi_1$ has positive values,

$$\int_K \frac{d\theta}{\psi_1(e^{i\theta})} < \tau_1.$$

Choose $\delta \in ]0, \min_{\mathcal{K}} \psi_1[$ such that

$$\int_K \frac{d\theta}{\psi_2(e^{i\theta}) + \delta} \geqslant \tau_2 - \varepsilon, \qquad \int_K \frac{d\theta}{\psi_1(e^{i\theta}) - \delta} \leqslant \tau_1. \tag{19}$$

By (16), there is an $\hat{R}_2 > 0$ such that, if $\lambda \geqslant \hat{R}_2$, then

$$\psi_1(w) - \delta \leqslant \left\langle \frac{Jf(t, \lambda w)}{\lambda}, w \right\rangle \leqslant \psi_2(w) + \delta, \quad \text{for every } t \in [0, T] \text{ and } w \in \mathcal{K}.$$

So, as long as

$$\bigl|u(t)\bigr| \geqslant \hat{R}_2 \quad \text{and} \quad \frac{u(t)}{|u(t)|} = e^{i\vartheta(t)} \in \mathcal{K},$$

we have

$$\psi_1\bigl(e^{i\vartheta(t)}\bigr) - \delta \leqslant -\vartheta'(t) \leqslant \psi_2\bigl(e^{i\vartheta(t)}\bigr) + \delta,$$

i.e.,

$$\frac{-\vartheta'(t)}{\psi_2(e^{i\vartheta(t)}) + \delta} \leqslant 1 \leqslant \frac{-\vartheta'(t)}{\psi_1(e^{i\vartheta(t)}) - \delta}.$$

Integrating, we see from (19) that, if $|u(t)| \geqslant \hat{R}_2$ for every $t \in [0, T]$, the time needed for $u(t)$ to cross $\Theta(\mathcal{K})$ lies between $\tau_2 - \varepsilon$ and $\tau_1$.

Summing up, setting $R = \max\{\hat{R}_1, \hat{R}_2\}$, we have that, if $u$ is solution of (1) such that $|u(t)| \geqslant R$ for every $t \in [0, T]$, the time needed to perform a complete rotation lies in $[\tau_2 - \varepsilon, \tau_1 + \varepsilon]$. So, in view of (18), such a solution cannot perform an integer number of rotations in the time $T$. Therefore, (H2) holds. $\square$

As straightforward consequences, we have the following.

**Corollary 2.9.** *If* (H1), (H4) *and* (H6) *hold, then Eq.* (1) *has a $T$-periodic solution.*

**Proof.** By Proposition 2.8, (H4) and (H6) imply (H2) and (H3). Hence, Theorem 2.3 applies. □

**Corollary 2.10.** *If* (H4), (H5) *and* (H6) *hold, then Eq.* (1) *has a $T$-periodic solution.*

**Proof.** By Proposition 2.5, (H4) and (H5) imply (H1). Hence, Corollary 2.9 applies. □

## 3. Some applications

In this section, we will illustrate some examples of applications of our main results. However, we will not look for the greatest generality, in order to keep the exposition at a rather simple level. For convenience, Eq. (1) will sometimes be written as

$$Ju' = g(t,u), \tag{20}$$

so that $Jf = g$.

### 3.1. Nonlinearities controlled by Hamiltonian functions

In this section, we deal with nonresonant problems where the nonlinearity is controlled by some positively homogeneous functions.

**Proposition 3.1.** *Let the following assumption hold*:

(H7) *There exist two continuous functions $H_1, H_2 : \mathbb{R}^2 \to \mathbb{R}$ with the following properties*:
 (i) *one has*

$$0 < H_j(\lambda v) = \lambda^2 H_j(v), \quad \textit{for every } v \neq 0 \textit{ and } \lambda > 0, \tag{21}$$

*for $j \in \{1,2\}$*;
 (ii) *there is a constant $c > 0$ such that*

$$2H_1(v) - c \leqslant \langle Jf(t,v), v\rangle \leqslant 2H_2(v) + c, \tag{22}$$

*for every $t \in [0,T]$ and $v \in \mathbb{R}^2$*;
 (iii) *setting*

$$\tau_1 = \int_0^{2\pi} \frac{d\theta}{2H_1(e^{i\theta})}, \qquad \tau_2 = \int_0^{2\pi} \frac{d\theta}{2H_2(e^{i\theta})}, \tag{23}$$

*one has that*

$$[\tau_2, \tau_1] \cap \left\{ \frac{T}{N} \colon N \in \mathbb{N}_0 \right\} = \varnothing. \tag{24}$$

*Then,* (H4) *and* (H6) *hold.*

**Proof.** Since $H_1$ has a positive minimum over $S^1$, by (21) and (22), we have that (H4) holds. Let $\psi_1(w) = 2H_1(w)$, and $\psi_2(w) = 2H_2(w)$, defined on the whole set $S^1$. Then, by (22),

$$2H_1(\lambda w) - c \leqslant \big\langle Jf(t, \lambda w), \lambda w \big\rangle \leqslant 2H_2(\lambda w) + c,$$

and using the positive homogeneity (21) of $H_1$, $H_2$,

$$\psi_1(w) - \frac{c}{\lambda^2} \leqslant \left\langle \frac{Jf(t, \lambda w)}{\lambda}, w \right\rangle \leqslant \psi_2(w) + \frac{c}{\lambda^2},$$

for every $w \in S^1$. Then, (H6) follows from (24). $\square$

By Corollaries 2.9 and 2.10, we immediately get the following consequences.

