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A Landesman–Lazer-type condition for asymptotically linear second-order equations with a singularity

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We consider the T-periodic problem

$$x'' + g(t,x) = 0,$$
$$x(0) = x(T), \qquad x'(0) = x'(T),$$

where $g\colon [0,T]\times\,]0,+\infty[\,\to \mathbb{R}$ exhibits a singularity of a repulsive type at the origin, and an asymptotically linear behaviour at infinity. In particular, for large x, $g(t,x)$ is controlled from both sides by two consecutive asymptotes of the T-periodic Fučik spectrum, with possible equality on one side. Using a suitable Landesman–Lazer-type condition, we prove the existence of a solution.

1. Introduction

Del Pino *et al*. [6] considered the problem of finding T-periodic positive solutions of the equation

$$x'' + g(t,x) = 0, \tag{1.1}$$

where $g\colon \mathbb{R}\times\,]0,+\infty[\,\to \mathbb{R}$ is continuous, T-periodic in its first variable and has a singularity of a repulsive type at the origin. They proved the following result.

Theorem 1.1 (Del Pino *et al*. [6, theorem 1]). *Assume that there exist an integer N and two constants $\hat\mu$, $\tilde\mu$ such that*

$$\left(\frac{N\pi}{T}\right)^2 < \hat\mu \leqslant \liminf_{x\to+\infty} \frac{g(t,x)}{x} \leqslant \limsup_{x\to+\infty} \frac{g(t,x)}{x} \leqslant \tilde\mu < \left(\frac{(N+1)\pi}{T}\right)^2, \tag{1.2}$$

uniformly for every $t \in [0,T]$. Moreover, suppose that there exist positive constants c', c'', δ and $\nu \geqslant 1$ such that

$$\frac{c'}{x^\nu} \leqslant -g(t,x) \leqslant \frac{c''}{x^\nu} \quad \textit{for every } t \in [0,T] \textit{ and every } x \in\,]0,\delta]. \tag{1.3}$$

Then equation (1.1) has a T-periodic solution.

Assumption (1.2) seems to be a kind of non-resonance condition with respect to the set

$$\Sigma = \left\{ \left(\frac{k\pi}{T}\right)^2,\ k \in \mathbb{N} \right\},$$

whose elements correspond to the heights of the asymptotes of the curves in the Fučik spectrum (see, for example, [10]). They also coincide with the eigenvalues of the associated Dirichlet problem on the interval $[0, T]$. As a direct consequence of theorem 1.1, the equation

$$x'' - \frac{1}{x^\nu} + \beta x = e(t), \tag{1.4}$$

where $\nu \geqslant 1$ and $e(t)$ is continuous and T-periodic, has a T-periodic solution, provided that $\beta > 0$ satisfies $\beta \notin \Sigma$.

We mention that the study of the existence of periodic solutions for equations with a singularity, like (1.1), was first considered by Lazer and Solimini in [14], and later investigated by many authors (see, for example, [2, 4, 5, 7–9, 11, 17, 19–23]).

A few lines below the statement of their main theorem, del Pino *et al*. raised the problem of finding sufficient conditions on $e(t)$ in order to ensure the existence of a T-periodic solution for equation (1.4) in the case when β belongs to the set Σ. As a matter of fact, this seems to be a delicate problem for the following reason. As already noted in [6] (see also [1, 16]), if $\nu = 3$, in the phase plane there is an isochronous centre for Steen's equation (see [18]):

$$x'' - \frac{1}{x^3} + \beta x = 0, \tag{1.5}$$

for any choice of $\beta > 0$, since all the positive solutions have minimal period equal to $\pi/\sqrt{\beta}$. This fact is crucial when dealing with the T-periodic problem associated with (1.5), in the case when $\beta \in \Sigma$. Indeed, it was shown in [16] using a phase plane analysis that, due to isochronicity, if one considers a forcing made by a T-periodic chain of Dirac deltas, then all the solutions of the forced equation are unbounded, both in the past and in the future. The same type of situation arises for the equation

$$x'' - \frac{1}{x^3} + \beta x = \varepsilon \sin(\sqrt{\beta} t),$$

for sufficiently small $|\varepsilon|$, as proved in [4, theorem 3].

However, in the case $\nu = 3$, defining the $(\pi/\sqrt{\beta})$-periodic function

$$\Phi(\theta) = \int_0^T e(t) |\sin(\sqrt{\beta}(t + \theta))| \, \mathrm{d}t,$$

it was shown in [4] that if Φ has only simple zeros, and their number in $[0, \pi/\sqrt{\beta}[$ is different from 2, then (1.4) has a T-periodic solution. In particular, this is true assuming that Φ has a constant sign, which, in this context, corresponds to the so-called Landesman–Lazer condition. The choice of $\nu = 3$ is crucial in [4] for exploiting the isochronicity of the homogeneous equation.

