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# Periodic Orbits of Radially Symmetric Systems with a Singularity: the Repulsive Case

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**Abstract**

We study radially symmetric systems with a singularity of repulsive type. In the presence of a radially symmetric periodic forcing, we show the existence of three distinct families of subharmonic solutions: One oscillates radially, one rotates around the origin with small angular momentum, and the third one with both large angular momentum and large amplitude. The proofs are carried out by the use of topological degree theory.

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# 1 Introduction

We are interested in proving the existence and multiplicity of periodic solutions for systems of the type

$$\ddot{x} = f(t, |x|)\, x\,. \tag{1.1}$$

Here, $f : \mathbb{R}\times\, ]0, +\infty[\, \to \mathbb{R}$ is $L^1$-Carathéodory and $T$-periodic with respect to its first variable, so that 0 is possibly a singularity point. We look for solutions $x(t) \in \mathbb{R}^N$ which never attain the singularity, in the sense that

$$x(t) \neq 0\,, \qquad \text{for every } t \in \mathbb{R}\,. \tag{1.2}$$

Since system (1.1) is radially symmetric, the orbit of a solution always lies on a plane, so we will assume, without loss of generality, that $N = 2$.

Let us consider, e.g., the model equation

$$\ddot{x} + c(t)\frac{x}{|x|^\gamma} = e(t)\frac{x}{|x|}\,. \tag{1.3}$$

Here, $\gamma > 1$, and $c, e \in L^1_{loc}(\mathbb{R})$ are $T$-periodic. Notice that, if $c$ and $e$ are constant functions, with $e(t)$ identically zero, and $\gamma = 3$, then (1.3) is the equation modelling the motion of an electrical charge in the field of another charge standing still at the origin. According to the sign of $c$, we can have both the cases of attractive and repulsive forces. In [8] we have treated the attractive case. In this paper, we are interested in the case of a repulsive force.

Writing the solutions in polar coordinates, we see that equation (1.3) is equivalent to the system

$$\begin{cases} \ddot{\rho} - \dfrac{\mu^2}{\rho^3} + \dfrac{c(t)}{\rho^{\gamma-1}} = e(t)\,, \\ \rho^2\dot{\varphi} = \mu\,, \end{cases} \tag{1.4}$$

where $\mu$ is the (scalar) angular momentum. Let us denote by $\bar{e}$ the mean value of the forcing term:

$$\bar{e} = \frac{1}{T}\int_0^T e(t)\,dt\,.$$

If $c(t)$ is strictly negative, integrating the first equation in (1.4), it is easily seen that a necessary condition for the existence of periodic solutions is that $\bar{e}$ be negative. On the other hand, when $\gamma \geq 2$ and $\mu = 0$, it was proved in [13, 12, 5, 7, 1] that, if $\bar{e}$ is negative, then system (1.4) has periodic radial solutions. Our interest is to find out whether, besides these radial solutions, there are periodic solutions rotating around the origin, as well.

In order to have an insight on the possible behaviour of the solutions, let us now give a description of the circular orbits of (1.3) in the case when $c(t) = \bar{c}$ and $e(t) = \bar{e}$ are negative constants. We have to look for a constant $\rho$ satisfying

$$-\frac{\mu^2}{\rho^3} + \frac{\bar{c}}{\rho^{\gamma-1}} = \bar{e}\,. \tag{1.5}$$

Defining

$$\hat{\rho} := \Big(\frac{\bar{c}}{\bar{e}}\Big)^{\frac{1}{\gamma-1}},$$

it is easily seen that, for every $\mu$, if $\rho$ satisfies (1.5), then $\rho \geq \hat{\rho}$, with equality only when $\mu = 0$. Hence, there is a circle of stationary points, centered at the origin, with radius $\hat{\rho}$, and there are no circular orbits inside that circle. The circular solutions with radius $\rho$ greater than $\hat{\rho}$ have angular momentum

$$\mu(\rho) = \sqrt{\bar{c}\rho^{4-\gamma} - \bar{e}\rho^3}\,.$$

Hence, the angular momentum is strictly increasing with the radius of the solutions, and

$$\lim_{\rho\to\hat{\rho}^+}\mu(\rho) = 0\,, \qquad \lim_{\rho\to+\infty}\mu(\rho) = +\infty\,.$$

In particular, the circular solutions can be parametrized by the angular momentum $\mu$, so that, in the plane $(\mu, \rho)$, we have a curve whose projection on the first component is the whole half-line $]0, +\infty[$ .

As for the period of the solutions, we have

$$\tau(\rho) = 2\pi \sqrt{\frac{\rho}{-\bar{e} + \bar{c}\rho^{1-\gamma}}} \,,$$

so that

$$\lim_{\rho \to \hat{\rho}^+} \tau(\rho) = +\infty \,, \qquad \lim_{\rho \to +\infty} \tau(\rho) = +\infty \,.$$

The function $\tau(\rho)$ has a minimum point at $\rho = \rho_{min} := \left(\frac{\bar{c}\gamma}{\bar{e}}\right)^{\frac{1}{\gamma-1}}$, with minimum value

$$\tau_{min} = \frac{2\pi}{\sqrt{\gamma - 1}} \left(\frac{-\bar{c}\,\gamma^\gamma}{(-\bar{e})^\gamma}\right)^{\frac{1}{2(\gamma-1)}} \,.$$

Moreover, $\tau(\rho)$ is strictly decreasing when $\rho < \rho_{min}$ and strictly increasing when $\rho > \rho_{min}$. Hence, for every $\tau$ greater than $\tau_{min}$, there are two circular orbits with minimal period $\tau$: one with a small, and the other with a large angular momentum.

In the case when $c(t)$ and $e(t)$ are not necessarily constant, but $T$-periodic, it is natural to investigate whether (1.3) still has periodic solutions, rotating around the origin, having a similar behaviour to the circular orbits described above. The aim of this paper is to prove that such solutions exist, even for much more general systems like (1.1). The idea is to look for periodic solutions $x(t)$ whose minimal periods are sufficiently large multiples of $T$, and whose angular momenta are sufficiently small, or sufficiently large, respectively. In order to prevent too large eccentricities of the orbits, we will impose the radial components of the solutions $x(t)$ to be $T$-periodic.

Let us consider the more general system (1.1). It is convenient to write it in the following form:

$$\ddot{x} = \Big(-\, h(t, |x|) + e(t)\Big)\frac{x}{|x|} \,. \tag{1.6}$$

We will prove the following two theorems. The first one deals with periodic solutions having a small angular momentum.

**Theorem 1.1** *Let the following three assumptions hold.*
(H1) *There is a function $\alpha \in L^1(0, T)$ such that*

$$\limsup_{r \to +\infty} \frac{h(t, r)}{r} \le \alpha(t) \,,$$

*uniformly for almost every $t \in ]0, T[$, and*

$$\alpha(t) \le \left(\frac{\pi}{T}\right)^2 , \tag{1.7}$$

*for almost every $t \in ]0, T[$, with strict inequality on a subset of $]0, T[$ having positive measure.*
(H2) *There exists a function $\eta \in L^1(0, T)$, with positive values, such that*

$$h(t, r) \le \eta(t) \,, \qquad \text{for every } r \in ]0, 1] \text{ and a.e. } t \in ]0, T[ \,,$$
$$h(t, r) \ge -\eta(t) \,, \qquad \text{for every } r \ge 1 \text{ and a.e. } t \in ]0, T[ \,,$$

*and*

$$\frac{1}{T}\int_0^T \limsup_{r\to 0^+} h(t,r)\,dt < \bar{e} < \frac{1}{T}\int_0^T \liminf_{r\to+\infty} h(t,r)\,dt\,.$$

(H3) *There is a constant* $\bar{\delta} > 0$ *and a differentiable function* $F : ]0, \bar{\delta}[ \to \mathbb{R}$ *such that*

$$h(t,r) \le F'(r), \qquad \text{for every } r \in ]0,\bar{\delta}[ \text{ and a.e. } t \in ]0,T[\,,$$

*and*

$$\lim_{r\to 0^+} F(r) = +\infty\,.$$

*Then, there exists a* $k_1 \ge 1$ *such that, for every integer* $k \ge k_1$*, equation (1.6) has a periodic solution* $x_k(t)$ *with minimal period* $kT$*, which makes exactly one revolution around the origin in the period time* $kT$*. Moreover, there is a constant* $C > 0$ *such that, for every* $k \ge k_1$

$$\frac{1}{C} < |x_k(t)| < C\,, \qquad \text{for every } t \in \mathbb{R}\,,$$

*and, if* $\mu_k$ *denotes the angular momentum associated to* $x_k(t)$*, then*

$$\lim_{k\to\infty} \mu_k = 0\,.$$

Let us make a brief comment on the hypotheses of Theorem 1.1. Assumption (H1), introduced in [15], can be interpreted as a nonresonance condition with respect to the Dancer-Fučik spectrum. Indeed, the number $(\frac{\pi}{T})^2$ is the value of the asymptote to the first curve of that spectrum. Assumption (H2) is the well-known Landesman-Lazer condition, a nonresonance condition with respect to the first eigenvalue of the differential operator. Assumption (H3) is introduced in order to control the solutions which approach the singularity, by the use of some energy estimates.