**Corollary 3.2.** *If* (H1) *and* (H7) *hold, then Eq.* (1) *has a $T$-periodic solution.*

**Corollary 3.3.** *If* (H5) *and* (H7) *hold, then Eq.* (1) *has a $T$-periodic solution.*

**Remark 3.4.** A result similar to Corollary 3.3 has been obtained in [2, Theorem 3], by a continuation approach, in the framework of Leray–Schauder degree theory, under the assumption that $f$ has an at most linear growth (which implies (H5), see Remark 2.6). In our framework, the linear growth assumption is unnecessary. Indeed, assume for example that $f$ satisfies (H7) and has an at most linear growth, and let

$$\tilde{f}(t, v) = f(t, v) + h\big(t, |v|\big)v,$$

where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and such that there is an $\bar{r} > 0$ for which

$$r \geqslant \bar{r} \quad \Rightarrow \quad h(t, r) \leqslant \ln(1 + r), \quad \text{for every } t \in [0, T].$$

Then, $\tilde{f}$ does not necessarily have an at most linear growth, but

$$\big\langle J\tilde{f}(t, v), v\big\rangle = \big\langle Jf(t, v), v\big\rangle,$$

so that $\tilde{f}$ verifies (H7), and, for $|v| > 1$ large enough,

$$\begin{aligned}\big\langle \tilde{f}(t, v), v\big\rangle &= \big\langle f(t, v), v\big\rangle + h\big(t, |v|\big)|v|^2 \\ &\leqslant C\big(1 + |v|\big)|v| + \ln\big(1 + |v|\big)|v|^2 \leqslant 2\ln\big(1 + |v|\big)|v|^2,\end{aligned}$$

so that $\tilde{f}$ verifies (H5), as well. Corollary 3.3 then applies to the equation

$$u' = \tilde{f}(t, u).$$

Notice also that we have only asked a one-sided control on the function $h$.

Consider now the case when $H_1$ and $H_2$ are continuously differentiable. Then, the Euler formula holds:

$$\langle \nabla H_j(v), v \rangle = 2H_j(v),$$

for every $v \in \mathbb{R}^2$, with $j \in \{1, 2\}$. It can be seen that, for the Hamiltonian systems

$$Ju' = \nabla H_1(u), \qquad Ju' = \nabla H_2(u), \tag{25}$$

the origin is an isochronous center, and the solutions have periods $\tau_1$ and $\tau_2$, respectively. This is the case described in [9, Theorem 5.2].

As a particular case of the above situation, we now want to deal with nonlinearities which are controlled, in some sense, by symmetric matrices. In what follows, we denote by $\mathcal{S}_{2\times 2}$ the set of $2 \times 2$ symmetric matrices, and we say that $\mathbb{A} \in \mathcal{S}_{2\times 2}$ is *positive definite* if

$$\langle \mathbb{A}v, v \rangle > 0, \quad \text{for every } v \in \mathbb{R}^2 \setminus \{0\}.$$

For two symmetric matrices $\mathbb{A}$ and $\mathbb{B}$, we write $\mathbb{A} \leqslant \mathbb{B}$ if $\langle \mathbb{A}v, v \rangle \leqslant \langle \mathbb{B}v, v \rangle$, for every $v \in \mathbb{R}^2$.

**Corollary 3.5.** *Let $\mathbb{A}$ and $\mathbb{B}$ be two positive definite symmetric $2 \times 2$ matrices, and $\Gamma : \mathbb{R} \times \mathbb{R}^2 \to \mathcal{S}_{2\times 2}$ be continuous, $T$-periodic in its first variable, and such that*

$$\mathbb{A} \leqslant \Gamma(t, v) \leqslant \mathbb{B}, \quad \textit{for every } t \in [0, T] \textit{ and } v \in \mathbb{R}^2.$$

*Moreover, let $r : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous and bounded function, $T$-periodic in its first variable. If*

$$[\det \mathbb{A}, \det \mathbb{B}] \cap \left\{ \left( \frac{2\pi N}{T} \right)^2 : N \in \mathbb{N} \right\} = \varnothing, \tag{26}$$

*then the equation*

$$Ju' = \Gamma(t, u)u + r(t, u)$$

*has a $T$-periodic solution.*

**Proof.** It is well known that the solutions of $Ju' = \mathbb{A}u$ and $Ju' = \mathbb{B}u$ have periods

$$\tau_1 = \frac{2\pi}{\sqrt{\det \mathbb{A}}}, \qquad \tau_2 = \frac{2\pi}{\sqrt{\det \mathbb{B}}},$$

respectively, corresponding to (23), with

$$H_1(v) = \frac{1}{2} \langle \mathbb{A}v, v \rangle, \qquad H_2(v) = \frac{1}{2} \langle \mathbb{B}v, v \rangle.$$

Taking $\varepsilon > 0$ small enough, and considering the matrices $\mathbb{A} - \varepsilon I$ and $\mathbb{B} + \varepsilon I$ instead of $\mathbb{A}$ and $\mathbb{B}$, respectively, the conclusion then follows from Corollary 3.3 and the observation concerning the Hamiltonian systems in (25), since the nonlinearity has, in this case, an at most linear growth. □

