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Radially symmetric systems with a singularity and asymptotically linear growth

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ABSTRACT

We prove the existence of infinitely many periodic solutions for radially symmetric systems with a singularity of repulsive type. The nonlinearity is assumed to have a linear growth at infinity, being controlled by two constants which have a precise interpretation in terms of the Dancer–Fučik spectrum. Our result generalizes an existence theorem by Del Pino et al. (1992) [4], obtained in the case of a scalar second order differential equation.

1. Introduction

We consider a radially symmetric system in $\mathbb{R}^N$ of the type

$$\ddot{x} = -h(t, |x|)\frac{x}{|x|}, \tag{1}$$

where $h : \mathbb{R}\times]0, +\infty[\to \mathbb{R}$ is continuous, T-periodic with respect to its first variable, and has a singularity of repulsive type at the origin in its second variable. We look for non-collision periodic solutions, i.e., solutions which never attain the singularity.

As a first approach, one could look for periodic solutions whose orbits always stay on a given radius. Setting $\rho(t) = |x(t)|$, the study of these solutions involves the scalar equation

$$\ddot{\rho} + h(t, \rho) = 0, \tag{2}$$

which has been first considered by Lazer and Solimini in [1] and later investigated by many authors; see, e.g., [2–11]. Roughly speaking, the singularity being of repulsive type, one can expect that T-periodic solutions exist, provided that the force is attractive at some distance from the origin. However, some care must be taken in order to avoid what seems to be a kind of *resonance* at infinity. This fact is put in evidence in the following result by Del Pino, Manásevich and Montero.

Theorem 1 ([4]). *Let the following two assumptions hold.*

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(H1) *There are an integer M and two constants α, β for which*

$$\left(\frac{M\pi}{T}\right)^2 < \alpha \le \liminf_{r\to+\infty} \frac{h(t,r)}{r} \le \limsup_{r\to+\infty} \frac{h(t,r)}{r} \le \beta < \left(\frac{(M+1)\pi}{T}\right)^2,$$

uniformly for every $t \in [0,T]$;

(H2) *there are positive constants c', c'', δ, and $\nu \ge 1$, such that*

$$\frac{c'}{r^\nu} \le -h(t,r) \le \frac{c''}{r^\nu}, \quad \textit{for every } t \in [0,T] \textit{ and every } r \in]0,\delta].$$

Then, Eq. (2) has a T-periodic solution.

The result in [4] is somehow related to a paper by Fabry and Habets [12], where a periodic problem without singularities is treated. Indeed, in [12] an asymmetric oscillator is considered, assuming, roughly speaking, that the nonlinearity at $+\infty$ asymptotically lies between the asymptotes of two consecutive curves of the Dancer–Fučik spectrum; cf. [13]. It is readily seen that the constants appearing in (H1) correspond to these asymptotes. The analogy between these results can be explained, see, e.g., [6], by the fact that the singularity provides for the solutions a similar behaviour as when a superlinear assumption on the nonlinearity at $-\infty$ is made. However, the methods of proof followed in [4] and [12] differ considerably, despite the fact they both use topological degree theory.

Condition (H2) in Theorem 1 has been improved in [9,10] at the expense of assuming $h(t,r)$ to be equal to $f(r)+e(t)$, the sum of two continuous functions. It was then asked that

$$\lim_{r\to 0^+} f(r) = -\infty, \qquad \int_0^1 f(r)\mathrm{d}r = -\infty. \tag{3}$$

Such an assumption was already considered, e.g., in [1].

In the recent papers [14–18], exploiting the radial symmetry of the system, it has been possible to prove the existence of periodic solutions of system (1) rotating around the origin, by a topological degree approach. The repulsive case was treated in [15], for nonlinearities with sublinear growth at infinity, and in [17], for those with superlinear growth. Since the solutions lie on a plane, the idea is to separate the radial and the angular component and, as a first step, to find T-periodic *radial* solutions, those for which the angular momentum is equal to zero. Those solutions for the radial component are then shown to survive when the angular momentum remains small enough, thus giving rise to slowly rotating solutions which, as a rule, are almost periodic, and become periodic for some suitable choices of the angular momenta.

In this paper, we will deal with the intermediate case of a nonlinearity with a linearly controlled growth at infinity, like in Theorem 1. We will generalize the result by Del Pino, Manásevich and Montero in two directions: first, by assuming only a one-sided inequality on the nonlinearity near the singularity, namely $h(t,r) \le f(r)$, with f satisfying a condition like (3). And, as a conclusion, by proving, besides the existence of a T-periodic solution for the scalar equation (2), the existence of infinitely many periodic solutions for system (1), rotating around the origin, as well.

Let us state our main result.

Theorem 2. *Let the following two assumptions hold.*

(H1) *There are an integer M and two constants α, β for which*

$$\left(\frac{M\pi}{T}\right)^2 < \alpha \le \liminf_{r\to+\infty} \frac{h(t,r)}{r} \le \limsup_{r\to+\infty} \frac{h(t,r)}{r} \le \beta < \left(\frac{(M+1)\pi}{T}\right)^2,$$

uniformly for every $t \in [0,T]$;

(H2′) *there are a positive constant δ and a continuous function $f :]0,\delta] \to \mathbb{R}$ such that*

$$h(t,r) \le f(r), \quad \textit{for every } t \in [0,T] \textit{ and every } r \in]0,\delta],$$

and

$$\lim_{r\to 0^+} f(r) = -\infty, \qquad \int_0^\delta f(r)\mathrm{d}r = -\infty.$$

Then, Eq. (2) has a T-periodic solution, and there exists a $k_1 \ge 1$ such that, for every integer $k \ge k_1$, Eq. (1) has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover, there is a constant $R > 0$ such that, for every $k \ge k_1$

$$\frac{1}{R} < |x_k(t)| < R, \quad \textit{for every } t \in \mathbb{R},$$

and, if μ_k denotes the angular momentum associated to $x_k(t)$, then

$$\lim_{k\to\infty} \mu_k = 0.$$

In the proof of Theorem 2, we will exploit the general approach developed in our previous papers, adapting moreover the method by Fabry and Habets to our situation. We thus also provide a different proof of the result by Del Pino, Manásevich and Montero, which becomes a corollary of Theorem 2. In particular, in Section 3 we will prove a general lemma ensuring a control on the solutions of the scalar differential equation (2) which, in the phase plane, perform a given number of rotations. We think that this result has an independent interest, and for this reason we have stated it in a more general framework.