As an immediate consequence of Theorem 1.1, in the particular case of system (1.3), we have the following.

**Corollary 1.1** *Assume that, for some negative constants* $c_1$ *and* $c_2$*,*

$$c_1 \le c(t) \le c_2 < 0\,, \qquad \text{for a.e. } t \in \mathbb{R}\,. \tag{1.8}$$

*If* $\gamma \ge 2$ *and* $\bar{e} < 0$*, then the same conclusion of Theorem 1.1 holds for system (1.3).*

Our second theorem deals with periodic solutions having a large angular momentum. It is closely related to [8, Theorem 4] and [9, Corollary 1.8], where the attractive case was studied.

**Theorem 1.2** *Let the following two assumptions hold.*

(H4)
$$\lim_{r\to+\infty} \frac{h(t,r)}{r} = 0\,,$$

*uniformly for almost every* $t \in ]0,T[$.

$(\mathrm{H2})_+$ *There exists a function* $\eta \in L^1(0,T)$*, with positive values, such that*

$$h(t,r) \ge -\eta(t)\,, \qquad \text{for every } r \ge 1 \text{ and a.e. } t \in ]0,T[\,,$$

*and*

$$\bar{e} < \frac{1}{T}\int_0^T \liminf_{r\to+\infty} h(t,r)\,dt\,.$$

*Then, there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, equation (1.6) has a periodic solution $x_k(t)$ with minimal period $kT$, which makes exactly one revolution around the origin in the period time $kT$. Moreover,*

$$\lim_{k\to\infty}(\min|x_k|) = +\infty\,, \qquad \lim_{k\to\infty}\frac{\min|x_k|}{\max|x_k|} = 1\,,$$

*and, if $\mu_k$ denotes the angular momentum associated to $x_k(t)$, then*

$$\lim_{k\to\infty}\mu_k = +\infty\,.$$

Notice that, in Theorem 1.2, we have no assumptions on the behaviour of the nonlinearity near the singularity. Indeed, $(H2)_+$ is a kind of Landesman-Lazer condition, but it is only assumed for large positive values of $r$. Assumption (H4) is a non-resonance condition which will be used to show that the angular velocity of the large-amplitude solutions can be arbitrarily small.

The following is a direct consequence of Theorem 1.2 concerning system (1.3).

**Corollary 1.2** *If $\gamma > 1$, $\bar{e} < 0$, and (1.8) holds, then the same conclusion of Theorem 1.2 holds for system (1.3).*

We notice that, in [7], assuming (H2), (H3) and (H4), it was proved that there is a family of subharmonic solutions with arbitrarily large minimal periods, which oscillate radially. We thus conclude with the following.

**Corollary 1.3** *Assume (H2), (H3) and (H4). Then, system (1.6) has three distinct families of subharmonic solutions, with the following distinct behaviour: one oscillates radially, one rotates with small angular momentum, and one rotates with large angular momentum and large amplitude.*

The above results should be compared with those contained in [16, 6] (see also [4, 11]), where systems of the type

$$\ddot{x} + \nabla V(x) = \mathbf{e}(t) \tag{1.9}$$

were considered, with $V : \mathbb{R}^N\setminus\{0\} \to \mathbb{R}$, a continuously differentiable function satisfying

$$\lim_{x\to 0} V(x) = +\infty\,, \tag{1.10}$$

and $\mathbf{e} : \mathbb{R} \to \mathbb{R}^N$, a locally integrable $T$-periodic vector-valued function. This type of problem is not radially symmetric, and the situation is substantially different from the one considered above. Let $\bar{\mathbf{e}}$ be the mean value of $\mathbf{e}(t)$:

$$\bar{\mathbf{e}} = \frac{1}{T}\int_0^T \mathbf{e}(t)\,dt\,.$$

The following result has been proved in [16, 6].

**Theorem 1.3** *Assume (1.10) and the following conditions:*

$$\limsup_{x\to 0} \langle \nabla V(x), x\rangle < 0\,,$$

$$\limsup_{|x|\to\infty} |\nabla V(x)| < |\bar{\mathbf{e}}|\,.$$

*Then, equation (1.9) has a $T$-periodic solution $x$, and a sequence $(x_k)_k$ of $kT$-periodic solutions, whose minimal periods tend to infinity.*

The above theorem applies to the equation

$$\ddot{x} + \bar{c}\,\frac{x}{|x|^\gamma} = \mathbf{e}(t)\,, \tag{1.11}$$

where $\gamma > 2$, $\bar{c}$ is a negative constant, and $\bar{\mathbf{e}} \neq 0$ . Notice that, in [16], this last assumption has been shown to be necessary for the existence of periodic solutions of (1.11), when $\mathbf{e}(t)$ is a small bounded function.

It should be observed that, assuming $N = 2$ in Theorem 1.3, the solutions could remain confined in a sector of the plane, and thus not necessarily rotate around the origin. See also [3, 10, 17, 18] for related results.

## 2 Proof of Theorem 1.1: solutions with small angular momentum

Let us first clarify our assumptions. The function $h$ is $L^1$-Carathéodory, i.e.,

- $h(\cdot, r)$ is measurable, for every $r > 0$;
- $h(t, \cdot)$ is continuous, for almost every $t \in\, ]0, T[$ ;
- for every compact interval $[a, b]$ in $]0, +\infty[$ , there exists $\ell_{a,b} \in L^1(0, T)$ such that

$$r \in [a, b] \quad\Rightarrow\quad |h(t, r)| \le \ell_{a,b}(t)\,, \quad \text{for a.e. } t \in\, ]0, T[\,.$$

Concerning (H1), we precisely ask that, for every $\varepsilon > 0$, there is an $R > 0$ and a zero measure set $\mathcal{N}$ in $]0, T[$ such that

$$\frac{h(t, r)}{r} \le \alpha(t) + \varepsilon\,, \qquad \text{for every } r \ge R \text{ and } t \in\, ]0, T[\, \setminus \mathcal{N}\,.$$

Clearly, we can assume that $\alpha(t) \ge 0$ for almost every $t \in\, ]0, T[$ . Hence, in particular, we assume $\alpha \in L^\infty(0, T)$.

In the sequel, we will implicitly assume that all functions defined (almost everywhere) on $]0, T[$ are extended by $T$-periodicity to the whole real line.

Let $H^1_0(0, T)$ denote the usual Sobolev space of functions satisfying the homogeneous Dirichlet boundary condition. Given a constant $\omega \in \mathbb{R}$, let $B_{\alpha,\omega} : H^1_0(0, T) \to \mathbb{R}$ be the quadratic form defined by

$$B_{\alpha,\omega}(v) = \int_0^T [\dot{v}^2(t) - \alpha(t + \omega)v^2(t)]\, dt\,.$$

We will need the following lemma, analogous to [15, Lemma 3].

**Lemma 2.1** *There is a constant $\bar{\varepsilon} > 0$ such that*

$$B_{\alpha,\omega}(v) \geq \bar{\varepsilon} \int_0^T \dot{v}^2(t)\,dt\,,$$

*for every $\omega \in \mathbb{R}$ and $v \in H_0^1(0,T)$.*

The above statement seems rather standard for Dirichlet problems. However, the fact that $\bar{\varepsilon}$ can be chosen independently of $\omega \in \mathbb{R}$ is not usually specified in the literature. This is why, for the reader's convenience, we provide below a complete proof of it.