**Proposition 3.6.** *Let $\mathbb{A}$ and $\mathbb{B}$ be two positive definite symmetric $2 \times 2$ matrices. Condition* (26) *is equivalent to*

$$\sigma\big((1-\lambda) J\mathbb{A} + \lambda J\mathbb{B}\big) \cap \frac{2\pi}{T} i\mathbb{Z} = \varnothing, \quad \textit{for every } \lambda \in [0,1]. \tag{27}$$

**Proof.** An elementary computation shows that, for the positive definite symmetric $2 \times 2$ matrix $\mathbb{A}$, the eigenvalues of $J\mathbb{A}$ are equal to $\pm i\sqrt{\det \mathbb{A}}$. Similarly,

$$\sigma\big((1-\lambda) J\mathbb{A} + \lambda J\mathbb{B}\big) = \Big\{\pm i\sqrt{\det\big((1-\lambda) J\mathbb{A} + \lambda J\mathbb{B}\big)}\Big\}.$$

Using linear algebra, one can show that, for positive definite symmetric matrices,

$$\det \mathbb{A} \leqslant \det\big((1-\lambda) J\mathbb{A} + \lambda J\mathbb{B}\big) \leqslant \det \mathbb{B},$$

for every $\lambda \in [0,1]$, and the dependence on $\lambda$ is continuous. The conclusion easily follows. □

Condition (27) was introduced in [10], in the framework of Hamiltonian systems in $\mathbb{R}^{2M}$ of the type

$$Ju' = \nabla_u H(t,u). \tag{28}$$

It is a simplification of a condition proposed by Amann in [1], in the abstract framework of operators in the Hilbert space $\mathcal{H} = L^2(0,T)$, which we now recall. Let $L : D(L) \subseteq \mathcal{H} \to \mathcal{H}$ be the self-adjoint differential operator defined by $Lu = Ju'$, where $D(L)$ includes the $T$-periodic conditions. Choose a positive constant $\beta \in \mathbb{R} \setminus \frac{2\pi}{T}\mathbb{Z}$ such that $-\beta I \leqslant \mathbb{A} \leqslant \mathbb{B} \leqslant \beta I$, and denote by $E$ the sum of the eigenspaces of $L$ belonging to the eigenvalues in $]-\beta, \beta[$. Amann then supposes that

$$\sigma(J\mathbb{A}) \cap \frac{2\pi}{T} i\mathbb{Z} = \varnothing = \sigma(J\mathbb{B}) \cap \frac{2\pi}{T} i\mathbb{Z},$$

and, concerning the Morse indices,

$$m\big((L-\mathbb{A})_{|E}\big) = m\big((L-\mathbb{B})_{|E}\big).$$

In our framework of planar equations, i.e. $M = 1$, the result in [1,10] for the Hamiltonian system (28) can then be stated as follows.

**Corollary 3.7.** *Let $\mathbb{A}$ and $\mathbb{B}$ be two positive definite symmetric $2 \times 2$ matrices, assume that $H(t,u)$ is twice continuously differentiable in u and*

$$\mathbb{A} \leqslant H_{uu}(t,u) \leqslant \mathbb{B}, \quad \textit{for every } t \in [0,T] \textit{ and } u \in \mathbb{R}^2.$$

*If*

$$[\det \mathbb{A}, \det \mathbb{B}] \cap \left\{ \left(\frac{2\pi N}{T}\right)^2 : N \in \mathbb{N}_0 \right\} = \varnothing,$$

*then Eq.* (28) *has a* unique *$T$-periodic solution.*

Let us remark that all the results of this subsection hold in the case of negative Hamiltonian functions, as well.

### 3.2. The Landesman–Lazer condition

Consider the system

$$Ju' = \nabla H(u) + r(t,u), \tag{29}$$

where $r : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded and continuous function, $T$-periodic in its first variable, and $H : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, and satisfies

$$0 < H(\lambda v) = \lambda^2 H(v), \quad \text{for every } v \neq 0 \text{ and } \lambda > 0. \tag{30}$$

The situation is thus similar to the one considered in Subsection 3.1, with $H_1 = H_2$. But, on the contrary, we assume now that

$$\int_0^{2\pi} \frac{d\theta}{2H(e^{i\theta})} = \frac{T}{N}, \quad \text{for some } N \in \mathbb{N}_0.$$

For any continuous function $u : [0,T] \to \mathbb{R}$, we use the notation

$$\mathcal{N}(u) = \sup\bigl\{\sqrt{2H\bigl(u(t)\bigr)}\colon t \in [0,T]\bigr\}.$$

It is easily seen from (30) that there are two positive constants $c_1$, $c_2$ such that

$$c_1 \|u\|_\infty \leqslant \mathcal{N}(u) \leqslant c_2 \|u\|_\infty,$$

for every such $u$. It will be useful to fix a $\varphi : \mathbb{R} \to \mathbb{R}^2$ such that

$$J\varphi'(t) = \nabla H\bigl(\varphi(t)\bigr), \qquad H\bigl(\varphi(t)\bigr) = \frac{1}{2}, \quad \text{for every } t \in [0,T].$$

Notice that $\varphi$ is periodic, with minimal period $\frac{T}{N}$, and $\mathcal{N}(\varphi) = 1$.