Concerning systems with repulsive singularities having no radial symmetry, existence results were obtained, e.g., in [19–27]. See also the recent monograph [28], and the references therein.

2. The main result

We look for periodic solutions of the system

$$\ddot{x} = -h(t, |x|)\frac{x}{|x|}. \tag{4}$$

The function $h : \mathbb{R} \times]0, +\infty[\to \mathbb{R}$, instead of being continuous, can more generally be assumed to be L^∞-Carathéodory, and T-periodic with respect to the first variable, i.e.,

- $h(\cdot, r)$ is measurable and T-periodic, for every $r > 0$;
- $h(t, \cdot)$ is continuous, for almost every $t \in]0, T[$;
- for every compact interval I in $]0, +\infty[$, there exists a constant $c_I \in \mathbb{R}$ such that

$$r \in I \Rightarrow |h(t, r)| \le c_I, \quad \text{for a.e. } t \in]0, T[.$$

Accordingly, the assumptions in (H1) and (H2′) are meant to be satisfied for almost every $t \in]0, T[$.

We look for solutions $x(t) \in \mathbb{R}^N$ which never attain the singularity, in the sense that

$$x(t) \neq 0, \quad \text{for every } t \in \mathbb{R}. \tag{5}$$

Since system (4) is radially symmetric, the orbit of a solution always lies on a plane, so we will assume, without loss of generality, that $N = 2$ (cf. [14, Appendix A]).

We may write the solutions of (4) in polar coordinates:

$$x(t) = \rho(t)(\cos\varphi(t), \sin\varphi(t)), \tag{6}$$

and (5) is satisfied if $\rho(t) > 0$, for every t. Eq. (4) is then equivalent to the system

$$(S)\ \begin{cases} \ddot{\rho} - \dfrac{\mu^2}{\rho^3} + h(t, \rho) = 0, \\ \rho^2\dot{\varphi} = \mu, \end{cases}$$

where μ is the (scalar) angular momentum of $x(t)$. Recall that μ is constant in time along any solution. In the following, when considering a solution of (S), we will always implicitly assume that $\rho > 0$.

We consider $\mu \ge 0$ as a parameter, and, by the use of degree theory, we will prove the existence of a T-periodic solution ρ of the first equation in (S). We thus deal now with the problem

$$(P_\mu)\ \begin{cases} \ddot{\rho} - \dfrac{\mu^2}{\rho^3} + h(t, \rho) = 0, \\ \rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T). \end{cases}$$

Let X be a Banach space of functions, such that

$$C^1([0, T]) \subseteq X \subseteq C([0, T]),$$

with continuous immersions, and set

$$X_+ = \{\rho \in X : \min \rho > 0\}.$$

Define the linear operator

$$L : D(L) \subset X \to L^1(0, T),$$
$$D(L) = \left\{\rho \in W^{2,1}(0, T) : \rho(0) = \rho(T), \dot{\rho}(0) = \dot{\rho}(T)\right\},$$
$$L\rho = \ddot{\rho},$$

and the Nemytskii operator

$$N_\mu : X_+ \to L^1(0, T),$$
$$(N_\mu\rho)(t) = \frac{\mu^2}{\rho^3(t)} - h(t, \rho(t)).$$

Problem (P_μ) is then equivalent to

$$L\rho = N_\mu \rho. \tag{7}$$

Taking $\sigma \in \mathbb{R}$ not belonging to the spectrum of L, we have that (7) can be translated to the fixed point problem

$$\rho = (L-\sigma I)^{-1}\left(N_\mu - \sigma I\right)\rho.$$

We will say that a set $\Omega \subseteq X$ is *uniformly positively bounded below* if there is a constant $\bar{\delta} > 0$ such that $\min \rho \geq \bar{\delta}$ for every $\rho \in \Omega$.

The following theorem has been proved in [16]. It provides a general degree assumption, only at $\mu = 0$, which automatically implies, for every $\mu > 0$ sufficiently small, the solvability of (S), with $\rho(t)$ a T-periodic function. Moreover, corresponding to an appropriate choice of μ, the function $\varphi(t)$ will satisfy the equality $\varphi(t+kT) = \varphi(t) + 2\pi$, for a sufficiently large integer k, thus providing a solution of (1) which rotates exactly once around the origin in the period time kT.

Theorem 3. *Let Ω be an open bounded subset of X, uniformly positively bounded below. Assume that there is no solution of (7), with $\mu = 0$, on the boundary $\partial\Omega$, and that*

$$\deg\left(I - (L-\sigma I)^{-1}(N_0 - \sigma I), \Omega\right) \neq 0. \tag{8}$$

Then, there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, system (4) has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. The function $|x_k(t)|$ is T-periodic and, when restricted to $[0, T]$, it belongs to Ω. Moreover, if μ_k denotes the angular momentum associated to $x_k(t)$, then

$$\lim_{k\to\infty} \mu_k = 0.$$

In order to apply the above theorem, we consider the case $\mu = 0$ and study, for $\lambda \in [0, 1]$, the problem

$$\begin{cases} \ddot{\rho} + g(t, \rho; \lambda) = 0, \\ \rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T), \end{cases} \tag{9}$$

where

$$g(t, r; \lambda) = (1-\lambda)\left(-\frac{1}{r} + \left(\frac{\left(M+\frac{1}{2}\right)\pi}{T}\right)^2 r\right) + \lambda h(t, r).$$

Notice that $g(t, r; 1) = h(t, r)$. We choose $X = C^1([0, T])$ and define, for $R > 1$, the open set

$$\Omega_R = \left\{\rho \in X : \frac{1}{R} < \rho(t) < R \text{ and } |\dot{\rho}(t)| < R, \text{ for every } t \in [0, T]\right\}.$$

Clearly, Ω_R is bounded, and uniformly positively bounded below. The important point to be proved is the following.