*Proof.* By (1.7) and the Poincaré inequality,

$$B_{\alpha,\omega}(v) \geq \int_0^T \Big[\dot{v}^2(t) - \Big(\frac{\pi}{T}\Big)^2 v^2(t)\Big]\,dt \geq 0\,, \tag{2.1}$$

for every $\omega \in \mathbb{R}$ and $v \in H_0^1(0,T)$. Let us see that

$$B_{\alpha,\omega}(v) = 0 \quad \Leftrightarrow \quad v = 0\,.$$

Indeed, by (2.1), $B_{\alpha,\omega}(v) = 0$ implies $v(t) = A\sin(\frac{\pi t}{T})$, for some constant $A \in \mathbb{R}$. Then,

$$\int_0^T \Big[A^2\Big(\frac{\pi}{T}\Big)^2 \cos^2\Big(\frac{\pi t}{T}\Big) - \alpha(t+\omega)A^2\sin^2\Big(\frac{\pi t}{T}\Big)\Big]\,dt = 0\,,$$

i.e., since $\int_0^T \cos^2(\frac{\pi t}{T})\,dt = \int_0^T \sin^2(\frac{\pi t}{T})\,dt$,

$$A^2 \int_0^T \Big[\Big(\frac{\pi}{T}\Big)^2 - \alpha(t+\omega)\Big]\sin^2\Big(\frac{\pi t}{T}\Big)\,dt = 0\,,$$

which implies $A = 0$, as $\alpha(\cdot) < (\frac{\pi}{T})^2$ on a subset of positive measure.

Assume now, by contradiction, that for every integer $n \geq 1$ there are $\omega_n \in [0,T]$ and $v_n \in H_0^1(0,T)$ such that

$$\int_0^T [\dot{v}_n^2(t) - \alpha(t+\omega_n)v_n^2(t)]\,dt < \frac{1}{n}\int_0^T \dot{v}_n^2(t)\,dt\,.$$

Let $z_n = v_n \,/\, \|v_n\|_{H_0^1}$, where

$$\|v_n\|_{H_0^1} = \Big(\int_0^T \dot{v}_n^2(t)\,dt\Big)^{1/2}$$

is the usual norm in $H_0^1(0,T)$. Then, $z_n \in H_0^1(0,T)$, $\|z_n\|_{H_0^1} = 1$, and

$$\int_0^T \alpha(t+\omega_n)z_n^2(t)\,dt > 1 - \frac{1}{n}\,.$$

Passing to subsequences, we can assume that $\omega_n \to \omega \in [0,T]$, $z_n \rightharpoonup z$ (weakly) in $H_0^1(0,T)$, and $z_n \to z$ uniformly. Then, $\|z\|_{H_0^1} \leq 1$, and, since by Lebesgue's Theorem $\alpha(\cdot+\omega_n) \to \alpha(\cdot+\omega)$ in $L^1(0,T)$,

$$\int_0^T \alpha(t+\omega)z^2(t)\,dt \geq 1\,. \tag{2.2}$$

Hence, $B_{\alpha,\omega}(z) \leq 0$, so that, by the above, it has to be $z = 0$. We thus get a contradiction with (2.2). ∎

Let us now begin the proof of Theorem 1.1. We may write the solutions of (1.6) in polar coordinates:

$$x(t) = \rho(t)(\cos\varphi(t), \sin\varphi(t)), \tag{2.3}$$

and (1.2) is satisfied if $\rho(t) > 0$, for every $t$. Equation (1.6) is then equivalent to the system

$$(S) \qquad \begin{cases} \ddot\rho - \dfrac{\mu^2}{\rho^3} + h(t,\rho) = e(t), \\ \rho^2\dot\varphi = \mu, \end{cases}$$

where $\mu$ is the (scalar) angular momentum of $x(t)$. Recall that $\mu$ is constant in time along any solution. In the following, when considering a solution of (S), we will always implicitly assume that $\rho > 0$.

Without loss of generality we assume that $e(t)$ has zero mean value, i.e.,

$$\bar e = 0. \tag{2.4}$$

Indeed, otherwise, we just replace $e(t)$ by $e(t) - \bar e$ and $h(t,\rho)$ by $h(t,\rho) - \bar e$. We consider $\mu \geq 0$ as a parameter, and, by the use of degree theory, we will prove the existence of a $T$-periodic solution $\rho$ of the first equation in (S). To this aim, for $\lambda \in [0, 1]$, we introduce the modified problem

$$\begin{cases} \ddot\rho - \dfrac{\mu^2}{\rho^3} + (1-\lambda)\left(1 - \dfrac{1}{\rho^3}\right) + \lambda h(t,\rho) = \lambda e(t), \\ \rho(0) = \rho(T), \ \dot\rho(0) = \dot\rho(T). \end{cases} \tag{2.5}$$

For some $r_0 \in \,]0, 1[$ , to be fixed later, we define the truncation $g_{\lambda,\mu,r_0} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, as follows:

$$g_{\lambda,\mu,r_0}(t,r) = \begin{cases} -\dfrac{\mu^2}{r^3} + (1-\lambda)\left(1 - \dfrac{1}{r^3}\right) + \lambda h(t,r) & \text{for } r \geq r_0 \\ -\dfrac{\mu^2}{r_0^3} + (1-\lambda)\left(1 - \dfrac{1}{r_0^3}\right) + \lambda h(t,r_0) & \text{for } r \leq r_0 . \end{cases}$$

This function is still $L^1$-Carathéodory. We consider the $T$-periodic problem

$$(P_{\lambda,\mu,r_0}) \qquad \begin{cases} \ddot\rho + g_{\lambda,\mu,r_0}(t,\rho) = \lambda e(t), \\ \rho(0) = \rho(T), \ \dot\rho(0) = \dot\rho(T), \end{cases}$$

and look for a priori bounds for the solutions $\rho$, for small values of $r_0$.

**Lemma 2.2** *Assume (H1)–(H3). Given $M > 0$, there exist $\bar r_0 > 0$ and $C > 0$ such that, if $\rho(t)$ is a solution of $(P_{\lambda,\mu,r_0})$, with $\lambda \in [0, 1]$, $\mu \in [0, M]$ and $r_0 \in \,]0, \bar r_0\,]$, then*

$$\frac{1}{C} < \rho(t) < C \qquad \textit{and} \qquad |\dot\rho(t)| < C,$$

*for every $t \in [0, T]$.*

*Proof.* By contradiction, assume that, for every $n \geq 1$, there are $\lambda_n \in [0, 1]$, $\mu_n \in [0, M]$, $r_{0,n} \in \,]0, \frac{1}{n}[$ , and a solution $\rho_n(t)$ of $(P_{\lambda_n,\mu_n,r_{0,n}})$, such that, either $\rho_n([0, T]) \not\subseteq \,]\frac{1}{n}, n[$, or $\|\dot\rho_n\|_{L^\infty} \geq n$. For simplicity we denote by $g_n$ the function $g_{\lambda_n,\mu_n,r_{0,n}}$. The remaining of the proof is divided into five steps.

Step 1. There exists $R_1 \geq 1$ such that $\min \rho_n \leq R_1$ , for every $n$.

Otherwise, there would exist a subsequence such that $\min \rho_n \to +\infty$. We may also assume that $\lambda_n \to \bar{\lambda} \in [0,1]$. Integrating the equation

$$\ddot{\rho}_n + g_n(t,\rho_n) = \lambda_n e(t)\,, \tag{2.6}$$

by (2.4) we have

$$\int_0^T g_n(t,\rho_n(t))\,dt = 0\,. \tag{2.7}$$

Hence, from the definition of $g_n$, since $\min \rho_n \to +\infty$,

$$(1-\bar{\lambda})T + \lim_n \int_0^T \lambda_n h(t,\rho_n(t))\,dt = 0\,.$$

Using Fatou's Lemma,

$$(1-\bar{\lambda})T + \bar{\lambda}\int_0^T \liminf_{r\to+\infty} h(t,r)\,dt \leq 0\,,$$

and we get a contradiction with (H2).

Step 2. There exists $R_2 > R_1$ such that $\max \rho_n \leq R_2$ , for every $n$.