**Theorem 3.8.** *In the above setting, assume that*

$$\int_0^T \liminf_{\substack{\rho\to+\infty \\ \omega\to\omega_0}} \bigl\langle r\bigl(t, \rho\varphi(t+\omega)\bigr), \varphi(t+\omega)\bigr\rangle\, dt > 0, \quad \textit{for every } \omega_0 \in [0,T]. \tag{31}$$

*Then, Eq.* (29) *has a $T$-periodic solution.*

**Proof.** We want to apply Corollary 2.7. As in the proof of Proposition 3.1, using the Euler formula, we immediately see that condition (H4) holds. Since $\nabla H(u)$ is positively homogeneous of degree 1 and $r(t,u)$ is bounded, the nonlinearity has an at most linear growth. Let us verify (H2). Assume by contradiction that there is a sequence $(u_n)_n$ of solutions such that $\min\{|u_n(t)|\colon t \in [0,T]\} \to +\infty$, and $u_n(T) = \lambda_n u_n(0)$, for some $\lambda_n \geqslant 1$. Set

$$v_n(t) = \frac{u_n(t)}{\mathcal{N}(u_n)}.$$

Clearly, $\mathcal{N}(v_n) = 1$, for every $n$, and

$$Jv_n'(t) = \nabla H\big(v_n(t)\big) + \frac{r(t, u_n(t))}{\mathcal{N}(u_n)}.$$

Since $(v_n)_n$ is uniformly bounded, we see that $(v_n')_n$ is uniformly bounded, as well. Hence, there is a $v$ such that, up to a subsequence, $(v_n)_n$ converges to $v$, weakly in $H^1(0,T)$, and uniformly in $[0,T]$. Then, $\mathcal{N}(v) = 1$. We then see from the equation that the convergence is indeed strong in $C^1([0,T])$, and $v$ satisfies

$$Jv' = \nabla H(v).$$

It is known that all solutions to this system are of the form $\rho\varphi(t+\omega)$, for some $\rho \geqslant 0$ and $\omega \in [0, \frac{T}{N}]$. Since $\mathcal{N}(\varphi) = 1$, it has to be $v(t) = \varphi(t+\omega_0)$, for some $\omega_0 \in [0, \frac{T}{N}]$. Let us switch to the generalized polar coordinates

$$u_n(t) = \rho_n(t)\varphi\big(t+\omega_n(t)\big), \qquad v_n(t) = \frac{\rho_n(t)}{\mathcal{N}(u_n)}\varphi\big(t+\omega_n(t)\big). \tag{32}$$

From the above discussion, it will be that

$$\rho_n(t) \to +\infty, \qquad \frac{\rho_n(t)}{\mathcal{N}(u_n)} \to 1, \qquad \omega_n(t) \to \omega_0, \tag{33}$$

uniformly in $t$. Computing $Ju_n'$ from (32), the differential equation becomes

$$\rho_n' J\varphi(t+\omega_n) + \rho_n\big(1+\omega_n'\big)J\varphi'(t+\omega_n) = \nabla H\big(\rho_n\varphi(t+\omega_n)\big) + r\big(t, \rho_n\varphi(t+\omega_n)\big).$$

A scalar product with $\varphi(t+\omega_n)$ yields

$$\omega_n' = \frac{1}{\rho_n}\big\langle r\big(t, \rho_n\varphi(t+\omega_n)\big), \varphi(t+\omega_n)\big\rangle.$$

Hence, since we are assuming by contradiction that $u_n(T) = \lambda_n u_n(0)$, and, for $n$ large enough, $v_n$ and $\varphi$ perform the same number of rotations around the origin in the time $T$,

$$0 = \omega_n(T) - \omega_n(0) = \int_0^T \frac{1}{\rho_n(t)}\big\langle r\big(t, \rho_n(t)\varphi\big(t+\omega_n(t)\big)\big), \varphi\big(t+\omega_n(t)\big)\big\rangle\, dt.$$

Consequently,

$$0 = \int_0^T \frac{\mathcal{N}(u_n)}{\rho_n(t)}\big\langle r\big(t, \rho_n(t)\varphi\big(t+\omega_n(t)\big)\big), \varphi\big(t+\omega_n(t)\big)\big\rangle\, dt.$$

Using Fatou's Lemma and the limits in (33), we have that

$$0 \geqslant \int_0^T \liminf_n \frac{\mathcal{N}(u_n)}{\rho_n(t)} \big\langle r\big(t, \rho_n(t)\varphi\big(t+\omega_n(t)\big)\big), \varphi\big(t+\omega_n(t)\big)\big\rangle\, dt$$

$$\geqslant \int_0^T \liminf_n \big\langle r\big(t, \rho_n(t)\varphi\big(t+\omega_n(t)\big)\big), \varphi\big(t+\omega_n(t)\big)\big\rangle\, dt$$

$$\geqslant \int_0^T \liminf_{\substack{\rho\to+\infty\\ \omega\to\omega_0}} \big\langle r\big(t, \rho\varphi(t+\omega)\big), \varphi(t+\omega)\big\rangle\, dt,$$

in contradiction with the hypothesis. $\square$

Clearly, the same type of result holds if the Hamiltonian function is negative or if, instead of (31), we assume the symmetrical condition

$$\int_0^T \limsup_{\substack{\rho\to+\infty\\ \omega\to\omega_0}} \big\langle r\big(t, \rho\varphi(t+\omega)\big), \varphi(t+\omega)\big\rangle\, dt < 0, \quad \text{for every } \omega_0 \in [0, T].$$

Assumptions like (31) and the above have been introduced in [9], where the double resonance case is also treated.