The presence of the repulsive singularity has some analogies with the problem of a bouncing particle, which was treated, for example, in [3, 12, 13, 15]. For instance, concerning the simple model

$$x'' + \beta x = e(t),$$
$$x(t_0) = 0 \implies x'(t_0^+) = -x'(t_0^-),$$

describing the behaviour of a particle bouncing elastically against the barrier $\{x = 0\}$, with $\beta > 0$ belonging to the set Σ, it was proved in [3] that, defining the function Φ as above, the same type of result holds. In particular, as also shown in [15, theorem 4.2], this is true if

$$\int_0^T e(t)|\sin(\sqrt{\beta}(t+\theta))|\,\mathrm{d}t > 0 \quad \text{for every } \theta \in [0, T]. \tag{1.6}$$

The aim of this paper is to propose a kind of Landesman–Lazer condition in the case of a nonlinearity with a singularity like the one considered in [6], with the strict inequalities in (1.2) possibly replaced by equalities. We will succeed in replacing the upper inequality with an equality by adding a suitable Landesman–Lazer-type condition on that side. Moreover, condition (1.3) will be replaced by a one-sided control on the function $g(t, x)$. We note that, under our assumptions, we are far from perturbing an isochronous oscillator like the one considered in [4].

Our situation seems to be more delicate than the one studied in [4], or, for the bouncing problem, in [3, 15]. For example, dealing with equation (1.4) with $\beta > 0$ belonging to Σ, and $\nu \neq 3$, it is not clear whether a condition like (1.6) is sufficient for the existence of T-periodic solutions. In order to explain this, we appeal to the analogy between the perturbed isochronous Steen equation and the bouncing problem. When looking for *a priori* estimates in the elastic bouncing problem, it is clear that the elasticity implies that all the bumps of the 'limit function' must have the same height. Unfortunately, this behaviour is not guaranteed for the limit function obtained for the singular periodic problem with lack of isochronicity (see remark 2.5). Hence, since we are unable to predict that the height of each bump will be the same, we will need a Landesman–Lazer condition that refers to each bump separately. Thus, defining

$$\psi(t) = \begin{cases} \sin(\sqrt{\beta}t) & \text{if } t \in \left[0, \dfrac{\pi}{\sqrt{\beta}}\right], \\ 0 & \text{if } t \in \left[\dfrac{\pi}{\sqrt{\beta}}, T\right], \end{cases}$$

instead of (1.6) we will ask a stronger condition, namely,

$$\int_0^T e(t)\psi(t+\theta)\,\mathrm{d}t > 0 \quad \text{for every } \theta \in [0, T]. \tag{1.7}$$

Under this assumption, we will prove that equation (1.4) has a T-periodic solution. However, the possibility of replacing condition (1.7) with (1.6) in our setting remains an open problem.

Note that, following the arguments in [8], under the assumptions of theorem 2.1 it is possible to prove that the radially symmetric system

$$\ddot{z} = -g(t, |z|)\frac{z}{|z|}, \quad z \in \mathbb{R}^M \setminus \{0\},$$

has infinitely many subharmonic solutions rotating around the origin. However, we will not discuss this argument further for reasons of brevity.

2. The main result

Let us consider the T-periodic problem

$$\left.\begin{array}{c} x'' + g(t,x) = 0, \\ x(0) = x(T), \qquad x'(0) = x'(T), \end{array}\right\} \tag{2.1}$$

where $g\colon [0,T]\times\,]0,+\infty[\, \to \mathbb{R}$ is an L^∞-Carathéodory function, namely,

- $t \mapsto g(t,x)$ is measurable for every $x > 0$,
- $x \mapsto g(t,x)$ is continuous, for almost every $t \in [0,T]$,
- for every compact interval $[a,b] \subset\,]0,+\infty[$ there exists a constant $C > 0$ such that

$$|g(t,x)| \leqslant C \quad \text{for almost every (a.e.) } t \in [0,T] \text{ and every } x \in [a,b].$$

Henceforth, when dealing with a solution $x(t)$ of (2.1), in the Carathéodory sense, we will implicitly assume $x(t) > 0$ for every $t \in [0,T]$. Moreover, we will assume that all the functions defined on $[0,T]$ are extended by T-periodicity to the whole real line.

Define, for a fixed non-negative integer N,

$$\mu_N = \left(\frac{N\pi}{T}\right)^2,$$

and set

$$\psi(t) = \begin{cases} \sin(\sqrt{\mu_{N+1}}\,t) & \text{if } t \in \left[0, \dfrac{T}{N+1}\right], \\ 0 & \text{if } t \in \left[\dfrac{T}{N+1}, T\right]. \end{cases}$$

Let us state our main result.

Theorem 2.1. *Assume the following hypotheses.*

(H1) *There exists a constant $\hat{\mu}$ such that*

$$\liminf_{x\to+\infty} \frac{g(t,x)}{x} \geqslant \hat{\mu} > \mu_N,$$

uniformly for almost every $t \in [0,T]$.

(H2) *There exists a constant $\hat{\eta}$ such that*

$$g(t,x) \leqslant \mu_{N+1}x + \hat{\eta} \quad \textit{for a.e. } t \in [0,T] \textit{ and every } x \geqslant 1.$$

Moreover, for every $\theta \in [0,T]$,

$$\int_0^T \limsup_{x\to+\infty}(g(t,x) - \mu_{N+1}x)\psi(t+\theta)\,\mathrm{d}t < 0. \tag{2.2}$$

(H3) *There exist a constant* $\delta > 0$ *and a continuous function* $f\colon\ ]0,\delta] \to \mathbb{R}$ *such that*

$$g(t,x) \leqslant f(x) \quad \text{for a.e. } t \in [0,T] \text{ and every } x \in\,]0,\delta],$$

with

$$\lim_{x\to 0^+} f(x) = -\infty, \qquad \int_0^\delta f(x)\,\mathrm{d}x = -\infty.$$

Then, problem (2.1) has a solution.