Proposition 1. *There is an $R > 1$ such that, for any $\lambda \in [0, 1]$, all the solutions of problem (9) belong to Ω_R.*

Assuming that Proposition 1 has been proved, and such an $R > 1$ has been determined, we have that, when considering the Nemytskii operator

$$\hat{N}_\lambda : X_+ \to L^1(0, T),$$
$$(\hat{N}_\lambda \rho)(t) = g(t, \rho(t); \lambda),$$

the degree $\deg(I - (L-\sigma I)^{-1}(\hat{N}_\lambda - \sigma I), \Omega_R)$ is independent of $\lambda \in [0, 1]$. Let us compute it for $\lambda = 0$. Writing the differential equation as the equivalent system

$$\begin{cases} \dot{u} = v \\ \dot{v} = \dfrac{1}{u} - \left(\dfrac{\left(M+\frac{1}{2}\right)\pi}{T}\right)^2 u, \end{cases}$$

and defining the function

$$F : \left[\frac{1}{R}, R\right] \times [-R, R] \to \mathbb{R}^2,$$

$$F(u, v) = \left(v, \frac{1}{u} - \left(\frac{\left(M+\frac{1}{2}\right)\pi}{T}\right)^2 u\right),$$

we apply a result of Capietto, Mawhin and Zanolin [29, Theorem 1]: the Leray–Schauder degree of $I-(L-\sigma I)^{-1}(\hat{N}_0-\sigma I)$ is equal to the Brouwer degree of F:

$$d_{LS}\left(I-(L-\sigma I)^{-1}\left(\hat{N}_0-\sigma I\right),\Omega_R\right)=d_B\left(F,\left]\frac{1}{R},R\right[\times]-R,R[\right).$$

The function F has a unique zero (u_0,v_0), necessarily belonging to the set $]\frac{1}{R},R[\times]-R,R[$, as $u_0=T/(M+\frac{1}{2})\pi$ is a constant solution of (9) with $\lambda=0$, and $v_0=0$. Since an explicit computation shows that the Jacobian matrix $J_F(u_0,v_0)$ has a positive determinant, the degree has to be equal to 1. Being $\hat{N}_1=N_0$, we conclude that $\deg(I-(L-\sigma I)^{-1}(N_0-\sigma I),\Omega_R)=1$, so that (8) is satisfied, and Theorem 3 applies, thus providing a proof of Theorem 2.

So, it remains to prove Proposition 1. To this aim, we need a preliminary lemma, to which the following section is devoted.

3. A preliminary lemma

In this section, we consider the general equation

$$\ddot{\rho}+g(t,\rho;\lambda)=0, \tag{10}$$

where $g:\mathbb{R}\times]a,b[\times[0,1]\to\mathbb{R}$, with $-\infty\le a<b\le+\infty$, is a L^∞-Carathéodory function, uniformly in $\lambda\in[0,1]$, and T-periodic in its first variable, i.e.,

- $g(\cdot,r;\lambda)$ is measurable and T-periodic, for every $(r,\lambda)\in]a,b[\times[0,1]$;
- $g(t,\cdot;\lambda)$ is continuous, for a.e. $t\in\mathbb{R}$ and every $\lambda\in[0,1]$;
- for every compact interval $[\hat{a},\hat{b}]$, contained in $]a,b[$, there is a constant $\hat{c}$ such that

$$|g(t,r;\lambda)|\le\hat{c},\quad \text{for a.e. } t\in\mathbb{R} \text{ and every } (r,\lambda)\in[\hat{a},\hat{b}]\times[0,1].$$

Moreover, we assume that there exist two constants $\bar{a},\bar{b}$, with $a<\bar{a}<\bar{b}<b$, and two continuous functions $g_1:]a,\bar{a}]\to\mathbb{R}$, $g_2:[\bar{b},b[\to\mathbb{R}$ such that

$$\begin{aligned} &g(t,r;\lambda)\le g_1(r)<0,\quad \text{for a.e. } t\in\mathbb{R},\ \text{every } r\in]a,\bar{a}]\ \text{and } \lambda\in[0,1],\\ &g(t,r;\lambda)\ge g_2(r)>0,\quad \text{for a.e. } t\in\mathbb{R},\ \text{every } r\in[\bar{b},b[\ \text{and } \lambda\in[0,1], \end{aligned} \tag{11}$$

and, if G_1 and G_2 are primitives of g_1, and g_2, respectively, then

$$\lim_{r\to a^+}G_1(r)=+\infty,\qquad \lim_{r\to b^-}G_2(r)=+\infty. \tag{12}$$

In the phase plane, writing the equivalent system

$$\begin{cases}\dot{u}=v\\ \dot{v}=-g(t,u;\lambda),\end{cases} \tag{13}$$

we work in the set

$$\mathcal{S}=\left\{(u,v)\in\mathbb{R}^2:a<u<b\right\},$$

and we introduce the function $\mathcal{N}:\mathcal{S}\to\mathbb{R}$, defined by

$$\mathcal{N}(u,v)=\begin{cases}\left(u^2+v^2\right)^{\frac12} & \text{if } a=-\infty,\ b=+\infty,\\ \left(\dfrac{1}{(u-a)^2}+u^2+v^2\right)^{\frac12} & \text{if } a>-\infty,\ b=+\infty,\\ \left(\dfrac{1}{(b-u)^2}+u^2+v^2\right)^{\frac12} & \text{if } a=-\infty,\ b<+\infty,\\ \left(\dfrac{1}{(u-a)^2}+\dfrac{1}{(b-u)^2}+v^2\right)^{\frac12} & \text{if } a>-\infty,\ b<+\infty.\end{cases}$$

Lemma 1. *In the above setting, let $\bar{r}\in]\bar{a},\bar{b}[$ be given and fix a positive integer $\bar{n}$. Then, for every $R_1>\mathcal{N}(\bar{r},0)$ there is a $R_2>R_1$ such that, for every $\lambda\in[0,1]$, if $\rho(t)$ is a solution of (10), and $t_1<t_2$ are such that*

$$\mathcal{N}(\rho(t_1),\dot{\rho}(t_1))=R_1,\qquad \mathcal{N}(\rho(t_2),\dot{\rho}(t_2))=R_2,$$

and

$$R_1<\mathcal{N}(\rho(t),\dot{\rho}(t))<R_2,\quad \text{for every } t\in]t_1,t_2[,$$

then $(\rho(t),\dot{\rho}(t))$ makes at least $\bar{n}$ clockwise rotations around the point $(\bar{r},0)$ in the time interval $[t_1,t_2]$.