As a particular case of Eq. (29), we now consider the system

$$\begin{cases} -y' = \mu x^+ - \nu x^- + r_1(t, x), \\ x' = y + r_2(t, y), \end{cases} \tag{34}$$

where $\mu, \nu$ are positive constants and $r_1, r_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are bounded continuous functions, $T$-periodic in their first variable. We assume that there is a positive integer $N$ such that

$$\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{N}.$$

As a direct consequence of Theorem 3.8, we have the following, where the classical Landesman–Lazer condition can easily be recognized (see, e.g. [6]).

**Corollary 3.9.** *In the above setting, assume that, for every nonzero solution $\phi(t)$ of the scalar equation $\phi'' + \mu\phi^+ - \nu\phi^- = 0$, we have*

$$\int_{\{\phi>0\}} \liminf_{x\to+\infty} r_1(t, x)\phi(t)\, dt + \int_{\{\phi<0\}} \limsup_{x\to-\infty} r_1(t, x)\phi(t)\, dt$$
$$+ \int_{\{\phi'>0\}} \liminf_{y\to+\infty} r_2(t, y)\phi'(t)\, dt + \int_{\{\phi'<0\}} \limsup_{y\to-\infty} r_2(t, y)\phi'(t)\, dt > 0.$$

*Then, system* (34) *has a $T$-periodic solution.*

### 3.3. One-sided superlinear growth

In this subsection, we consider a special case of Eq. (20), i.e., a Hamiltonian system of the type

$$\begin{cases} -y' = g_1(t,x), \\ x' = g_2(t,y), \end{cases} \tag{35}$$

where $g_1, g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, and $T$-periodic in their first variable. Notice that, here,

$$g(t,x,y) = \big(g_1(t,x), g_2(t,y)\big).$$

We assume that, for $i, j \in \{1,2\}$, there are some $\mu_{i,j}, \nu_{i,j} \in \,]0,+\infty]$ such that

$$\mu_{1,1} \leqslant \liminf_{x\to+\infty} \frac{g_1(t,x)}{x} \leqslant \limsup_{x\to+\infty} \frac{g_1(t,x)}{x} \leqslant \mu_{1,2}, \tag{36}$$

$$\nu_{1,1} \leqslant \liminf_{x\to-\infty} \frac{g_1(t,x)}{x} \leqslant \limsup_{x\to-\infty} \frac{g_1(t,x)}{x} \leqslant \nu_{1,2}, \tag{37}$$

$$\mu_{2,1} \leqslant \liminf_{y\to+\infty} \frac{g_2(t,y)}{y} \leqslant \limsup_{y\to+\infty} \frac{g_2(t,y)}{y} \leqslant \mu_{2,2}, \tag{38}$$

$$\nu_{2,1} \leqslant \liminf_{y\to-\infty} \frac{g_2(t,y)}{y} \leqslant \limsup_{y\to-\infty} \frac{g_2(t,y)}{y} \leqslant \nu_{2,2}. \tag{39}$$

With the usual convention that $\frac{1}{+\infty} = 0$, let

$$\tau_j = \frac{\pi}{2}\left(\frac{1}{\sqrt{\mu_{1,j}\,\mu_{2,j}}} + \frac{1}{\sqrt{\nu_{1,j}\,\mu_{2,j}}} + \frac{1}{\sqrt{\nu_{1,j}\,\nu_{2,j}}} + \frac{1}{\sqrt{\mu_{1,j}\,\nu_{2,j}}}\right), \tag{40}$$

for $j \in \{1,2\}$.

**Theorem 3.10.** *Assume that all the constants in* (36)–(39) *are finite, and*

$$[\tau_2, \tau_1] \cap \left\{ \frac{T}{N} \colon N \in \mathbb{N}_0 \right\} = \varnothing. \tag{41}$$

*Then, system* (35) *has a $T$-periodic solution. The same is true if one of the constants $\mu_{1,2}$, $\nu_{1,2}$, $\mu_{2,2}$, $\nu_{2,2}$ is equal to $+\infty$, all the others being finite.*

**Proof.** In the case where all the constants in (36)–(39) are finite, we will apply Corollary 2.10. Condition (H5) holds, since the nonlinearities have an at most linear growth. Modifying slightly the constants in (36)–(39), without affecting (41), we can assume without loss of generality that there is an $R > 0$ such that, for every $t \in [0,T]$,

$$x \geqslant R \quad \Rightarrow \quad \mu_{1,1} \leqslant \frac{g_1(t,x)}{x} \leqslant \mu_{1,2},$$

$$x \leqslant -R \quad \Rightarrow \quad \nu_{1,1} \leqslant \frac{g_1(t,x)}{x} \leqslant \nu_{1,2},$$

$$y \geqslant R \quad \Rightarrow \quad \mu_{2,1} \leqslant \frac{g_2(t,y)}{y} \leqslant \mu_{2,2},$$
$$y \leqslant -R \quad \Rightarrow \quad \nu_{2,1} \leqslant \frac{g_2(t,y)}{y} \leqslant \nu_{2,2}. \tag{42}$$

Moreover, we have the existence of a constant $C > 0$ such that

$$|x| \leqslant R \quad \Rightarrow \quad \big|g_1(t,x)x\big| \leqslant C,$$
$$|y| \leqslant R \quad \Rightarrow \quad \big|g_2(t,y)y\big| \leqslant C. \tag{43}$$