Let $\bar{\varepsilon} > 0$ be as in Lemma 2.1. By (H1), there is a $\tilde{R}_1 > R_1$ such that

$$h(t,r) \leq \Big(\alpha(t) + \frac{\bar{\varepsilon}}{2}\Big(\frac{\pi}{T}\Big)^2\Big)r\,, \quad \text{for every } r \geq \tilde{R}_1 \text{ and a.e. } t \in \mathbb{R}\,. \tag{2.8}$$

By contradiction, we assume that $\max \rho_n \to +\infty$. Then, for $n$ large enough, there exists an interval $[a_n, b_n]$, with $b_n - a_n \leq T$, such that

$$\rho_n(a_n) = \tilde{R}_1 = \rho_n(b_n)\,,$$

and

$$\rho_n(t) > \tilde{R}_1\,, \qquad \text{for every } t \in\, ]a_n, b_n[\,,$$

and there exists $t_n \in\, ]a_n, b_n[$ such that $\max \rho_n = \rho_n(t_n)$. Let $u_n(t) = \rho_n(t) - \tilde{R}_1$, so that $u_n(a_n) = 0 = u_n(b_n)$. Recall the Poincaré inequality

$$\int_{a_n}^{b_n} u_n^2(t)\,dt \leq \left(\frac{b_n - a_n}{\pi}\right)^2 \int_{a_n}^{b_n} \dot{u}_n^2(t)\,dt\,. \tag{2.9}$$

Define $v_n : [0,T] \to \mathbb{R}$ as follows:

$$v_n(t) = \begin{cases} u_n(t+a_n) & \text{if } t \in [0, b_n - a_n]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, $v_n \in H_0^1(0,T)$ and, by Lemma 2.1,

$$B_{\alpha,a_n}(v_n) \geq \bar{\varepsilon}\int_0^T \dot{v}_n^2(t)\,dt\,,$$

for every $n$, i.e.,

$$\int_{a_n}^{b_n} [\dot u_n^2(t) - \alpha(t) u_n^2(t)]\, dt \geq \bar\varepsilon \int_{a_n}^{b_n} \dot u_n^2(t)\, dt\,. \tag{2.10}$$

Multiplying by $u_n$ in equation (2.6) and integrating between $a_n$ and $b_n$, by (2.8) we obtain

$$\begin{aligned}\int_{a_n}^{b_n} \dot u_n^2 &= \int_{a_n}^{b_n} u_n[g_n(t,\rho_n) - \lambda_n e(t)]\, dt \\ &\leq \int_{a_n}^{b_n} u_n[1 + \lambda_n(h(t,\rho_n) - e(t))]\, dt \\ &\leq \int_{a_n}^{b_n} u_n\Big[1 + \Big(\alpha(t) + \frac{\bar\varepsilon}{2}\Big(\frac{\pi}{T}\Big)^2\Big)(u_n + \tilde R_1) + |e(t)|\Big]\, dt \\ &\leq \int_{a_n}^{b_n} \Big[\Big(\alpha(t) + \frac{\bar\varepsilon}{2}\Big(\frac{\pi}{T}\Big)^2\Big)u_n^2 + \gamma(t)\, u_n\Big]\, dt\,,\end{aligned}$$

where

$$\gamma(t) = \tilde R_1\Big(\alpha(t) + \frac{\bar\varepsilon}{2}\Big(\frac{\pi}{T}\Big)^2\Big) + |e(t)| + 1\,.$$

Then, using (2.10),

$$\bar\varepsilon \int_{a_n}^{b_n} \dot u_n^2 \leq \frac{\bar\varepsilon}{2}\Big(\frac{\pi}{T}\Big)^2 \int_{a_n}^{b_n} u_n^2 + \|\gamma\|_{L^2}\Big(\int_{a_n}^{b_n} u_n^2\Big)^{\frac{1}{2}}$$

so that, by (2.9), as $b_n - a_n \leq T$, the sequence $\Big(\int_{a_n}^{b_n} \dot u_n^2\Big)_n$ has to be bounded. Since

$$\max u_n \leq \int_{a_n}^{b_n} |\dot u_n| \leq \sqrt{T}\left(\int_{a_n}^{b_n} \dot u_n^2\right)^{\frac{1}{2}},$$

we get a contradiction with the fact that $\max \rho_n \to +\infty$.

Step 3. There exists $r_1 \in ]0,1[$ such that $\max \rho_n \geq r_1$, for every $n$.

Otherwise, there would exist a subsequence such that $\max \rho_n \to \bar\rho \in [-\infty, 0]$. We may also assume that $\lambda_n \to \bar\lambda \in [0,1]$ and $\mu_n \to \bar\mu \in [0, M]$. Set

$$\tilde\rho_n(t) = \max\{\rho_n(t), r_{0,n}\}\,. \tag{2.11}$$

Notice that

$$g_n(t,\rho_n(t)) = g_n(t,\tilde\rho_n(t))\,,$$

for almost every $t$. Since $r_{0,n} \leq \tilde\rho_n(t) \leq 1$ for $n$ large enough, by (H2),

$$\begin{aligned}g_n(t,\rho_n(t)) &= -\frac{\mu_n^2}{\tilde\rho_n^3(t)} + (1-\lambda_n)\left(1 - \frac{1}{\tilde\rho_n^3(t)}\right) + \lambda_n h(t,\tilde\rho_n(t)) \\ &\leq 1 + \lambda_n h(t,\tilde\rho_n(t)) \\ &\leq 1 + \eta(t)\,.\end{aligned}$$

Hence, by Fatou's Lemma and (2.7), we have

$$\int_0^T \limsup_n g_n(t,\rho_n(t))\,dt \ge \limsup_n \int_0^T g_n(t,\rho_n(t))\,dt = 0\,,$$

and then

$$\int_0^T \limsup_n \left[(1-\lambda_n)\left(1-\frac{1}{\tilde\rho_n^3(t)}\right)+\lambda_n h(t,\tilde\rho_n(t))\right]dt \ge 0\,.$$

Since $\tilde\rho_n(t) > 0$ for every $t$ and $\max\tilde\rho_n(t) \to 0$, we easily get a contradiction with (H2).

Step 4. Proof of the estimate on the derivative.

By (H2), we have

$$g_n(t,r) \le \eta(t)\,, \qquad \text{for every } r \le 1 \text{ and a.e. } t \in\, ]0,T[\,.$$

Then, using (2.7),

$$\begin{aligned}
\int_{\{\rho_n\le 1\}} |g_n(t,\rho_n(t))|\,dt &\le \int_{\{\rho_n\le 1\}} |g_n(t,\rho_n(t))-\eta(t)|\,dt + \|\eta\|_{L^1}\\
&= \int_{\{\rho_n\le 1\}} (-g_n(t,\rho_n(t))+\eta(t))\,dt + \|\eta\|_{L^1}\\
&\le -\int_{\{\rho_n\le 1\}} g_n(t,\rho_n(t))\,dt + 2\|\eta\|_{L^1}\\
&= \int_{\{\rho_n> 1\}} g_n(t,\rho_n(t))\,dt + 2\|\eta\|_{L^1}\\
&\le \int_{\{\rho_n> 1\}} |g_n(t,\rho_n(t))|\,dt + 2\|\eta\|_{L^1}\,,
\end{aligned}$$

so that

$$\begin{aligned}
\int_0^T |g_n(t,\rho_n(t))|\,dt &= \left(\int_{\{\rho_n\le 1\}} + \int_{\{\rho_n> 1\}}\right)|g_n(t,\rho_n(t))|\,dt\\
&\le 2\int_{\{\rho_n> 1\}} |g_n(t,\rho_n(t))|\,dt + 2\|\eta\|_{L^1}\,.
\end{aligned}$$

As proved in Step 2, we have $\rho_n(t) \le R_2$, for every $t$. So, there is a constant $c_1 > 0$ for which

$$\int_{\{\rho_n> 1\}} |g_n(t,\rho_n(t))|\,dt \le c_1\,,$$

for every $n$. Consequently,

$$\int_0^T |\ddot\rho_n(t)|\,dt \le \int_0^T |g_n(t,\rho_n(t))|\,dt + \|e\|_{L^1} \le 2c_1 + 2\|\eta\|_{L^1} + \|e\|_{L^1}\,.$$

Since, due to the periodicity of $\rho_n$, its derivative must vanish somewhere, we have that $\|\dot\rho_n\|_{L^\infty} \le \|\ddot\rho_n\|_{L^1}$. Setting $C_1 = 2c_1 + 2\|\eta\|_{L^1} + \|e\|_{L^1}$, we thus have

$$\|\dot\rho_n\|_{L^\infty} \le C_1\,, \tag{2.12}$$

for every $n$.

Step 5. Conclusion of the proof.