Remark 2.2. Theorem 2.1 clearly generalizes theorem 1.1, since (1.2) implies (H1) and (H2), while (1.3) implies (H3), being, in this case, the left-hand side of (2.2) equal to $-\infty$.

Let us introduce, for $\lambda \in [0,1]$, the family of functions

$$g_\lambda(t,x) = \lambda g(t,x) + (1-\lambda)\biggl(-\frac{1}{x^3} + \tfrac{1}{2}(\mu_N + \mu_{N+1})x\biggr). \tag{2.3}$$

Moreover, for $R > 1$, let

$$\Lambda_R = \biggl\{ x \in C([0,T]) \biggm| \frac{1}{R} < x(t) < R \text{ for every } t \in [0,T] \biggr\}.$$

Following the arguments in [8, § 2], it is sufficient to prove that there exists an $R > 1$ such that, for any $\lambda \in [0,1]$, every solution of the problem

$$\left.\begin{aligned} x'' + g_\lambda(t,x) = 0,\\ x(0) = x(T), \qquad x'(0) = x'(T), \end{aligned}\right\} \tag{2.4}$$

belongs to Λ_R. We will prove this using the following two lemmas, the first of which gives the estimate for the maxima, and the second of which focuses on minima.

Lemma 2.3. *There exists a constant* $R > 1$ *such that, for any* $\lambda \in [0,1]$*, if* $x(t)$ *is a solution of (2.4), then*

$$\max_{t\in[0,T]} x(t) < R.$$

Proof. By contradiction, assume that there exist $(\lambda_n)_n \subset [0,1]$ and a sequence $(x_n)_n$ satisfying

$$\left.\begin{aligned} x_n'' + g_{\lambda_n}(t,x_n) = 0,\\ x_n(0) = x_n(T), \qquad x_n'(0) = x_n'(T), \end{aligned}\right\} \tag{2.5}$$

with $\|x_n\|_\infty \to +\infty$. Up to a subsequence, we can assume $\lambda_n \to \bar\lambda \in [0,1]$. We will prove the following claims.

Claim 1. There exists $M > 1$ such that

$$\min_{t\in[0,T]} x_n(t) \leqslant M \quad \text{for every } n \in \mathbb{N}.$$

Proof of claim 1. By contradiction, assume that this is not the case. Namely, there exists a subsequence, still denoted by $(x_n)_n$, such that $x_n(t) \to +\infty$ uniformly in $t \in [0,T]$. Integrating (2.5), we see that

$$\int_0^T g_{\lambda_n}(t, x_n(t))\,\mathrm{d}t = 0.$$

On the other hand, in view of (H1), there exists $d > 0$ such that

$$g_{\lambda_n}(t,x) \geqslant \tfrac{1}{2}\hat{\mu}x \quad \text{for a.e. } t \in [0,T] \text{ and every } x \geqslant d.$$

Since, for n large, $x_n(t) \geqslant d$ for every $t \in [0,T]$, this is clearly impossible. □

Claim 2. For large n, $(x_n(t), x_n'(t))$ makes exactly $N+1$ clockwise revolutions around the point $(1,0)$ when t varies from 0 to T.

Proof of claim 2. We will use some arguments from [8], introducing, with the same notation, the function $\mathcal{N}\colon\ ]0,+\infty[\to \mathbb{R}$, defined as

$$\mathcal{N}(u,v) = \left(\frac{1}{u^2} + u^2 + v^2\right)^{1/2}.$$

Let $\bar{\epsilon} > 0$ be such that

$$\bar{\epsilon} < \frac{T}{(N+1)(N+2)}. \tag{2.6}$$

It is possible to see [8, lemma 2] that there exist $\zeta > 0$ and a sufficiently large $R_1 > M$ with the following property.

If $\tau_1 < \tau_2$ are such that $(x_n(t), x_n'(t))$ makes exactly one rotation in the phase plane around the point $(1,0)$ when t varies from τ_1 to τ_2, and $\mathcal{N}(x_n(t), x_n'(t)) \geqslant R_1$ for every $t \in [\tau_1, \tau_2]$, then

$$\frac{T}{N+1} - \bar{\epsilon} \leqslant \tau_2 - \tau_1 \leqslant \zeta < \frac{T}{N}. \tag{2.7}$$

Now choose a positive integer $\bar{n}$ such that

$$\bar{n}\left(\frac{T}{N+1} - \bar{\epsilon}\right) > T. \tag{2.8}$$

Using [8, lemma 1], there exists $R_2 > R_1$ such that if, for some $t_1 < t_2$,

$$\mathcal{N}(x_n(t_1), x_n'(t_1)) = R_1, \qquad \mathcal{N}(x_n(t_2), x_n'(t_2)) = R_2,$$

and

$$R_1 < \mathcal{N}(x_n(t), x_n'(t)) < R_2 \quad \text{for every } t \in \,]t_1, t_2[\,,$$

then $(x_n(t), x_n'(t))$ makes at least $\bar{n}$ clockwise turns in the phase plane around $(1,0)$ when t varies from t_1 to t_2. Since $\|x_n\|_\infty \to +\infty$, for every large n we will have

$$\max\{\mathcal{N}(x_n(t), x_n'(t)) \mid t \in [0,T]\} > R_2.$$

Consequently, $x_n(t)$ being T-periodic, for sufficiently large n, it will be $\mathcal{N}(x_n(t), x_n'(t)) > R_1$ for every $t \in [0,T]$, otherwise $(x_n(t), x_n'(t))$ would make at least $\bar{n}$ turns around $(1,0)$ in the time period T, which is impossible due to (2.7) and (2.8).