Proof. First of all, notice that it is not restrictive to assume that the strict inequalities hold in (11), taking if necessary $\frac{1}{2}g_1$ and $\frac{1}{2}g_2$ instead of g_1 and g_2, respectively, so that (12) is still satisfied. We begin by defining two continuous functions $\tilde{g}_1, \tilde{g}_2 :]a, b[\to \mathbb{R}$ such that

$$\tilde{g}_2(r) < g(t, r; \lambda) < \tilde{g}_1(r), \quad \text{for a.e. } t \in [0, T], \text{ and every } (r, \lambda) \in]a, b[\times [0, 1],$$

and, if $\tilde{G}_1$ and $\tilde{G}_2$ are primitives of $\tilde{g}_1$, and $\tilde{g}_2$, respectively, then

$$\lim_{r \to a^+} \tilde{G}_1(r) = \lim_{r \to b^-} \tilde{G}_1(r) = +\infty, \qquad \lim_{r \to a^+} \tilde{G}_2(r) = \lim_{r \to b^-} \tilde{G}_2(r) = +\infty. \tag{14}$$

In order to define $\tilde{g}_1$, let $(b_n)_n$ be a strictly increasing sequence such that $b_0 = \bar{a}$, $b_1 = \bar{b}$, and $\lim_n b_n = b$. Let, for every $n \geq 1$,

$$c_n = \operatorname{esssup}\{g(t, r; \lambda) : t \in [0, T], r \in [\bar{a}, b_n], \lambda \in [0, 1]\} + 1.$$

Fix $\hat{a} \in]a, \bar{a}[$ and set

$$\tilde{g}_1(r) = \begin{cases} g_1(r) & \text{if } r \leq \hat{a}, \\ \max\left\{ \dfrac{c_1 - g_1(\hat{a})}{b_0 - \hat{a}}(r - \hat{a}) + g_1(\hat{a}), g_1(r) \right\} & \text{if } \hat{a} < r \leq b_0, \\ \dfrac{c_{n+2} - c_{n+1}}{b_{n+1} - b_n}(r - b_n) + c_{n+1} & \text{if } b_n < r \leq b_{n+1}. \end{cases}$$

Notice that $\tilde{g}_1$ is piecewise linear on $[\bar{a}, b[$, and $\tilde{g}_1(b_n) = c_{n+1}$ for every $n \in \mathbb{N}$. It is easy to see that $\tilde{g}_1$ fulfills the above requirements. The function $\tilde{g}_2$ is defined analogously.

We now construct a curve Γ which will control the evolution of the solutions of (10) in the phase plane. To simplify the exposition, in the rest of the proof we will assume the function g to be continuous. Let

$$V_1(u, v) = \frac{1}{2}v^2 + \tilde{G}_1(u), \qquad V_2(u, v) = \frac{1}{2}v^2 + \tilde{G}_2(u).$$

Set $D_1 = \max\{\tilde{G}_2(u) : u \in [\bar{a}, \bar{b}]\} + 1$ and define the line

$$\mathcal{L}_1 = \{(u, v) \in \mathcal{S} : V_2(u, v) = D_1, v \geq 0\}.$$

Notice that $\mathcal{L}_1$ is the graph of the function $F_1(u) = \sqrt{2(D_1 - \tilde{G}_2(u))}$. By (14), there are u_1^-, u_1^+, with $a < u_1^- < \bar{a} < \bar{b} < u_1^+ < b$, such that

$$F_1(u_1^-) = F_1(u_1^+) = 0, \quad \text{and} \quad F_1(u) > 0 \quad \text{for every } u \in]u_1^-, u_1^+[.$$

Remark that, if $\rho(\cdot)$ is a solution of (10) and $w(\tau) = (\rho(\tau), \dot{\rho}(\tau))$ belongs to $\mathcal{L}_1$, for some τ, and $\dot{\rho}(\tau) > 0$, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V_2(w(\tau)) &= \nabla V_2(w(\tau)) \cdot \dot{w}(\tau) \\ &= (\tilde{g}_2(\rho(\tau)), \dot{\rho}(\tau)) \cdot (\dot{\rho}(t), \ddot{\rho}(\tau)) \\ &= \dot{\rho}(\tau)(\tilde{g}_2(\rho(\tau)) - g(\tau, \rho(\tau); \lambda)) < 0, \end{aligned}$$

thus showing that, roughly speaking, in the phase plane, the solution cannot cross the line $\mathcal{L}_1$ from the inside to the outside. More precisely, if $\rho(\cdot)$ is a solution of (10) and $V_2(\rho(\tau), \dot{\rho}(\tau)) = D_1$, for some τ, with $\dot{\rho}(\tau) > 0$, then there is a $\delta > 0$ such that $V_2(\rho(t), \dot{\rho}(t)) > D_1$ for every $t \in]\tau - \delta, \tau[$ and $V_2(\rho(t), \dot{\rho}(t)) < D_1$ for every $t \in]\tau, \tau + \delta[$.

Set now $D_2 = \max\{\tilde{G}_1(u) : u \in [u_1^-, u_1^+]\} + 1$ and define the line

$$\mathcal{L}_2 = \{(u, v) \in \mathcal{S} : V_1(u, v) = D_2, v \leq 0\}.$$

Notice that $\mathcal{L}_2$ is the graph of the function $F_2(u) = -\sqrt{2(D_2 - \tilde{G}_1(u))}$. By (14), there are u_2^-, u_2^+, with $a < u_2^- < u_1^- < u_1^+ < u_2^+ < b$, such that

$$F_2(u_2^-) = F_2(u_2^+) = 0, \quad \text{and} \quad F_2(u) < 0 \quad \text{for every } u \in]u_2^-, u_2^+[.$$

A computation similar to the above shows that the solutions of (10) cannot cross the line $\mathcal{L}_2$, in the phase plane, from the inside to the outside: if $\rho(\cdot)$ is a solution of (10) and $V_1(\rho(\tau), \dot{\rho}(\tau)) = D_2$, for some τ, with $\dot{\rho}(\tau) < 0$, then there is a $\delta > 0$ such that $V_1(\rho(t), \dot{\rho}(t)) > D_2$ for every $t \in]\tau - \delta, \tau[$ and $V_1(\rho(t), \dot{\rho}(t)) < D_2$ for every $t \in]\tau, \tau + \delta[$.