Consequently, if $(x,y) \neq (0,0)$, in the four different quadrants we have that:

(I) If $x \geqslant 0$ and $y \geqslant 0$, then

$$\mu_{1,1}x^2 + \mu_{2,1}y^2 - 2C \leqslant \big\langle g(t,x,y),(x,y)\big\rangle \leqslant \mu_{1,2}x^2 + \mu_{2,2}y^2 + 2C.$$

(II) If $x \leqslant 0$ and $y \geqslant 0$, then

$$\nu_{1,1}x^2 + \mu_{2,1}y^2 - 2C \leqslant \big\langle g(t,x,y),(x,y)\big\rangle \leqslant \nu_{1,2}x^2 + \mu_{2,2}y^2 + 2C.$$

(III) If $x \leqslant 0$ and $y \leqslant 0$, then

$$\nu_{1,1}x^2 + \nu_{2,1}y^2 - 2C \leqslant \big\langle g(t,x,y),(x,y)\big\rangle \leqslant \nu_{1,2}x^2 + \nu_{2,2}y^2 + 2C.$$

(IV) If $x \geqslant 0$ and $y \leqslant 0$, then

$$\mu_{1,1}x^2 + \nu_{2,1}y^2 - 2C \leqslant \big\langle g(t,x,y),(x,y)\big\rangle \leqslant \mu_{1,2}x^2 + \nu_{2,2}y^2 + 2C.$$

The left-hand side inequalities imply that (H4) holds, with

$$\eta = \frac{1}{2}\min\{\mu_{1,1}, \nu_{1,1}, \mu_{2,1}, \nu_{2,1}\}.$$

In order to verify (H6), we take a compact subset

$$\mathcal{K} \subseteq S^1 \setminus \big\{e^0, e^{i\pi/2}, e^{i\pi}, e^{i3\pi/2}\big\}.$$

Without loss of generality, we can assume it to be of the form $\mathcal{K} = \{e^{i\theta}\colon \theta \in K\}$, with

$$K = \left[\alpha, \frac{\pi}{2} - \alpha\right] \cup \left[\frac{\pi}{2} + \alpha, \pi - \alpha\right] \cup \left[\pi + \alpha, \frac{3\pi}{2} - \alpha\right] \cup \left[\frac{3\pi}{2} + \alpha, 2\pi - \alpha\right],$$

for some $\alpha \in \, ]0, \frac{\pi}{2}[$. We define

$$\psi_1\big(e^{i\theta}\big) = \begin{cases} \mu_{1,1}\cos^2\theta + \mu_{2,1}\sin^2\theta, & \text{if } \theta \in \, ]0, \frac{\pi}{2}[, \\ \nu_{1,1}\cos^2\theta + \mu_{2,1}\sin^2\theta, & \text{if } \theta \in \, ]\frac{\pi}{2}, \pi[, \\ \nu_{1,1}\cos^2\theta + \nu_{2,1}\sin^2\theta, & \text{if } \theta \in \, ]\pi, \frac{3\pi}{2}[, \\ \mu_{1,1}\cos^2\theta + \nu_{2,1}\sin^2\theta, & \text{if } \theta \in \, ]\frac{3\pi}{2}, 2\pi[, \end{cases}$$

and

$$\psi_2(e^{i\theta}) = \begin{cases} \mu_{1,2}\cos^2\theta + \mu_{2,2}\sin^2\theta, & \text{if } \theta \in \left]0, \frac{\pi}{2}\right[, \\ \nu_{1,2}\cos^2\theta + \mu_{2,2}\sin^2\theta, & \text{if } \theta \in \left]\frac{\pi}{2}, \pi\right[, \\ \nu_{1,2}\cos^2\theta + \nu_{2,2}\sin^2\theta, & \text{if } \theta \in \left]\pi, \frac{3\pi}{2}\right[, \\ \mu_{1,2}\cos^2\theta + \nu_{2,2}\sin^2\theta, & \text{if } \theta \in \left]\frac{3\pi}{2}, 2\pi\right[. \end{cases}$$

Condition (41) then implies that (H6) holds (see [11] for the computations). Corollary 2.10 can thus be applied, and the proof is completed in this case.

Assume now, for instance, that $\nu_{1,2} = +\infty$, all the other constants being finite. In this case, we will apply Corollary 2.9. Indeed, condition (H4) still holds, since it follows from the left-hand side estimates above. Condition (H6) can also be proved similarly as above. In this case, we will have that

$$\psi_2(e^{i\theta}) = +\infty, \quad \text{for every } \theta \in \left]\frac{\pi}{2}, \pi\right[ \cup \left]\pi, \frac{3\pi}{2}\right[.$$

We now need to verify (H1), showing that an admissible clockwise rotating regular spiral exists. Using (37), it is possible to construct two continuous functions $h_1, h_2 : ]-\infty, -R] \to \mathbb{R}$ such that

$$h_1(x) < g_1(t, x) < h_2(x) < 0, \quad \text{for every } x \leqslant -R,$$

and whose primitive functions $H_1, H_2$ satisfy

$$\lim_{x \to -\infty} H_1(x) = \lim_{x \to -\infty} H_2(x) = +\infty.$$

In order to construct the admissible regular spiral we consider four different regions in the plane:

$$E = [-R, +\infty[ \times \mathbb{R},$$
$$SW = ]-\infty, -R] \times ]-\infty, -R],$$
$$W = ]-\infty, -R] \times [-R, R],$$
$$NW = ]-\infty, -R] \times [R, +\infty[.$$

The regular spiral will be constructed by gluing together pieces of curves belonging to each of these regions. Concerning the region $E$, we easily construct the curve $\gamma$ like in the proof of Proposition 2.5 (or like a logarithmic spiral, see Remark 2.6).