Using Step 2 and Step 4, if $n$ is sufficiently large we have that $\rho_n(t) < n$ and $\|\dot\rho_n(t)\| < n$, for every $t$. Therefore, it has to be $\min \rho_n \leq \frac{1}{n}$, for $n$ large enough. Let $r_1 \in ]0,1[$ be as in Step 3, and set $\tilde r_1 = \min\{r_1, \bar\delta\}$. We can assume $\tilde r_1 > \frac{1}{n}$. Then, there is an interval $[\gamma_n, \delta_n]$ such that

$$\rho_n(\gamma_n) = \tilde r_1\,, \qquad \rho_n(\delta_n) = \tfrac{1}{n}\,,$$

and

$$\tfrac{1}{n} < \rho_n(t) < \tilde r_1\,, \qquad \text{for every } t \in ]\gamma_n, \delta_n[\,.$$

Recall that $r_{0,n} < \frac{1}{n}$. So, for $t \in [\gamma_n, \delta_n]$, we have that $\rho_n(t) = \tilde\rho_n(t)$. We define

$$\tilde h(t,r) = h(t,r) - \eta(t)\,, \quad \tilde e(t) = e(t) - \eta(t)\,, \quad \tilde f(r) = \min\{F'(r), 0\}\,,$$

and let $\tilde F : ]0, \bar\delta[ \to \mathbb{R}$ be a primitive of $\tilde f$, i.e., $\tilde F'(r) = \tilde f(r)$, for every $r$. Then,

$$\tilde h(t,r) \leq \tilde F'(r), \qquad \text{for every } r \in ]0,\bar\delta[ \text{ and a.e. } t \in \mathbb{R}\,,$$

and

$$\lim_{r\to 0^+} \tilde F(r) = +\infty\,. \tag{2.13}$$

We can write (2.6) as

$$\ddot\rho_n + \tilde g_n(t,\rho_n) = \lambda_n \tilde e(t)\,, \tag{2.14}$$

where $\tilde g_n(t,r)$ is defined as $g_n(t,r)$, with $\tilde h(t,r)$ instead of $h(t,r)$. Notice that

$$\tilde g_n(t,\rho_n(t)) = -\frac{\mu_n^2}{\rho_n^3(t)} + (1-\lambda_n)\left(1 - \frac{1}{\rho_n^3(t)}\right) + \lambda_n \tilde h(t,\rho_n(t)) \leq 0\,,$$

for every $t \in ]\gamma_n, \delta_n[$. Let $C_1$ be the constant defined in Step 4, for which (2.12) holds. Multiplying in (2.14) by $(\dot\rho_n - C_1)$ and integrating on $[\gamma_n, \delta_n]$,

$$\begin{aligned}\frac{1}{2}[(\dot\rho_n - C_1)^2]_{\gamma_n}^{\delta_n} + \int_{\gamma_n}^{\delta_n} \tilde g_n(t,\rho_n)(\dot\rho_n - C_1)\,dt &= \int_{\gamma_n}^{\delta_n} \lambda_n \tilde e(t)(\dot\rho_n - C_1)\,dt \\ &\leq 2C_1(\|e\|_{L^1} + \|\eta\|_{L^1})\,.\end{aligned}$$

Hence, there is a constant $c_2 > 0$ for which

$$\int_{\gamma_n}^{\delta_n} \tilde g_n(t,\rho_n)(\dot\rho_n - C_1)\,dt \leq c_2\,, \tag{2.15}$$

for every $n$. On the other hand, since $\dot\rho_n - C_1 \le 0$, and $\tilde F'(r) \le 0$ for every $r$,

$$\begin{aligned}
\int_{\gamma_n}^{\delta_n} \tilde g_n(t,\rho_n)(\dot\rho_n - C_1)\, dt \ge\\
&\ge \int_{\gamma_n}^{\delta_n} \left((1-\lambda_n)\left(1-\frac{1}{\rho_n^3}\right) + \lambda_n \tilde h(t,\rho_n)\right)(\dot\rho_n - C_1)\, dt\\
&\ge \int_{\gamma_n}^{\delta_n} \left((1-\lambda_n)\left(1-\frac{1}{\rho_n^3}\right) + \lambda_n \tilde F'(\rho_n)\right)(\dot\rho_n - C_1)\, dt\\
&\ge \int_{\gamma_n}^{\delta_n} \left((1-\lambda_n)\left(1-\frac{1}{\rho_n^3}\right) + \lambda_n \tilde F'(\rho_n)\right)\dot\rho_n\, dt\\
&= \left[(1-\lambda_n)\left(\rho_n + \frac{2}{\rho_n^2}\right) + \lambda_n \tilde F(\rho_n)\right]_{\gamma_n}^{\delta_n}\\
&\ge (1-\lambda_n)\left(\frac{1}{n} + 2n^2\right) + \lambda_n \tilde F\Big(\frac{1}{n}\Big) - c_3\,,
\end{aligned}$$

for some constant $c_3 > 0$. Using (2.13), when $n$ tends to infinity we get a contradiction with (2.15), thus ending the proof of the lemma. ∎

We now fix $M > 0$ and take $\bar r_0 > 0$ and $C > 0$ as given by Lemma 2.2. Let us also fix $r_0 = \min\{\bar r_0, \frac{1}{C}\}$. As a consequence of Lemma 2.2, if $\lambda \in [0,1]$ and $\mu \in [0,M]$, any solution $\rho(t)$ of $(P_{\lambda,\mu,r_0})$ is also a solution of (2.5). In particular, if $\lambda = 1$, any solution of $(P_{1,\mu,r_0})$ is a $T$-periodic solution of the first equation in (S).

Notice that, viceversa, once $M > 0$ is fixed, every $T$-periodic solution of the first equation in (S) with $\mu \in [0,M]$ satisfies $(P_{1,\mu,\hat r_0})$ for sufficiently small $\hat r_0 > 0$, so that it also verifies the estimates given by Lemma 2.2.

In the following, we denote by $C^1_T$ the set of $T$-periodic $C^1$-functions, with the usual norm of $C^1([0,T])$.

**Lemma 2.3** *Given $M > 0$, there is a continuum $\mathcal{C}$ in $[0,M]\times C^1_T$, connecting $\{0\}\times C^1_T$ with $\{M\}\times C^1_T$, whose elements $(\mu,\rho)$ satisfy the first equation in (S).*

*Proof.* In order to apply degree theory, let us define the following operators:

$$\begin{aligned}
&L : D(L) \subset C^1([0,T]) \to L^1(0,T)\,,\\
&D(L) = \{\rho \in W^{2,1}(0,T) : \rho(0) = \rho(T)\,, \dot\rho(0) = \dot\rho(T)\}\,,\\
&L\rho = \ddot\rho - \rho\,,
\end{aligned}$$

and, for $\lambda \in [0,1]$,

$$\begin{aligned}
&N_\lambda : [0,M] \times C^1([0,T]) \to L^1(0,T)\,,\\
&N_\lambda(\mu,\rho)(t) = -g_{\lambda,\mu,r_0}(t,\rho(t)) + \lambda e(t) - \rho(t)\,.
\end{aligned}$$

Problem $(P_{\lambda,\mu,r_0})$ is thus equivalent to

$$L\rho = N_\lambda(\mu,\rho)\,.$$

Since $L$ is invertible, we can write equivalently

$$\rho - L^{-1}N_\lambda(\mu,\rho) = 0\,. \tag{2.16}$$

Let $C > 0$ be the constant given by Lemma 2.2 and define $\Omega$ to be the following open subset of $C^1([0,T])$:

$$\Omega = \Big\{\rho \in C^1([0,T]) : \frac{1}{C} < \rho(t) < C \ \text{ and } \ |\dot\rho(t)| < C\,,\ \text{for every } t \in [0,T]\Big\}.$$

By Lemma 2.2, equation (2.16) has no solutions $\rho$ on $\partial\Omega$, for any $\lambda \in [0,1]$ and $\mu \in [0,M]$. Since $L^{-1}N_\lambda(\mu,\cdot)$ is a compact operator, by the global continuation principle of Leray-Schauder (see e.g. [19, Theorem 14.C]), the lemma will be proved if we show that the degree is nonzero for some $(\lambda,\mu) \in [0,1]\times[0,M]$. Indeed, since the degree is the same for every $(\lambda,\mu) \in [0,1]\times[0,M]$, then, it will be nonzero when $\lambda = 1$, for every $\mu \in [0,M]$.