We will now show how to pass from $\mathcal{L}_1$ to $\mathcal{L}_2$ in order to obtain a clockwise rotating path which cannot be crossed by the solutions from the inside to the outside. Following an idea introduced in [30], we consider the straight line $\mathcal{R}_1$ passing through the point $(\frac{1}{2}(u_1^+ + u_2^+), 0)$ with a sufficiently small negative slope m_1 (see Fig. 1).

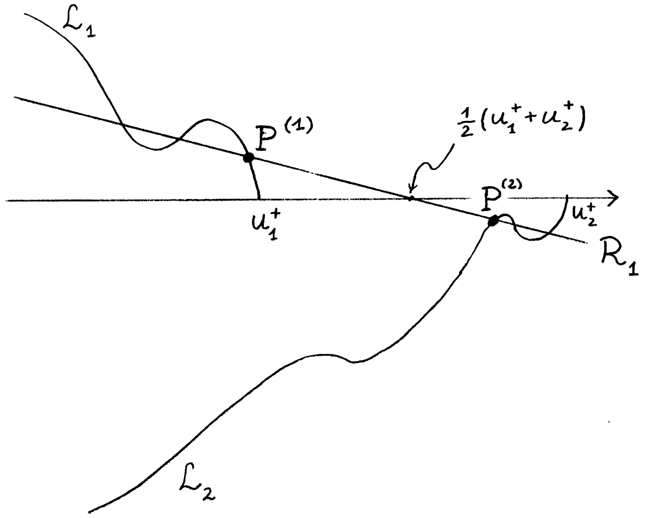

Fig. 1. The connection between $\mathcal{L}_1$ and $\mathcal{L}_2$.

Let $P^{(1)} = (p_1^{(1)}, p_2^{(1)})$ be the first point where the line $\mathcal{R}_1$ meets $\mathcal{L}_1$, starting from $(\frac{1}{2}(u_1^+ + u_2^+), 0)$, and similarly, let $P^{(2)} = (p_1^{(2)}, p_2^{(2)})$ be the first point where $\mathcal{R}_1$ meets $\mathcal{L}_2$. If m_1 is sufficiently small, such points exist and, by continuity, they are arbitrarily close to $(u_1^+, 0)$ and $(u_2^+, 0)$, respectively. In particular, we may assume that $p_1^{(1)} \geq \bar{b}$, and $-1 \leq p_2^{(2)} < p_2^{(1)} \leq 1$. Setting

$$\nu_1 = \min \left\{g_2(r) : r \in \left[\bar{b}, u_2^+\right]\right\},$$

we have

$$g(t, r; \lambda) \geq \nu_1 > 0, \quad \text{for every } t \in \mathbb{R},\ r \in [\bar{b}, u_2^+] \text{ and } \lambda \in [0, 1].$$

Hence, if $\rho(\cdot)$ is a solution of (10) such that $w(\tau) = (\rho(\tau), \dot{\rho}(\tau))$ belongs to the segment joining $P^{(1)}$ and $P^{(2)}$, for some τ, with $|\dot{\rho}(\tau)| \leq 1$, and, moreover, $|m_1| < \nu_1$, we get

$$(-m_1, 1) \cdot \dot{w}(\tau) = -m_1\dot{\rho}(\tau) - g(\tau, \rho(\tau); \lambda) \leq |m_1| - \nu_1 < 0.$$

Therefore, the solutions cannot cross the segment from the inner region to the outer one: if $\rho(\cdot)$ is a solution of (10) and $\dot{\rho}(\tau) = m_1(\rho(\tau) - \frac{1}{2}(u_1^+ + u_2^+))$, for some τ, with $|\dot{\rho}(\tau)| \leq 1$, there is a $\delta > 0$ such that $\dot{\rho}(t) > m_1(\rho(t) - \frac{1}{2}(u_1^+ + u_2^+))$ for every $t \in]\tau - \delta, \tau[$ and $\dot{\rho}(t) < m_1(\rho(t) - \frac{1}{2}(u_1^+ + u_2^+))$ for every $t \in]\tau, \tau + \delta[$.

Continuing this construction, we set now $D_3 = \max\{\tilde{G}_2(u) : u \in [u_2^-, u_2^+]\} + 1$, and define the line

$$\mathcal{L}_3 = \{(u, v) \in \mathcal{S} : V_2(u, v) = D_3, v \geq 0\},$$

which is the graph of the function $F_3(u) = \sqrt{2(D_3 - \tilde{G}_2(u))}$. Again by (14), there are u_3^-, u_3^+, with $a < u_3^- < u_2^- < u_2^+ < u_3^+ < b$, such that

$$F_3(u_3^-) = F_3(u_3^+) = 0, \quad \text{and} \quad F_3(u) > 0 \quad \text{for every } u \in]u_3^-, u_3^+[.$$

As above, one can see that the solutions of (10) cannot cross the line $\mathcal{L}_3$, in the phase plane, from the inside to the outside (in the above sense).