In the region $SW$, the regular spiral is built as a level curve of the Hamiltonian function

$$\mathcal{H}_{SW}(x, y) = \frac{1}{2}\nu_{2,2}y^2 + H_2(x).$$

For a solution of (35) which intersects a level curve in this region, at a time $t$, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{H}_{SW}\big(x(t), y(t)\big) &= -\nu_{2,2}y(t)g_1\big(t, x(t)\big) + h_2\big(x(t)\big)g_2\big(t, y(t)\big) \\ &\leqslant \nu_{2,2}y(t)\big(h_2\big(x(t)\big) - g_1\big(t, x(t)\big)\big) < 0, \end{aligned}$$

so that (7) holds.

In the region $W$, we build the curve as a straight line with a negative slope $-m$, with $m>0$ sufficiently small. Let $\tilde{C}>0$ be such that

$$\left|g_2(t,y)\right| \leqslant \tilde{C}, \quad \text{if } t\in[0,T] \text{ and } |y|\leqslant R.$$

Being $x\leqslant -R$, $|y|\leqslant R$, and since $\dot{\gamma}$ has the direction of $(-1,m)$, using (42) and (43) we have

$$-g_1(t,x)+mg_2(t,y)\geqslant \nu_{1,1}R-m\tilde{C}>0,$$

provided that $m<\nu_{1,1}R/\tilde{C}$. Hence, (7) holds in this region.

In the region $NW$, the regular spiral is built as a level curve of the Hamiltonian function

$$\mathcal{H}_{NW}(x,y)=\frac{1}{2}\mu_{2,1}y^2+H_1(x).$$

For a solution of (35) which intersects a level curve in this region, at a time $t$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_{NW}\big(x(t),y(t)\big)&=-\mu_{2,1}y(t)g_1\big(t,x(t)\big)+h_1\big(x(t)\big)g_2\big(t,y(t)\big)\\&\leqslant \mu_{2,1}y(t)\big(h_1\big(x(t)\big)-g_1\big(t,x(t)\big)\big)<0,\end{aligned}$$

so that (7) holds.

In order to be sure that the curve grows towards infinity, we will be careful in choosing, in the region $W$, the slope $m$ small enough, so that at every turn the curve gets larger and larger. In this way, (H1) is verified, and Corollary 2.9 applies, so that the proof is completed. □

Theorem 3.10 partially generalizes the existence results obtained in [5,7] for the scalar equation (2), for which $\mu_{2,1}=\mu_{2,2}=\nu_{2,1}=\nu_{2,2}=1$. Indeed, the conditions in [7] were more subtle, involving some integrals over $t$. For briefness, we prefer not entering in these details.

Let us state the following corollary, where $\nu_{1,2}=+\infty$ and $\nu_{1,1}$ can be chosen to be arbitrarily large.

**Corollary 3.11.** *Assume that*

$$\lim_{x\to-\infty}\frac{g_1(t,x)}{x}=+\infty,$$

*and that* (36), (38) *and* (39) *hold. If there is a positive integer N such that*

$$\frac{2T}{(N+1)\pi}<\frac{1}{\sqrt{\mu_{1,2}\,\mu_{2,2}}}+\frac{1}{\sqrt{\mu_{1,2}\,\nu_{2,2}}}\leqslant\frac{1}{\sqrt{\mu_{1,1}\,\mu_{2,1}}}+\frac{1}{\sqrt{\mu_{1,1}\,\nu_{2,1}}}<\frac{2T}{N\pi},$$

*then system* (35) *has a T-periodic solution.*

**Remark 3.12.** We may repeat the arguments in this subsection for a more general system like

$$\begin{cases}-y'=g_1(t,x)+\beta y+r_1(t,x,y),\\ x'=\beta x+g_2(t,y)+r_2(t,x,y),\end{cases}$$

where $\beta$ is such that

$$\beta^2<\min\{\mu_{1,1}\mu_{2,1},\,\mu_{1,1}\nu_{2,1},\,\nu_{1,1}\mu_{2,1},\,\nu_{1,1}\nu_{2,1}\},$$

and $r_1, r_2$ are two continuous functions, $T$-periodic in their first variable, such that

$$\lim_{\lambda \to +\infty} \frac{r_i(t, \lambda\cos\theta, \lambda\sin\theta)}{\lambda} = 0, \quad i \in \{1, 2\},$$

uniformly for $t \in [0, T]$ and $\theta \in [0, 2\pi]$. In this case, the definition of $\tau_j$ in (40) should be changed, taking into account the presence of the new constant $\beta$. We will have

$$\tau_j = \Psi(\mu_{1,j}, \mu_{2,j}, -1) + \Psi(\mu_{1,j}, \nu_{2,j}, +1) + \Psi(\nu_{1,j}, \nu_{2,j}, -1) + \Psi(\nu_{1,j}, \mu_{2,j}, +1),$$

for $j \in \{1, 2\}$, where

$$\Psi(\xi_1, \xi_2, \kappa) = \frac{1}{\sqrt{\xi_1\xi_2 - \beta^2}}\left[\frac{\pi}{2} + \kappa \arctan\left(\frac{\beta}{\sqrt{\xi_1\xi_2 - \beta^2}}\right)\right].$$

We refer to [11] for the corresponding computations.