Let us then take $\mu = \lambda = 0$ and define the function

$$\Psi : \Big[\,\frac{1}{C}\,, C\,\Big] \times [-C, C\,] \to \mathbb{R}^2\,,$$
$$\Psi(u,v) = \Big(v\,,\ \frac{1}{u^3} - 1\Big).$$

By a result of Capietto, Mawhin and Zanolin [2, Theorem 1], one can compute the Leray-Schauder degree of $I - L^{-1}N_0(0,\cdot)$ as the Brouwer degree of $\Psi$:

$$d_{LS}(I - L^{-1}N_0(0,\cdot),\Omega) = d_B\Big(\Psi,\ \Big]\,\frac{1}{C}\,, C\,\Big[\times\,]-C, C\,[\,\Big).$$

Since $\Psi$ has the unique zero $(1,0)$, and the jacobian matrix $J_\Psi(1,0)$ has a positive determinant, we conclude that the degree has to be equal to 1. ■

Notice that, if $(\mu,\rho) \in \mathcal{C}$ then $\rho$ is $T$-periodic and the first equation in (S) is satisfied. Let us consider the function $\Phi : \mathcal{C} \to \mathbb{R}$, defined by

$$\Phi(\mu,\rho) = \int_0^T \frac{\mu}{\rho^2(t)}\,dt\,.$$

It is continuous and defined on a compact and connected domain, so its image is a compact interval. Since $\Phi(0,\rho) = 0$ and $\Phi$ is not identically zero, this interval is of the type $[0,\overline{\theta}]$ for some $\overline{\theta} > 0$.

**Lemma 2.4** *For every $\theta \in [0,\overline{\theta}]$, there are $(\mu,\rho,\varphi)$, verifying system (S), for which $(\mu,\rho) \in \mathcal{C}$, and*

$$\rho(t+T) = \rho(t)\,, \qquad \varphi(t+T) = \varphi(t) + \theta\,,$$

*for every $t \in \mathbb{R}$.*

*Proof.* Given $\theta \in [0,\overline{\theta}]$, there are $(\mu,\rho) \in \mathcal{C}$ such that

$$\int_0^T \frac{\mu}{\rho^2(t)}\,dt = \theta\,.$$

As noticed above, the first equation in (S) is satisfied. Moreover, defining

$$\varphi(t) = \int_0^t \frac{\mu}{\rho^2(s)}\, ds\,,$$

the second equation in (S) is also satisfied and

$$\varphi(t+T) - \varphi(t) = \int_t^{t+T} \frac{\mu}{\rho^2(s)}\, ds = \int_0^T \frac{\mu}{\rho^2(s)}\, ds = \theta\,.$$

■

We are going to complete now the proof of Theorem 1.1. For every $\theta \in [0, \bar{\theta}]$, the solution of system (S) found in Lemma 2.4 provides, through (2.3), a solution to equation (1.6) such that

$$x(t+T) = e^{i\theta} x(t)\,,$$

for every $t \in \mathbb{R}$ (for briefness we used here the complex notation).

In particular, if $\theta = \frac{2\pi}{k}$ for some integer $k \geq 1$, then $x(t)$ is periodic with minimal period $kT$, and rotates exactly once around the origin in the period time $kT$. Hence, for every integer $k \geq 2\pi/\bar{\theta}$, we have such a $kT$-periodic solution, which we denote by $x_k(t)$. Let $(\rho_k(t), \varphi_k(t))$ be its polar coordinates, and $\mu_k$ be its angular momentum. By the above construction, $(\mu_k, \rho_k, \varphi_k)$ verify system (S), $(\mu_k, \rho_k) \in \mathcal{C}$, and

$$\int_0^T \frac{\mu_k}{\rho_k^2(t)}\, dt = \frac{2\pi}{k}\,.$$

Since $\mu_k \in [0, M]$, by Lemma 2.2 we have that

$$\frac{1}{C} < \rho_k(t) < C \qquad \text{and} \qquad |\dot{\rho}_k(t)| < C\,,$$

for every $t \in [0, T]$. Hence,

$$\frac{2\pi}{k} = \int_0^T \frac{\mu_k}{\rho_k^2(t)}\, dt > T\,\frac{\mu_k}{C^2}\,,$$

so that

$$\lim_{k\to\infty} \mu_k = 0\,.$$

The proof is thus completed. ■

## 3 Proof of Theorem 1.2: solutions with large angular momentum

The proof of Theorem 1.2 follows closely both the proof of [8, Theorem 4], where the attractive case has been considered, and the proof of Theorem 1.1.

As in the previous section, without loss of generality, we assume (2.4), i.e.,

$$\bar{e} = 0\,.$$

We consider $\mu \geq 0$ as a parameter, and, by the use of degree theory, we will prove the existence of a $T$-periodic solution $\rho$ of the first equation in (S). To this aim, for $\lambda \in [0,1]$, we introduce the modified problem (2.5). We define the truncation at $r_0 = 1$, i.e., $g_{\lambda,\mu} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, as follows:

$$g_{\lambda,\mu}(t,r) = \begin{cases} -\dfrac{\mu^2}{r^3} + (1-\lambda)\left(1 - \dfrac{1}{r^3}\right) + \lambda h(t,r) & \text{for } r \geq 1 \\ -\mu^2 + \lambda h(t,1) & \text{for } r \leq 1\,. \end{cases}$$

This function is still $L^1$-Carathéodory. We consider the $T$-periodic problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} \ddot{\rho} + g_{\lambda,\mu}(t,\rho) = \lambda e(t)\,, \\ \rho(0) = \rho(T)\,,\ \dot{\rho}(0) = \dot{\rho}(T)\,, \end{cases}$$

and look for a priori bounds for the positive solutions $\rho$. This time, we will consider solutions with a large angular momentum.

**Lemma 3.1** *For every $\Gamma > 1$ there exists $\mu(\Gamma) \geq 1$ such that, if $\mu \geq \mu(\Gamma)$, $\lambda \in [0,1]$, and $\rho$ is a positive solution of $(P_{\lambda,\mu})$, then $\|\rho\|_{L^\infty} \geq \Gamma$.*

*Proof.* Assume by contradiction that there are $\Gamma > 1$, and sequences $(\lambda_n)_n$, $(\mu_n)_n$, and $(\rho_n)_n$ such that $\lambda_n \in [0,1]$, $\lim_n \mu_n = +\infty$, and $\rho_n$ is a positive solution of $(P_{\lambda_n,\mu_n})$, with $\|\rho_n\|_{L^\infty} < \Gamma$. Integrating the equation, since $\bar{e} = 0$, we obtain

$$\int_0^T g_{\lambda_n,\mu_n}(t,\rho_n(t))\,dt = 0\,.$$

Hence, setting $\tilde{\rho}_n(t) = \max\{\rho_n(t), 1\}$,

$$\mu_n^2 \int_0^T \frac{1}{\tilde{\rho}_n^3(t)}\,dt = (1-\lambda_n)\int_0^T \left(1 - \frac{1}{\tilde{\rho}_n^3(t)}\right)dt + \lambda_n \int_0^T h(t,\tilde{\rho}_n(t))\,dt\,.$$

Since $1 \leq \tilde{\rho}_n \leq \Gamma$, using the $L^1$-Carathéodory condition we see that the right-hand side is bounded, and we get a contradiction with the assumption that $(\mu_n)_n$ tends to $+\infty$. ■

**Lemma 3.2** *There exists a constant $\overline{C} > 0$ such that, if $\mu \geq 1$, $\lambda \in [0,1]$, and $\rho$ is a positive solution of $(P_{\lambda,\mu})$, then*

$$\min \rho \geq \frac{1}{2}\|\rho\|_{L^\infty} - \overline{C}\,.$$

*Proof.* By (H4), we can fix $\overline{r} \geq 1$ such that

$$r \geq \overline{r} \quad \Rightarrow \quad |h(t,r)| \leq \frac{r}{4T^2} \quad \text{for a.e. } t \in \,]0,T[\,.$$