We now consider the straight line $\mathcal{R}_2$ passing through the point $(\frac{1}{2}(u_2^- + u_3^-), 0)$ with a sufficiently small negative slope m_2. Let $P^{(3)} = (p_1^{(3)}, p_2^{(3)})$ be the first point where the line $\mathcal{R}_2$ meets $\mathcal{L}_2$, starting from $(\frac{1}{2}(u_2^- + u_3^-), 0)$, and let $P^{(4)} = (p_1^{(4)}, p_2^{(4)})$ be the first point where $\mathcal{R}_2$ meets $\mathcal{L}_3$. If m_2 is sufficiently small, such points exist and, by continuity, they are arbitrarily close to $(u_2^-, 0)$ and $(u_3^-, 0)$, respectively. In particular, we may assume that $p_1^{(3)} \leq u_1^-$, and $-1 \leq p_2^{(3)} < p_2^{(4)} \leq 1$. Setting

$$\nu_2 = \max \left\{g_1(r) : r \in [u_3^-, u_1^-]\right\},$$

we have

$$g(t, r; \lambda) \leq \nu_2 < 0, \quad \text{for every } t \in \mathbb{R},\ r \in [u_3^-, u_1^-] \text{ and } \lambda \in [0, 1].$$

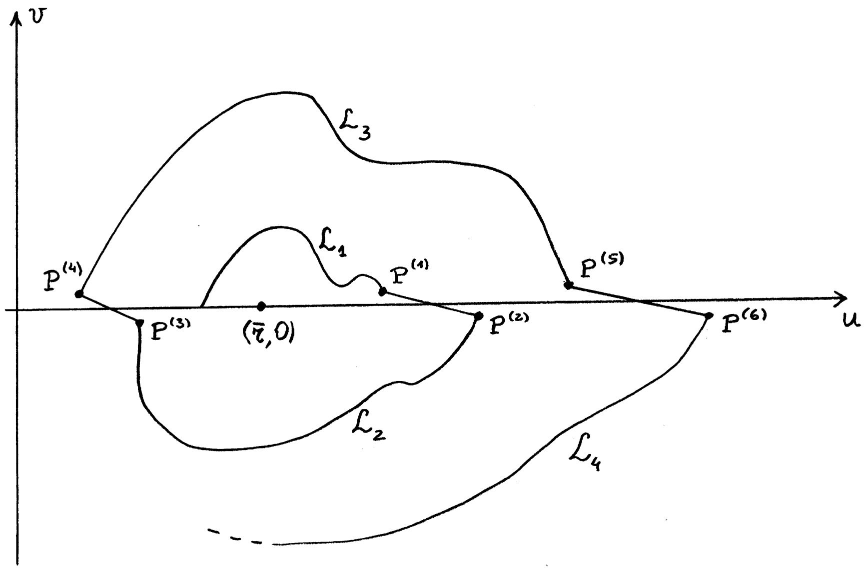

Fig. 2. The curve Γ.

Hence, if $\rho(\cdot)$ is a solution of (10) such that $w(\tau) = (\rho(\tau), \dot\rho(\tau))$ belongs to the segment joining $P^{(3)}$ and $P^{(4)}$, for some τ, with $|\dot\rho(\tau)| \le 1$, and, moreover, $|m_2| < |\nu_2|$, we get

$$(-m_2, 1) \cdot \dot w(\tau) = -m_2\dot\rho(\tau) - g(\tau, \rho(\tau); \lambda) \ge -|m_2| - \nu_2 > 0.$$

Hence, again the solutions cannot cross the segment from the inner region to the outer one: if $\rho(\cdot)$ is a solution of (10) and $\dot\rho(\tau) = m_2(\rho(\tau) - \frac{1}{2}(u_2^- + u_3^-))$, for some τ, with $|\dot\rho(\tau)| \le 1$, there is a $\delta > 0$ such that $\dot\rho(t) < m_2(\rho(t) - \frac{1}{2}(u_2^- + u_3^-))$ for every $t \in]\tau - \delta, \tau[$ and $\dot\rho(t) > m_2(\rho(t) - \frac{1}{2}(u_2^- + u_3^-))$ for every $t \in]\tau, \tau + \delta[$.

The curve Γ is thus constructed (see Fig. 2): it starts following $\mathcal{L}_1$ in clockwise direction till the point $P^{(1)}$ is reached; then, it continues along the segment from $P^{(1)}$ to $P^{(2)}$; from there, it follows $\mathcal{L}_2$ till reaching the point $P^{(3)}$; then, it coincides with the segment from $P^{(3)}$ to $P^{(4)}$; once arrived there, it starts again along $\mathcal{L}_3$, and so on, recursively.

We parametrize the curve Γ by a continuous function $\gamma : [0, +\infty[\to \mathbb{R}^2$. Notice that, by construction, for every compact set $\mathcal{K}$ contained in $]a, b[\times\mathbb{R}$ there is a $s_{\mathcal{K}} \ge 0$ such that $\gamma(s) \notin \mathcal{K}$ for every $s > s_{\mathcal{K}}$.

Let $R_1 > \mathcal{N}(\bar r, 0)$ be given. Let $s_1 \ge 0$ be such that $\mathcal{N}(\gamma(s)) > R_1$, for every $s \ge s_1$. There is a $s_2 > s_1$ such that $\gamma(s)$ makes exactly $\bar n + 1$ rotations around $(\bar r, 0)$ as s varies from s_1 to s_2. We take $R_2 > R_1$ in such a way that $\mathcal{N}(\gamma(s)) < R_2$ for every $s \in [s_1, s_2]$.

Let $\rho(t)$ be a solution of (10), and let $t_1 < t_2$ be such that $\mathcal{N}(\rho(t_1), \dot\rho(t_1)) = R_1$, $\mathcal{N}(\rho(t_2), \dot\rho(t_2)) = R_2$, and $R_1 < \mathcal{N}(\rho(t), \dot\rho(t)) < R_2$ for every $t \in]t_1, t_2[$. By construction, $(\rho(t), \dot\rho(t))$ cannot cross the curve Γ from the inside to the outside. Moreover, looking at the equivalent system (13), since $\dot u(t)$ has to be positive in the upper half-plane and negative in the lower half-plane, we see that $(\rho(t), \dot\rho(t))$ can never rotate around $(\bar r, 0)$ in counter-clockwise direction, not even once. Therefore, it has to make at least $\bar n$ clockwise rotations around the point $(\bar r, 0)$ in the time interval $[t_1, t_2]$. The proof is thus completed. □

4. Proof of Proposition 1

In this final section, we provide a proof of Proposition 1 which, as shown in Section 2, will complete the proof of Theorem 2. In the following lemma, we estimate the time needed for a large amplitude solution to perform, in the phase plane, a complete rotation around the point $(\bar r, 0)$. For simplicity, we will assume the starting point to be on the horizontal axis.