Since $\mathcal{N}(x_n(t), x_n'(t)) > R_1$ for every $t \in [0, T]$, enlarging R_1 if necessary, as shown in the proof of [8, lemma 2], $(x_n(t), x_n'(t))$ clockwise rotates at least once around $(1, 0)$ in the time T. Taking into account (2.6) and (2.7), then, for sufficiently large n, $(x_n(t), x_n'(t))$ will make, in the phase plane, strictly more than N and strictly less than $N+2$ turns around $(1, 0)$ in time T. Since $x_n(t)$ is T-periodic, this implies that $(x_n(t), x_n'(t))$ will make exactly $N+1$ such turns in the time T. □

Claim 3. Setting

$$v_n = \frac{x_n}{\|x_n\|_\infty},$$

the sequence $(v_n)_n$ converges uniformly, up to a subsequence, to a continuous function v. This function is not identically zero and it is twice continuously differentiable, except on a finite subset of $[0, T]$. Moreover, v satisfies $v''(t) + \mu_{N+1} v(t) = 0$ for almost every $t \in [0, T]$.

Proof of claim 3. For every n, the T-periodic function v_n solves

$$v_n'' + \frac{g_{\lambda_n}(t, x_n(t))}{\|x_n\|_\infty} = 0. \tag{2.9}$$

Multiplying both sides of equation (2.9) by v_n and integrating between 0 and T, we have

$$\int_0^T v_n'(t)^2 \, \mathrm{d}t = \int_0^T \frac{g_{\lambda_n}(t, x_n(t))}{\|x_n\|_\infty} v_n(t) \, \mathrm{d}t.$$

By (H3), we can choose $\bar{\bar{\delta}} \in \,]0, 1[$ such that, for every n,

$$g_{\lambda_n}(t, x) \leqslant 0 \quad \text{for a.e. } t \in [0, T] \text{ and every } x \in \,]0, \bar{\bar{\delta}}].$$

For large n, it follows that

$$\begin{aligned}
\int_0^T \frac{g_{\lambda_n}(t, x_n(t))}{\|x_n\|_\infty} v_n(t) \, \mathrm{d}t &= \int_{\{x_n \leqslant \bar{\bar{\delta}}\}} \cdots + \int_{\{\bar{\bar{\delta}} < x_n < 1\}} \cdots + \int_{\{x_n \geqslant 1\}} \cdots \\
&\leqslant CT + \int_{\{x_n \geqslant 1\}} \frac{\mu_{N+1} x_n(t) + \hat{\eta}}{\|x_n\|_\infty} \, \mathrm{d}t \\
&\leqslant (C + \mu_{N+1})T + 1,
\end{aligned}$$

for a suitable constant $C > 0$ coming from the Carathéodory condition. This immediately gives an estimate of the L^2-norm of v_n' which is independent of n. It follows that $(v_n)_n$ is bounded in $H^1(0, T)$, so there exists $v \in H^1(0, T)$ such that, up to a subsequence, $v_n \rightharpoonup v$ (weakly) in $H^1(0, T)$ and $v_n \to v$ uniformly. Let us still denote such a subsequence by $(v_n)_n$. Since $\|v_n\|_\infty = 1$ for every n, we have that $\|v\|_\infty = 1$. Hence, v is not identically zero. Moreover, it is clear that, since v is the uniform limit of positive continuous T-periodic functions, it is a non-negative continuous T-periodic function.

We now prove that $v''(t) + \mu_{N+1} v(t) = 0$ almost everywhere in $[0, T]$. To this end, define the open set

$$\Omega^+ = \{t \in \mathbb{R} \mid v(t) > 0\},$$

which is an at most countable union of open intervals, and consider a C^1-function φ with compact support K_φ contained in Ω^+. Multiplying equation (2.9) by φ and integrating on K_φ, we obtain

$$\int_{K_\varphi} v_n'(t)\varphi'(t)\,\mathrm{d}t = \int_{K_\varphi} \frac{g_{\lambda_n}(t,x_n(t))}{\|x_n\|_\infty}\varphi(t)\,\mathrm{d}t = \int_{K_\varphi} \frac{g_{\lambda_n}(t,x_n(t))}{x_n(t)} v_n(t)\varphi(t)\,\mathrm{d}t. \tag{2.10}$$

Observe now that, in view of the continuity of v (which is strictly greater than 0 on K_φ) and recalling that K_φ is compact and strictly contained in Ω^+, we have that $x_n(t) \to +\infty$ uniformly for every $t \in K_\varphi$. By hypotheses (H1) and (H2), for every positive integer m, there exists a sufficiently large n_m such that

$$\mu_N < \hat{\mu} - \frac{1}{m} \leqslant \frac{g_{\lambda_{n_m}}(t,x_{n_m}(t))}{x_{n_m}(t)} \leqslant \mu_{N+1} + \frac{1}{m}, \tag{2.11}$$

for almost every $t \in K_\varphi$. We can assume the sequence $(n_m)_m$ to be strictly increasing. Considering the subsequence

$$\left(\frac{g_{\lambda_{n_m}}(t,x_{n_m}(t))}{x_{n_m}(t)}\right)_m,$$

it then turns out that it is bounded in $L^2(K_\varphi)$, so that (up to a further subsequence) it converges weakly in $L^2(K_\varphi)$ to a function $\gamma(t)$ which, in view of (2.11), satisfies

$$\mu_N < \hat{\mu} \leqslant \gamma(t) \leqslant \mu_{N+1} \quad \text{for a.e. } t \in K_\varphi. \tag{2.12}$$