### *3.4. Nonlinearities with a singularity*

As already mentioned in the Introduction, we can adapt our results to the case where $f : \mathbb{R} \times \mathcal{A} \to \mathbb{R}^2$, where $\mathcal{A}$ is, e.g., a star-shaped subset of $\mathbb{R}^2$. In this case, instead of (4), the regular spiral $\gamma(s)$ will accordingly be asked to exit any given compact subset in $\mathcal{A}$, when $s$ is sufficiently large. Even more general subsets $\mathcal{A}$ could be considered, of course, but we will not enter into details. We just illustrate below a case when $\mathcal{A}$ is the right half-plane.

Let $g_1 : \mathbb{R} \times ]0, +\infty[ \to \mathbb{R}$ and $g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous, and $T$-periodic with respect to their first variable.

**Corollary 3.13.** *Assume that there are a constant $\delta > 0$ and a continuous function $\hat{g}_1 : ]0, \delta[ \to \mathbb{R}$ such that*

$$g_1(t, x) \leqslant \hat{g}_1(x), \quad \text{for every } t \in [0, T] \text{ and } x \in ]0, \delta[,$$

*and*

$$\lim_{x \to 0^+} \hat{g}_1(x) = -\infty, \qquad \int_0^\delta \hat{g}_1(x)\, dx = -\infty.$$

*If moreover* (36), (38) *and* (39) *hold, and there is a positive integer $N$ such that*

$$\frac{2T}{(N+1)\pi} < \frac{1}{\sqrt{\mu_{1,2}\,\mu_{2,2}}} + \frac{1}{\sqrt{\mu_{1,2}\,\nu_{2,2}}} \leqslant \frac{1}{\sqrt{\mu_{1,1}\,\mu_{2,1}}} + \frac{1}{\sqrt{\mu_{1,1}\,\nu_{2,1}}} < \frac{2T}{N\pi},$$

*then system* (35) *has a $T$-periodic solution.*

**Proof.** We apply our general theorem, adapted to this situation. The construction of the admissible curve follows closely the one provided in [8, Section 3], gluing together level lines of the appropriate Hamiltonian functions, as in the proof of Theorem 3.10, and straight lines having a sufficiently small slope. Concerning the estimates of the time needed for a large amplitude solution to make a rotation around, say, the point $(1, 0)$, we refer to [8, Section 4]. □

The above corollary generalizes the existence results obtained in [4] and [8] for the scalar equation (2), for which $\mu_{2,1} = \mu_{2,2} = \nu_{2,1} = \nu_{2,2} = 1$.

## References

[1] H. Amann, On the unique solvability of semilinear operator equations in Hilbert spaces, J. Math. Pures Appl. 61 (1982) 149–175.
[2] A. Capietto, J. Mawhin, F. Zanolin, A continuation approach to superlinear periodic boundary value problems, J. Differential Equations 88 (1990) 347–395.
[3] E.N. Dancer, Boundary-value problems for weakly nonlinear ordinary differential equations, Bull. Aust. Math. Soc. 15 (1976) 321–328.
[4] M. Del Pino, R. Manásevich, A. Montero, $T$-periodic solutions for some second order differential equations with singularities, Proc. Roy. Soc. Edinburgh Sect. A 120 (1992) 231–243.
[5] P. Drabek, S. Invernizzi, On the periodic boundary vale problem for forced Duffing equations with jumping nonlinearity, Nonlinear Anal. 10 (1986) 643–650.
[6] C. Fabry, Landesman–Lazer conditions for periodic boundary value problems with asymmetric nonlinearities, J. Differential Equations 116 (1995) 405–418.
[7] C. Fabry, P. Habets, Periodic solutions of second order differential equations with superlinear asymmetric nonlinearities, Arch. Math. 60 (1993) 266–276.
[8] A. Fonda, R. Toader, Radially symmetric systems with a singularity and asymptotically linear growth, Nonlinear Anal. 74 (2011) 2485–2496.
[9] A. Fonda, M. Garrione, Double resonance with Landesman–Lazer conditions for planar systems of ordinary differential equations, J. Differential Equations 250 (2011) 1052–1082.
[10] A. Fonda, J. Mawhin, An iterative method for the solvability of semilinear equations in Hilbert spaces and applications, in: J. Wiener, J.K. Hale (Eds.), Partial Differential Equations and Other Topics, Longman, London, 1992, pp. 126–132.
[11] A. Fonda, J. Mawhin, Planar differential systems at resonance, Adv. Differential Equations 11 (2006) 1111–1133.
[12] S. Fučík, Solvability of Nonlinear Equations and Boundary Value Problems, Reidel, Boston, 1980.
[13] M.A. Krasnosel'skii, The Operator of Translation along the Trajectories of Differential Equations, Transl. Math. Monogr., vol. 19, Amer. Math. Soc., Providence, RI, 1968.
[14] J. Mawhin, Topological Degree Methods in Nonlinear Boundary Value Problems, CBMS Reg. Conf. Ser. Math., vol. 40, Amer. Math. Soc., Providence, RI, 1979.
[15] F. Zanolin, Continuation theorems for the periodic problem via the translation operator, Rend. Semin. Mat. Univ. Politec. Torino 54 (1996) 1–23.