Lemma 2. *There is a $R_1 > \mathcal{N}(\bar r, 0)$ such that, if $\rho(t)$ is a solution of the differential equation in (9) and $\tau_1 < \tau_2$ are such that $(\rho(t), \dot\rho(t))$ makes exactly one rotation around $(\bar r, 0)$ as t varies in $[\tau_1, \tau_2]$, with $\rho(\tau_1) > \bar r$, $\dot\rho(\tau_1) = 0$, and*

$$\mathcal{N}(\rho(t), \dot\rho(t)) \ge R_1, \quad \textit{for every } t \in [\tau_1, \tau_2],$$

then

$$\frac{T}{M+1} < \tau_2 - \tau_1 < \frac{T}{M}.$$

Proof. We need to estimate the time spent by the solution in different regions of the plane. Without loss of generality, let us assume that $\bar{r} = 1$. We start by noticing that assumption (H1) implies the existence of three real numbers α_1, β_1 and d, with

$$\left(\frac{M\pi}{T}\right)^2 < \alpha_1 < \alpha, \qquad \beta < \beta_1 < \left(\frac{(M+1)\pi}{T}\right)^2, \quad d > 1,$$

such that, for every $\lambda \in [0, 1]$,

$$r \geq d \Rightarrow \alpha_1(r-1) \leq g(t, r; \lambda) \leq \beta_1(r-1).$$

Taking R_1 sufficiently large, we may assume that $\rho(\tau_1) > d$. Set $\hat{t}_0 = \tau_1$, and let $\hat{t}_1 > \hat{t}_0$ be the first instant when $\rho(\hat{t}_1) = d$ (see Fig. 3). We want to estimate the time $\hat{t}_1 - \hat{t}_0$. Writing the differential equation in (9) as the equivalent system (13), and passing to polar coordinates

$$u = 1 + \xi\cos\theta, \qquad v = \xi\sin\theta,$$

we have that

$$-\dot{\theta}(t) = \frac{\dot{u}(t)v(t) - \dot{v}(t)(u(t)-1)}{(u(t)-1)^2 + v^2(t)} = \frac{v^2(t) + g(t, u(t); \lambda)(u(t)-1)}{(u(t)-1)^2 + v^2(t)}. \tag{15}$$

So, if $t \in [\hat{t}_0, \hat{t}_1]$,

$$\frac{v^2(t) + \alpha_1(u(t)-1)^2}{(u(t)-1)^2 + v^2(t)} \leq -\dot{\theta}(t) \leq \frac{v^2(t) + \beta_1(u(t)-1)^2}{(u(t)-1)^2 + v^2(t)},$$

i.e.,

$$\frac{-\dot{\theta}(t)}{\sin^2\theta(t) + \beta_1\cos^2\theta(t)} \leq 1 \leq \frac{-\dot{\theta}(t)}{\sin^2\theta(t) + \alpha_1\cos^2\theta(t)}.$$

Since, for any $\omega > 0$,

$$\int \frac{\mathrm{d}\theta}{\sin^2\theta + \omega\cos^2\theta} = \frac{1}{\sqrt{\omega}}\arctan\left(\frac{\tan\theta}{\sqrt{\omega}}\right) + c,$$

integrating, we have

$$\frac{1}{\sqrt{\beta_1}}\left[\arctan\left(\frac{1}{\sqrt{\beta_1}}\tan\theta(t)\right)\right]_{\hat{t}_1}^{\hat{t}_0} \leq \hat{t}_1 - \hat{t}_0 \leq \frac{1}{\sqrt{\alpha_1}}\left[\arctan\left(\frac{1}{\sqrt{\alpha_1}}\tan\theta(t)\right)\right]_{\hat{t}_1}^{\hat{t}_0}.$$

Let us define

$$\eta = \min\left\{\frac{\pi}{2\sqrt{\beta_1}} - \frac{T}{2(M+1)}, \frac{T}{M} - \frac{\pi}{\sqrt{\alpha_1}}\right\}.$$

Taking R_1 sufficiently large, since $\theta(\hat{t}_0) = 0$ and $\theta(\hat{t}_1)$ is near $-\frac{\pi}{2}$, we thus have

$$\frac{T}{2(M+1)} < \frac{\pi}{2\sqrt{\beta_1}} - \frac{\eta}{2} < \hat{t}_1 - \hat{t}_0 < \frac{\pi}{2\sqrt{\alpha_1}} < \frac{T}{2M}.$$

Using assumption (H2′), let $d' \in]0, 1[$ be such that, for every $\lambda \in [0, 1]$,

$$r \in]0, d'] \Rightarrow g(t, r; \lambda) \leq -\left(\frac{3\pi}{\eta}\right)^2.$$

Following the solution, there is a first instant $\hat{t}_2 > \hat{t}_1$ in which $\rho(\hat{t}_2) = d'$. Let $\hat{t}_3 > \hat{t}_2$ be such that $\rho(\hat{t}_3) = d'$ again, and $\rho(t) < d'$ for every $t \in]\hat{t}_2, \hat{t}_3[$ (see Fig. 3). Then, if $t \in]\hat{t}_2, \hat{t}_3[$, using (15),

$$-\dot{\theta}(t) \geq \frac{v^2(t) + (\frac{3\pi}{\eta})^2(1-u(t))}{(u(t)-1)^2 + v^2(t)} \geq \frac{v^2(t) + (\frac{3\pi}{\eta})^2(1-u(t))^2}{(u(t)-1)^2 + v^2(t)},$$

so that

$$-\dot{\theta}(t) \geq \sin^2\theta(t) + \left(\frac{3\pi}{\eta}\right)^2\cos^2\theta(t),$$