Since $v_n \to v$ uniformly and $v_n \rightharpoonup v$ in $H^1(0,T)$, passing to the limit in (2.10) then yields

$$\int_{K_\varphi} v'(t)\varphi'(t)\,\mathrm{d}t = \int_{K_\varphi} \gamma(t)v(t)\varphi(t)\,\mathrm{d}t.$$

By the uniqueness of the weak limit, $\gamma(t)$ can be uniquely extended on the whole set Ω^+, hence, we have

$$\int_{\Omega^+} v'(t)\varphi'(t)\,\mathrm{d}t = \int_{\Omega^+} \gamma(t)v(t)\varphi(t)\,\mathrm{d}t$$

for every C^1-function φ with compact support in Ω^+. This implies that $v(t)$ is a *weak* solution of the equation

$$v'' + \gamma(t)v = 0 \tag{2.13}$$

in Ω^+. Hence, $v \in H^2_{\mathrm{loc}}(\Omega^+)$, and satisfies (2.13) for almost every $t \in \Omega^+$. Moreover, $v(t)$ is continuously differentiable at any point $t \in \Omega^+$.

We now aim to prove that $\gamma(t) = \mu_{N+1}$ for almost every $t \in \Omega^+$. Observe first that, writing the equation as a first-order system, and taking into account that, for every n, the angular velocity of (v_n, v_n') has to be non-positive (see [8]), the set $\{t \in [0,T] \mid x_n(t) = R_1\}$ (which is non-empty since $R_1 > M$) is discrete. Moreover, returning to claim 2, we know that, in the phase plane, $(x_n(t), x_n'(t))$ has to perform exactly $N+1$ turns around the point $(1,0)$ in time T. Then, in the interval $[0,2T]$, we can find

$$\alpha_1^n < \beta_1^n < \alpha_2^n < \beta_2^n < \cdots < \alpha_{N+1}^n < \beta_{N+1}^n < \alpha_1^n + T$$

such that, setting $\alpha_{N+2}^n = \alpha_1^n + T$,

$$x_n(t) > R_1 \quad \text{if } t \in \,]\alpha_r^n, \beta_r^n[\,, \qquad r = 1, \dots, N+1,$$

and

$$x_n(t) < R_1 \quad \text{if } t \in \,]\beta_r^n, \alpha_{r+1}^n[\,, \quad r = 1, \dots, N+1.$$

Since both the sequences $(\beta_r^n)_n$ and $(\alpha_r^n)_n$ are bounded, there exist ξ_r^- and ξ_r^+, with $r = 1, \dots, N+1$ such that, up to subsequences,

$$\alpha_r^n \to \xi_r^-, \quad \beta_r^n \to \xi_r^+, \quad r = 1, \dots, N+1,$$

with

$$\xi_1^- \leqslant \xi_1^+ \leqslant \xi_2^- \leqslant \xi_2^+ \leqslant \cdots \leqslant \xi_{N+1}^- \leqslant \xi_{N+1}^+ \leqslant \xi_1^- + T. \tag{2.14}$$

By the estimates in [8, lemma 2], given any $\epsilon > 0$, for sufficiently large n we have

$$\beta_r^n - \alpha_r^n \geqslant \frac{T}{N+1} - \epsilon, \quad r = 1, \dots, N+1.$$

Passing to the limit, since ϵ is arbitrary, we deduce

$$\xi_r^+ - \xi_r^- \geqslant \frac{T}{N+1}, \quad r = 1, \dots, N+1.$$

Since, by (2.14), $\xi_{N+1}^+ - \xi_1^- \leqslant T$, this implies

$$\xi_r^+ = \xi_r^- + \frac{T}{N+1} \quad \text{and} \quad \xi_{r+1}^- = \xi_r^+, \quad r = 1, \dots, N+1.$$

For simplicity of notation, we set $\xi_r = \xi_r^-$ for $r = 1, \dots, N+1$. Recalling that

$$v_n(\alpha_r^n) = v_n(\beta_r^n) = \frac{R_1}{\|x_n\|_\infty},$$

we deduce that $v(\xi_r) = 0$ for every $r = 1, \dots, N+1$, since v is continuous and $v_n \to v$ uniformly.

Let us now focus on an interval $[\tilde{\alpha}, \tilde{\beta}]$ such that $v > 0$ on $]\tilde{\alpha}, \tilde{\beta}[$, with $v(\tilde{\alpha}) = v(\tilde{\beta}) = 0$. Since, on $]\tilde{\alpha}, \tilde{\beta}[$, v is continuously differentiable and satisfies (2.13) almost everywhere, writing

$$v(t) = \rho(t)\cos(\theta(t)), \qquad v'(t) = \rho(t)\sin(\theta(t)),$$

we have

$$-\theta'(t) = \frac{\gamma(t)v(t)^2 + v'(t)^2}{v(t)^2 + v'(t)^2},$$

which yields, in view of (2.12),

$$\frac{-\theta'(t)}{\mu_{N+1}\cos^2\theta(t) + \sin^2\theta(t)} \leqslant 1 < \frac{-\theta'(t)}{\hat{\mu}\cos^2\theta(t) + \sin^2\theta(t)}.$$