Multiplying in $(P_{\lambda,\mu})$ by $\rho$ and integrating we get, writing $\tilde{\rho} = \max\{\rho, 1\}$,

$$\begin{aligned}\int_0^T \dot{\rho}^2 &= -\int_0^T \frac{\mu^2}{\tilde{\rho}^3}\rho + (1-\lambda)\int_0^T \left(1 - \frac{1}{\tilde{\rho}^3}\right)\rho + \lambda \int_0^T h(t,\tilde{\rho})\rho - \lambda \int_0^T e\rho \\ &\leq T\|\rho\|_{L^\infty} + \int_{\{1\leq\tilde{\rho}<\bar{r}\}} |h(t,\tilde{\rho})|\rho + \int_{\{\tilde{\rho}\geq\bar{r}\}} |h(t,\tilde{\rho})|\rho + \|e\|_{L^1}\|\rho\|_{L^\infty} \\ &\leq \frac{1}{4T}\|\rho\|^2_{L^\infty} + (T + \|\ell_{1,\bar{r}}\|_{L^1} + \|e\|_{L^1})\|\rho\|_{L^\infty} \,.\end{aligned}$$

Setting

$$\overline{C} := (T + \|\ell_{1,\bar{r}}\|_{L^1} + \|e\|_{L^1})T \,,$$

we have

$$T\|\dot{\rho}\|^2_{L^2} \leq \frac{1}{4}(\|\rho\|^2_{L^\infty} + 4\overline{C}\|\rho\|_{L^\infty}) \leq \frac{1}{4}(\|\rho\|_{L^\infty} + 2\overline{C})^2 \,.$$

So,

$$\max\rho - \min\rho \leq \sqrt{T}\|\dot{\rho}\|_{L^2} \leq \frac{1}{2}\|\rho\|_{L^\infty} + \overline{C} \,,$$

thus proving the lemma. ∎

Let us now fix $\overline{\Gamma} > 2(1+\overline{C})$, where $\overline{C}$ is given by Lemma 3.2. Correspondingly, let $\bar{\mu} := \mu(\overline{\Gamma})$, with $\mu(\overline{\Gamma}) \geq 1$ as in Lemma 3.1.

**Lemma 3.3** *Given $A, B$, with $\bar{\mu} \leq A \leq B$, there is a constant $C > 1$ such that, if $\mu \in [A, B]$, $\lambda \in [0,1]$, and $\rho$ is a positive solution of $(P_{\lambda,\mu})$, then*

$$1 < \rho(t) < C \,, \quad |\dot{\rho}(t)| < C \,,$$

*for every $t \in [0, T]$.*

*Proof.* By contradiction, assume that, for every $n \geq 1$ there are $\lambda_n \in [0,1]$, $\mu_n \in [A, B]$ and a positive solution $\rho_n(t)$ of the $T$-periodic problem $(P_{\lambda_n,\mu_n})$ such that, either $\rho_n([0,T]) \not\subseteq ]1, n[$ or $\|\dot{\rho}_n\|_{L^\infty} \geq n$. By the choice of $\bar{\mu}$ and Lemma 3.1, it has to be $\|\rho_n\|_{L^\infty} \geq \overline{\Gamma}$, and by Lemma 3.2,

$$\min\rho_n \geq \frac{1}{2}\|\rho_n\|_{L^\infty} - \overline{C} \geq \frac{1}{2}\overline{\Gamma} - \overline{C} > 1 \,.$$

Arguing as in Step 1 of the proof of Lemma 2.2 we show that there exists a $R_1 \geq 1$ such that $\min\rho_n \leq R_1$ for every $n$. By Lemma 3.2 we deduce that

$$\|\rho_n\|_{L^\infty} \leq 2(\min\rho_n + \overline{C}) < 2(R_1 + \overline{C}) \,.$$

Set $b = 2(R_1 + \overline{C})$. From the equation in $(P_{\lambda_n,\mu_n})$ and the Carathéodory conditions, recalling that $\rho_n(t) > 1$, since $\dot{\rho}_n(t)$ vanishes somewhere, we obtain

$$\|\dot{\rho}_n\|_{L^\infty} \leq \|\ddot{\rho}_n\|_{L^1} \leq \int_0^T \frac{\mu_n^2}{\rho_n^3} + \int_0^T |h(t,\rho_n)| + \|e\|_{L^1} < B^2 T + \|\ell_{1,b}\|_{L^1} + \|e\|_{L^1} \,,$$

thus ariving at a contradiction. ∎

As a consequence of Lemma 3.3, if $\mu \geq \overline{\mu}$ and $\rho(t)$ is a $T$-periodic solution of $(P_{1,\mu})$ then $\rho(t)$ also satisfies the first equation in (S). The following lemma gives an important information concerning the $T$-periodic solutions of that equation.

**Lemma 3.4** *Given $A, B$, with $\overline{\mu} \leq A \leq B$, there is a continuum $C_{A,B}$ in $[A,B] \times C^1_T$, connecting $\{A\} \times C^1_T$ with $\{B\} \times C^1_T$, whose elements $(\mu,\rho)$ satisfy the first equation in (S), being $\rho(t) > 1$ for every $t$.*

*Proof.* We proceed as in the proof of Lemma 2.3. Let $L$ be the differential operator defined there, and, for $\lambda \in [0,1]$,

$$N_\lambda : [A,B] \times C^1([0,T]) \to L^1(0,T)\,,$$
$$N_\lambda(\mu,\rho)(t) = -g_{\lambda,\mu}(t,\rho(t)) + \lambda e(t) - \rho(t)\,.$$

Problem $(P_{\lambda,\mu})$ is thus equivalent to (2.16). Let $C > 0$ be the constant given by Lemma 3.3 and define $\Omega$ to be the following open subset of $C^1([0,T])$:

$$\Omega = \Big\{\rho \in C^1([0,T]) : 1 < \rho(t) < C \text{ and } |\dot\rho(t)| < C\,, \text{ for every } t \in [0,T]\Big\}\,.$$

By Lemma 3.3, equation (2.16) has no solutions $\rho$ on $\partial\Omega$, for any $\lambda \in [0,1]$ and $\mu \in [A,B]$. Since $L^{-1}N_\lambda(\mu,\cdot)$ is a compact operator, by the global continuation principle of Leray-Schauder, the lemma will be proved if we show that the degree is nonzero for some $(\lambda,\mu) \in [0,1] \times [A,B]$.

Let us then take $\lambda = 0$, $\mu = A$, and define the function

$$\Psi : [1\,,C] \times [-C,C] \to \mathbb{R}^2\,,$$
$$\Psi(u,v) = \left(v\,,\ \frac{A^2+1}{u^3} - 1\right).$$

Since

$$d_B\Big(\Psi,\ ]1\,,C[\times]-C,C[\Big) = 1\,,$$

we conclude as in the proof of Lemma 2.3. ∎

Using classical arguments from the theory of global continuation (see, e.g., [14, Lemma 2.3]), we can deduce from Lemmas 3.3 and 3.4 that there is a connected set $\mathcal{C}$, contained in $[\overline{\mu},+\infty[\times C^1_T$, which connects $\{\overline{\mu}\} \times C^1_T$ with $\{\mu^*\} \times C^1_T$, for every $\mu^* > \overline{\mu}$, whose elements $(\mu,\rho)$ satisfy the first equation in (S).

**Lemma 3.5** *For every $\varepsilon > 0$, there exists $\mu_\varepsilon \geq \overline{\mu}$ such that, if $(\mu,\rho) \in \mathcal{C}$ with $\mu \geq \mu_\varepsilon$, then*

$$\int_0^T \frac{\mu}{\rho^2(t)}\,dt \leq \varepsilon\,.$$

*Proof.* Given $\varepsilon > 0$, set

$$\varepsilon' := \left(\frac{\varepsilon}{36\,T}\right)^2\,.$$

Let $\overline{C}$ be as in Lemma 3.2. By (H4), there exists $r' \geq \max\{3\overline{C}, 1\}$ such that

$$r \geq r' \quad \Rightarrow \quad |h(t,r)| \leq \varepsilon' r\,, \qquad \text{for a.e. } t \in ]0,T[\,.$$

For $\Gamma := 2(r' + \overline{C})$, let $\mu(\Gamma)$ be as in Lemma 3.1. Set $\mu_\varepsilon := \max\{\mu(\Gamma), \overline{\mu}\}$.