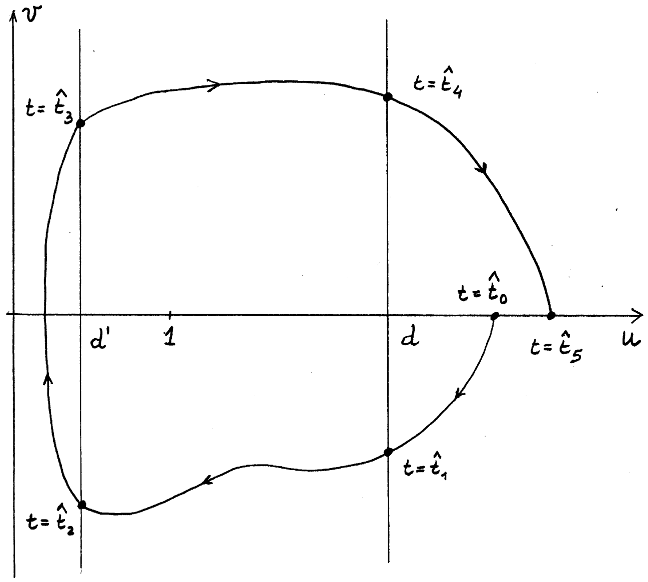

Fig. 3. Computing the time needed for a rotation.

i.e.,

$$1 \le \frac{-\dot\theta(t)}{\sin^2\theta(t) + (\frac{3\pi}{\eta})^2\cos^2\theta(t)}.$$

Integrating, we get

$$\hat t_3 - \hat t_2 \le \frac{\eta}{3\pi}\left[\arctan\left(\frac{\eta}{2\pi}\tan\theta(t)\right)\right]_{\hat t_3}^{\hat t_2} \le \frac{\eta}{3}.$$

Using the Carathéodory conditions, let $\tilde C > 0$ be such that, for every almost every $t \in [0, T]$, every $r \in [d', d]$ and $\lambda \in [0, 1]$,

$$|g(t, r; \lambda)| \le \tilde C.$$

Then, if $t \in]\hat t_1, \hat t_2[$, using (15),

$$-\dot\theta(t) \ge \frac{v^2(t) - \tilde C|1 - u(t)|}{(u(t) - 1)^2 + v^2(t)} \ge \frac{v^2(t) - \tilde C(d - d')}{(d - d')^2 + v^2(t)}.$$

Therefore, if R_1 is sufficiently large, since we are considering $u(t)$ in the bounded interval $[d', d]$, it will be that $|v(t)|$ is so large that $-\dot\theta(t) \ge \frac{1}{2}$, for every $t \in]\hat t_1, \hat t_2[$, and the total angle $\theta(\hat t_1) - \theta(\hat t_2)$ is smaller than $\frac{\eta}{6}$. By Lagrange Theorem, for some $\xi \in]\hat t_1, \hat t_2[$,

$$\hat t_2 - \hat t_1 = \frac{1}{-\dot\theta(\xi)}\left(\theta(\hat t_1) - \theta(\hat t_2)\right) < 2\frac{\eta}{6} = \frac{\eta}{3}.$$

If we now continue following the solution, there will be a first instant $\hat t_4 > \hat t_3$ such that $\rho(\hat t_4) = d$ (see Fig. 3 again). As for the estimate of $\hat t_2 - \hat t_1$, in the same way we can prove that

$$\hat t_4 - \hat t_3 < \frac{\eta}{3}.$$

Finally, there is a first instant $\hat t_5 > \hat t_4$ for which $\dot\rho(\hat t_5) = 0$. Clearly, $\hat t_5 = \tau_2$. In a similar way as for the estimate of $\hat t_1 - \hat t_0$, we can prove that

$$\frac{T}{2(M+1)} < \hat t_5 - \hat t_4 < \frac{\pi}{2\sqrt{\alpha_1}}.$$

We therefore conclude that

$$\hat{t}_5 - \hat{t}_0 = \sum_{j=1}^{5} \left(\hat{t}_j - \hat{t}_{j-1}\right) < 2\frac{\pi}{2\sqrt{\alpha_1}} + 3\frac{\eta}{3} \le \frac{T}{M},$$

and

$$\hat{t}_5 - \hat{t}_0 > \left(\hat{t}_1 - \hat{t}_0\right) + \left(\hat{t}_5 - \hat{t}_4\right) > 2\frac{T}{2(M+1)} = \frac{T}{M+1}. \quad \square$$

We now conclude the proof of Proposition 1. Let $R_1 > 0$ be given by Lemma 2. Notice that the assumptions of Lemma 1 are satisfied if we choose $a = 0$, $b = +\infty$, the function g_1 equal to the function f introduced in assumption (H2′), and $g_2(r) = (\frac{M\pi}{T})^2(r-1)$. Choose $\bar{n} = M + 1$ and let $R_2 > R_1$ be given by Lemma 1. We will show that all the solutions $\rho(t)$ of (9) have to satisfy

$$\mathcal{N}(\rho(t), \dot{\rho}(t)) < R_2, \quad \text{for every } t \in [0, T].$$

Indeed, assume by contradiction that there is a $\bar{t}$ for which $\mathcal{N}(\rho(\bar{t}), \dot{\rho}(\bar{t})) \ge R_2$. By Lemma 2, it cannot be that $\mathcal{N}(\rho(t), \dot{\rho}(t)) \ge R_1$, for every $t \in [0, T]$, since, in the period time T, the solution could not perform an integer number of rotations around $(\bar{r}, 0)$. Hence, by the T-periodicity of the solution, there must exist $t_1 < t_2$, with $t_2 - t_1 < T$, such that

$$\mathcal{N}(\rho(t_1), \dot{\rho}(t_1)) = R_1, \qquad \mathcal{N}(\rho(t_2), \dot{\rho}(t_2)) = R_2,$$

and

$$R_1 < \mathcal{N}(\rho(t), \dot{\rho}(t)) < R_2, \quad \text{for every } t \in]t_1, t_2[.$$

By Lemma 1, $(\rho(t), \dot{\rho}(t))$ makes at least $\bar{n} = M+1$ clockwise rotations around the point $(\bar{r}, 0)$ in the time interval $[t_1, t_2]$. By Lemma 2,

$$t_2 - t_1 > \bar{n}\frac{T}{M+1} = T,$$

a contradiction. Choosing any $R \ge R_2$, since

$$\mathcal{N}(u, v) < R_2 \Rightarrow \frac{1}{R} < u < R \quad \text{and} \quad |v| < R,$$

we have that all the solutions of (9) belong to Ω_R, and the proof of Proposition 1 is thus completed.

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