Consequently, integrating between $\tilde{\alpha}$ and $\tilde{\beta}$, we infer, taking into account that $\hat{\mu} > \mu_N$,

$$\frac{T}{N+1} \leqslant \tilde{\beta} - \tilde{\alpha} < \frac{T}{N}.$$

However, since $v(\xi_r) = 0$ for every $r = 1, \ldots, N+1$, and the points ξ_r are equally distributed at a distance $T/(N+1)$ from each other, then it must be the case that $\tilde{\beta} - \tilde{\alpha} = T/(N+1)$. This means that, whenever v becomes positive, it is forced to remain positive for a time exactly equal to $T/(N+1)$. Recalling that v solves $v'' + \gamma(t)v = 0$ on $]\tilde{\alpha}, \tilde{\beta}[$, we now pass to generalized polar coordinates by writing

$$v(t) = \frac{1}{\sqrt{\mu_{N+1}}}\hat{\rho}(t)\cos(\hat{\theta}(t)), \qquad v'(t) = \hat{\rho}(t)\sin(\hat{\theta}(t)).$$

Integrating $\hat{\theta}'(t)$ on $[\tilde{\alpha}, \tilde{\beta}]$, we have

$$\pi = \sqrt{\mu_{N+1}} \int_{\tilde{\alpha}}^{\tilde{\beta}} \frac{\gamma(t)v(t)^2 + v'(t)^2}{\mu_{N+1}v(t)^2 + v'(t)^2}\,\mathrm{d}t \leqslant \sqrt{\mu_{N+1}}\frac{T}{N+1} = \pi,$$

from which we deduce that

$$\gamma(t) = \mu_{N+1} \quad \text{for a.e. } t \in [\tilde{\alpha}, \tilde{\beta}].$$

This means, in particular, that Ω^+ is the union of some intervals of the type $]\xi_r, \xi_{r+1}[$, on which we have

$$v(t) = c_r \sin(\sqrt{\mu_{N+1}}(t - \xi_r)),$$

where, for every r, the constants c_r are in $[0, 1]$, and at least one of them is equal to 1. The claim is thus proved. □

Conclusion of the proof of lemma 2.3. Let us focus again on an interval $[\tilde{\alpha}, \tilde{\beta}]$ such that $v(\tilde{\alpha}) = v(\tilde{\beta}) = 0$, and $v > 0$ on $]\tilde{\alpha}, \tilde{\beta}[$. Note first that, since $\gamma(t) = \mu_{N+1}$ almost everywhere in Ω^+, it must be the case that $\lambda_n \to 1$. Indeed, if it were $\lambda_n \to \bar{\lambda} < 1$, taking any interval $I \subset\,]\tilde{\alpha}, \tilde{\beta}[$, for every $t \in I$ it would be, for large n,

$$\frac{g_{\lambda_n}(t, x_n(t))}{x_n(t)} \leqslant \tilde{\zeta} < \mu_{N+1},$$

for a suitable constant $\tilde{\zeta}$ and, taking the weak limit, we would reach the contradiction $\gamma(t) < \mu_{N+1}$ for almost every $t \in I$. By the previous discussion, we know that $\tilde{\beta} - \tilde{\alpha} = T/(N+1)$ and that, on $]\tilde{\alpha}, \tilde{\beta}[$, v solves the Dirichlet problem

$$\begin{aligned} v'' + \mu_{N+1}v &= 0, \\ v(\tilde{\alpha}) = 0 &= v(\tilde{\beta}). \end{aligned}$$

Explicitly, for some $C > 0$,

$$v(t) = C\sin\left(\frac{(N+1)\pi}{T}(t - \tilde{\alpha})\right). \tag{2.15}$$

Let $(\phi_k)_{k\geqslant 1}$ be the orthonormal $L^2(\tilde{\alpha},\tilde{\beta})$-basis made of the solutions of the Dirichlet problem

$$\begin{gathered}\phi_k''+\mu_k\phi_k=0,\\ \phi_k(\tilde{\alpha})=0=\phi_k(\tilde{\beta}),\end{gathered}$$

where μ_k is the kth eigenvalue. Let us denote by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\tilde{\alpha},\tilde{\beta})$ and by $\|\cdot\|_2$ the $L^2(\tilde{\alpha},\tilde{\beta})$-norm. We write, for every n, the Fourier series of x_n,

$$x_n=\sum_{k=1}^{+\infty}\langle x_n,\phi_k\rangle\phi_k.$$

It is useful to split such an expression as follows:

$$x_n=x_n^0+x_n^{\perp},$$

where

$$x_n^0=\langle x_n,\phi_{N+1}\rangle\phi_{N+1},$$

while

$$x_n^{\perp}=\sum_{k\neq N+1}\langle x_n,\phi_k\rangle\phi_k.$$

It is known that $x_n^{\perp}$ is L^2-orthogonal to x_n^0 in view of the properties of the eigenfunctions ϕ_k. Dividing x_n by $\|x_n\|_\infty$, we have $v_n=v_n^0+v_n^{\perp}$ with

$$v_n^0=\frac{x_n^0}{\|x_n\|_\infty},\qquad v_n^{\perp}=\frac{x_n^{\perp}}{\|x_n\|_\infty}.$$