Let $(\mu, \rho)$ be an element of $\underline{C}$, with $\mu \geq \mu_\varepsilon$. By Lemma 3.1, $\|\rho\|_{L^\infty} \geq \Gamma$, and by Lemma 3.2, $\rho(t) \geq \frac{1}{2}\|\rho\|_{L^\infty} - \overline{C} \geq r'$, for every $t \in \mathbb{R}$. Hence,

$$\frac{1}{T}\int_0^T h(t,\rho) \leq \frac{1}{T}\int_0^T \varepsilon'\rho \leq \varepsilon'\|\rho\|_{L^\infty}\,.$$

Integrating the first equation in (S), since $\bar{e} = 0$, we have

$$\frac{1}{T}\int_0^T \frac{\mu^2}{\rho^3} = \frac{1}{T}\int_0^T h(t,\rho) \leq \varepsilon'\|\rho\|_{L^\infty}\,.$$

On the other hand,

$$\frac{1}{T}\int_0^T \frac{\mu^2}{\rho^3} \geq \frac{\mu^2}{\|\rho\|^3_{L^\infty}}\,,$$

so that

$$\frac{\mu^2}{\|\rho\|^4_{L^\infty}} \leq \varepsilon'\,.$$

Then, using again Lemma 3.2,

$$\begin{aligned}
\int_0^T \frac{\mu}{\rho^2(t)}\,dt &\leq T\frac{\mu}{(\min\rho)^2} \leq T\frac{4\mu}{(\|\rho\|_{L^\infty} - 2\overline{C})^2} \\
&= T\frac{\mu}{\|\rho\|^2_{L^\infty}}\left(\frac{2\|\rho\|_{L^\infty}}{\|\rho\|_{L^\infty} - 2\overline{C}}\right)^2 \leq T\sqrt{\varepsilon'}\left(\frac{2r'}{r' - 2\overline{C}}\right)^2 \\
&\leq T\sqrt{\varepsilon'}\left(\frac{6\overline{C}}{3\overline{C} - 2\overline{C}}\right)^2 = 36\,T\sqrt{\varepsilon'} = \varepsilon\,,
\end{aligned}$$

thus proving the lemma. ■

Since the function

$$(\mu,\rho) \mapsto \int_0^T \frac{\mu}{\rho^2(t)}\,dt$$

is continuous from $\underline{C}$ to $\mathbb{R}$, and $\underline{C}$ is connected, its image is an interval. By Lemmas 3.4 and 3.5 this interval is of the type $]0, \overline{\theta}]$ for some $\overline{\theta} > 0$. The analogue of Lemma 2.4 then holds in this case, too, with $\theta \in ]0, \overline{\theta}]$.

The proof of Theorem 1.2 can now be completed as in the previous section (see also [8, Theorem 4]).

# 4 Remarks on the continuum of solutions and multiplicity of periodic solutions

Assume (H1), (H2) and (H3). By Lemma 2.2 we have that, for every $M > 0$, there exists a constant $C > 0$ such that, for any $\mu \in [0, M]$, if $\rho(t)$ is a $T$-periodic solution of the first equation in (S), then

$$\frac{1}{C} < \rho(t) < C \qquad \text{and} \qquad |\dot{\rho}(t)| < C\,,$$

for every $t \in [0, T]$. Moreover, as seen in Lemma 2.3, for every $\mu$ the associated degree is constantly equal to 1. Hence, by the global continuation principle of Leray-Schauder (see, e.g., [14, Lemma 2.3]), there is a closed connected set $C$, contained in $[0, +\infty[\, \times C^1_T$, which connects $\{0\} \times C^1_T$ with $\{\mu^*\} \times C^1_T$, for every $\mu^* > 0$, whose elements $(\mu, \rho)$ satisfy the first equation in (S).

Now consider the function $\hat{\varphi} : C \to C^2(\mathbb{R})$, defined by

$$\hat{\varphi}(\mu, \rho)(t) = \int_0^t \frac{\mu}{\rho^2(\tau)}\, d\tau\,.$$

Since this function is continuous, its graph is a closed connected subset $\hat{C}_+$ of $[0, +\infty[\, \times C^1_T \times C^2(\mathbb{R})$. Moreover, the projection of $\hat{C}_+$ on its first component is the whole half-line $[0, +\infty[$ .

It is easily seen that the same arguments used till now symmetrically hold in the case of negative angular momenta $\mu$. In particular, the above a priori bounds hold for every $\mu \in [-M, M]$, once $M > 0$ has been fixed. We then have the following result.

**Lemma 4.1** *Let assumptions (H1)-(H3) hold. Then, there is a closed connected subset $\hat{C}$ of $\mathbb{R} \times C^1_T \times C^2(\mathbb{R})$ whose elements $(\mu, \rho, \varphi)$ solve system (S), and whose projection on its first component is the whole real line $]-\infty, +\infty[$ . Moreover, for any sufficiently large integer k, this set $\hat{C}$ contains solutions for which $\mu$ is small and $\varphi(t + kT) = \varphi(t) + 2\pi$, for every t.*

If, instead of (H1), the stronger assumption (H4) holds, then we can repeat the estimates made in Lemmas 3.1 - 3.5 to deduce that, along the connected set $\hat{C}$ given by Lemma 4.1, there also are solutions with large angular momenta and amplitudes, for which $\varphi(t + kT) = \varphi(t) + 2\pi$, for every $t$.

Recalling that (S) represents, through

$$x(t) = \rho(t)(\cos\varphi(t), \sin\varphi(t))\,,$$

the solutions of (1.6) having angular momentum $\mu$, we can conclude as follows.

**Corollary 4.1** *Let assumptions (H2), (H3) and (H4) hold. Then, there is a closed connected subset of $\mathbb{R} \times C^1(\mathbb{R}, \mathbb{R}^2 \setminus \{0\})$ whose elements $(\mu, x)$ are such that $x(t)$ is a solution of system (1.6) with angular momentum $\mu$, and $|x(t)|$ is T-periodic. The projection of this set on its first component is the whole real line $]-\infty, +\infty[$ . Moreover, this set contains infinitely many periodic solutions, rotating around the origin, with arbitrarily large periods and amplitudes.*

Till now, we always considered periodic solutions making exactly one revolution around the origin in their period time. Concerning the existence of periodic solutions making a higher number of revolutions around the origin, we have the following result.

**Theorem 4.1** *Let $e(t)$ be continuous, with minimal period T, and negative mean value. Let $h(t, r)$ be continuous, and assume that*

$$\text{(H5)} \qquad \lim_{r \to +\infty} h(t, r) = 0\,,$$

*uniformly for every t. Then, for every integer $m \geq 1$, there exists a $k_m \geq 1$ such that, for every integer $k \geq k_m$, if k and m are relatively prime, equation (1.6) has a periodic solution $x_{k,m}(t)$ with minimal period kT, which makes exactly m revolutions around the origin in the period time kT. Moreover,*

$$\lim_{k \to \infty} (\min |x_{k,m}|) = +\infty\,,$$

*and*

$$\lim_{k\to\infty} \frac{\min |x_{k,m}|}{\max |x_{k,m}|} = 1 .$$

We omit the proof, since it is exactly the same as that of [8, Theorem 7]. To conclude, let us state the following multiplicity result.

**Theorem 4.2** *Let $e(t)$ be continuous, with minimal period $T$, and negative mean value. Let $h(t, r)$ be continuous, and assume (H5). Then, for every $n \in \mathbb{N}$, there is a $k(n) \geq 1$ such that, for every prime integer $k \geq k(n)$, equation (1.6) has at least $n$ geometrically distinct periodic solutions with minimal period $kT$.*

*Proof.* Given $n \geq 1$, let $p_1, p_2, \ldots, p_n$ be the first $n$ prime numbers. Correspondingly, let $k_{p_1}, k_{p_2}, \ldots, k_{p_n}$ be as in Theorem 4.1. Define

$$k(n) = \max\{k_{p_1}, k_{p_2}, \ldots, k_{p_n}\} .$$

By Theorem 4.1, for every prime number $k \geq k(n)$, equation (1.6) has periodic solutions $x_{k,p_1}$, $x_{k,p_2}, \ldots, x_{k,p_n}$, with minimal period $kT$, which make exactly $p_1, p_2, \ldots, p_n$ rotations around the origin, respectively, in the period time $kT$. ■

We immediately deduce the following corollary concerning equation (1.3).

**Corollary 4.2** *Let $c(t)$ and $e(t)$ be continuous. Assume $e(t)$ has minimal period $T$, and $\bar{e} < 0$. If $\gamma > 1$ and (1.8) holds, then the same conclusions of Theorems 4.1 and 4.2 hold for system (1.3).*

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