Since $v=v^0$, we have

$$v_n^0=\frac{\langle v_n,v\rangle}{\|v\|_2}v\to v\quad\text{uniformly in } t\in[\tilde{\alpha},\tilde{\beta}].$$

As

$$\begin{aligned}x_n''&=\sum_{k=1}^{+\infty}\langle x_n,\phi_k\rangle\phi_k''\\ &=-\sum_{k=1}^{+\infty}\langle x_n,\phi_k\rangle\mu_k\phi_k,\end{aligned}$$

we have that

$$(x_n'')^0=(x_n^0)'',\qquad (x_n'')^{\perp}=(x_n^{\perp})''.$$

Multiplying equation (2.5) by v_n^0 and integrating between $\tilde{\alpha}$ and $\tilde{\beta}$, we then have

$$-\int_{\tilde{\alpha}}^{\tilde{\beta}}(x_n^0)''(t)v_n^0(t)\,\mathrm{d}t=\int_{\tilde{\alpha}}^{\tilde{\beta}}g_{\lambda_n}(t,x_n(t))v_n^0(t)\,\mathrm{d}t. \tag{2.16}$$

Integrating twice by parts, we obtain

$$\int_{\tilde\alpha}^{\tilde\beta} (x_n^0)''(t) v_n^0(t)\,\mathrm{d}t = \int_{\tilde\alpha}^{\tilde\beta} x_n^0(t)(v_n^0)''(t)\,\mathrm{d}t = -\int_{\tilde\alpha}^{\tilde\beta} \mu_{N+1} x_n^0(t) v_n^0(t)\,\mathrm{d}t. \tag{2.17}$$

On the other hand, note that, by (H3), there exists $\omega \in\,]0, \min\{\delta, 1\}[$ such that

$$g(t,x) \leqslant f(x) \leqslant 0 \quad \text{for every } x \in\,]0, \omega[\,.$$

Define the two L^∞-Carathéodory functions

$$\sigma(t,x) = \begin{cases} g(t,x) & \text{if } x \in\,]0, \frac{1}{2}\omega[, \\ \dfrac{2}{\omega} g(t,x)(\omega - x) & \text{if } x \in [\frac{1}{2}\omega, \omega], \\ 0 & \text{if } x \in\,]\omega, +\infty[\,, \end{cases}$$

and

$$r(t,x) = g(t,x) - \mu_{N+1} x - \sigma(t,x).$$

Using (H2), together with the fact that $\sigma(t,x) \leqslant 0$, there exists a positive constant $\tilde\eta$ such that

$$r(t,x) \leqslant \tilde\eta \quad \text{for a.e. } t \in [0,T] \text{ and every } x > 0. \tag{2.18}$$

As a consequence,

$$\begin{aligned} &\int_{\tilde\alpha}^{\tilde\beta} g_{\lambda_n}(t, x_n(t)) v_n^0(t)\,\mathrm{d}t \\ &\qquad = \lambda_n \int_{\tilde\alpha}^{\tilde\beta} \sigma(t, x_n(t)) v_n^0(t)\,\mathrm{d}t + \lambda_n \int_{\tilde\alpha}^{\tilde\beta} \mu_{N+1} x_n(t) v_n^0(t)\,\mathrm{d}t \\ &\qquad\quad + \lambda_n \int_{\tilde\alpha}^{\tilde\beta} r(t, x_n(t)) v_n^0(t)\,\mathrm{d}t \\ &\qquad\quad + (1-\lambda_n) \int_{\tilde\alpha}^{\tilde\beta} \left(-\frac{1}{x_n(t)^3} + \tfrac{1}{2}(\mu_N + \mu_{N+1}) x_n(t) \right) v_n^0(t)\,\mathrm{d}t, \end{aligned}$$

so that

$$\int_{\tilde\alpha}^{\tilde\beta} g_{\lambda_n}(t, x_n(t)) v_n^0(t)\,\mathrm{d}t \leqslant \int_{\tilde\alpha}^{\tilde\beta} \mu_{N+1} x_n^0(t) v_n^0(t)\,\mathrm{d}t + \lambda_n \int_{\tilde\alpha}^{\tilde\beta} r(t, x_n(t)) v_n^0(t)\,\mathrm{d}t,$$

in view of the fact that x_n and v_n^0 are positive. Using (2.16), (2.17) and the fact that $\lambda_n \to 1$, we deduce that, for sufficiently large n,

$$\int_{\tilde\alpha}^{\tilde\beta} r(t, x_n(t)) v_n^0(t)\,\mathrm{d}t \geqslant 0.$$

As $v_n^0 \to v$ uniformly, by (2.18) the integrand is bounded from above by a positive constant. It is then possible to apply Fatou's lemma, so that

$$\int_{\tilde{\alpha}}^{\tilde{\beta}} \limsup_{n\to+\infty} r(t,x_n(t))v_n^0(t)\,\mathrm{d}t \geqslant 0,$$

whence, since $\lim_{n\to+\infty} v_n^0(t) = v(t)$ for every $t \in [\tilde{\alpha},\tilde{\beta}]$,

$$\int_{\tilde{\alpha}}^{\tilde{\beta}} \limsup_{n\to+\infty} r(t,x_n(t))v(t)\,\mathrm{d}t \geqslant 0.$$

As

$$r(t,x) = g(t,x) - \mu_{N+1}x \quad \text{for a.e. } t \in [0,T] \text{ and every } x \geqslant 1,$$

and $x_n(t) \to +\infty$ for every $t \in \,]\tilde{\alpha},\tilde{\beta}[\,$, we deduce that

$$\int_{\tilde{\alpha}}^{\tilde{\beta}} \limsup_{x\to+\infty}(g(t,x) - \mu_{N+1}x)v(t)\,\mathrm{d}t \geqslant 0.$$

Since, in view of (2.15), $v(t) = C\psi(t-\tilde{\alpha})$ for every $t \in [\tilde{\alpha},\tilde{\beta}]$, this contradicts (2.2). □

We now prove the counterpart of lemma 2.3 for minima.

LEMMA 2.4. *There exists a constant* $R' \in \,]0,1[$ *such that, for any* $\lambda \in [0,1]$, *if* $x(t)$ *is a solution of (2.4), then*

$$\min_{t\in[0,T]} x(t) > R'.$$

Proof. By contradiction, assume that there exist two sequences $(\lambda_n)_n \subset [0,1]$ and $(x_n)_n$, with $x_n(t)$ solving (2.4) for $\lambda = \lambda_n$, such that $\min_{t\in[0,T]} x_n(t) \to 0$. Let $R > 1$ be as in lemma 2.3 and fix a sufficiently small $\epsilon > 0$. As we have already recalled, by [8, lemma 2] there exists a sufficiently large $R_1 > \mathcal{N}(R,0)$ such that if $(x_n(t),x_n'(t))$ makes exactly one rotation around the point $(1,0)$ when t varies from τ_1 to τ_2, and $\mathcal{N}(x_n(t),x_n'(t)) \geqslant R_1$ for every $t \in [\tau_1,\tau_2]$, then

$$\tau_2 - \tau_1 \geqslant \frac{T}{N+1} - \epsilon.$$

Now choose a positive integer $\bar{n}$ such that

$$\bar{n}\left(\frac{T}{N+1} - \epsilon\right) > T.$$

Recalling again [8, lemma 1], as used in the proof of lemma 2.3, there exists $R_2 > R_1$ such that if

$$\mathcal{N}(x_n(t_1),x_n'(t_1)) = R_1, \qquad \mathcal{N}(x_n(t_2),x_n'(t_2)) = R_2,$$

for some $t_1 < t_2$, and

$$R_1 < \mathcal{N}(x_n(t),x_n'(t)) < R_2 \quad \text{for every } t \in \,]t_1,t_2[\,,$$

then $(x_n(t), x_n'(t))$ turns at least $\bar{n}$ times around $(1, 0)$ in the phase plane when t varies from t_1 to t_2. As $\min_{t\in[0,T]} x_n(t) \to 0$, for every large n, we will have

$$\max\{\mathcal{N}(x_n(t), x_n'(t)) \mid t \in [0, T]\} > R_2.$$

Since $x_n(t)$ is T-periodic, and in view of the continuity of $\mathcal{N}(x_n(t), x_n'(t))$, for sufficiently large n it follows that $\mathcal{N}(x_n(t), x_n'(t)) > R_1$ for every $t \in [0, T]$, otherwise $(x_n(t), x_n'(t))$ would turn at least $\bar{n}$ times around $(1, 0)$ in time T, which is impossible. Choosing $\bar{t}_n$ such that $x_n(\bar{t}_n) = \max_{t\in[0,T]} x_n(t)$, being $x_n'(\bar{t}_n) = 0$ it would then be, for large n,

$$\mathcal{N}(x_n(\bar{t}_n), 0) > R_1,$$

which, since $x_n(\bar{t}_n) < R$, contradicts the inequality $\mathcal{N}(R, 0) < R_1$. □

REMARK 2.5. As mentioned in § 1, for the special case of equation (1.4), condition (2.2) reads as (1.7). Let us try to explain why we are not able to replace (1.7) with the more general assumption (1.6). We recall that, in [8], a curve $\Gamma\colon [0, +\infty[\to \mathbb{R}^2$ was constructed in order to control the solutions in the phase plane. One could try to estimate the height of the bumps of $v(t)$ by using such a curve for the solutions $x_n(t)$. However, consider, for example, equation (1.5) and define $g_\lambda(t, x)$ as in (2.3). We would then have

$$g_\lambda(t, x) = -\frac{1}{x^3} + (\tfrac{1}{2}(\mu_N + \mu_{N+1}) + \tfrac{1}{2}\lambda(\mu_{N+1} - \mu_N))x.$$

Hence, the curve $\Gamma(t)$, as it is constructed in [8], should use the level lines of the functions

$$V_1(u, v) = \frac{1}{2}\left(\frac{1}{u^2} + \mu_{N+1}u^2 + v^2\right), \qquad V_2(u, v) = \frac{1}{2}\left(\frac{1}{u^2} + \tfrac{1}{2}(\mu_N + \mu_{N+1})u^2 + v^2\right).$$

A straightforward computation shows, however, that, denoting by $P_m = (u_m, 0)$ and $P_{m+1} = (u_{m+1}, 0)$ two consecutive intersections of $\Gamma(t)$ with the half-line $\{(u, v) \in \mathbb{R}^2 \mid u \geqslant 1, v = 0\}$, it is

$$\liminf_{m\to+\infty} \frac{u_{m+1}}{u_m} > 1.$$

This fact makes us unable to prove that the height of the bumps of $v(t)$, which is obtained as the limit of the normalized sequence $(v_n)_n$, is always the same. This is why we did not manage, in our setting, to assume a Landesman–Lazer condition like (1.6).

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