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Periodic orbits of radially symmetric Keplerian-like systems: A topological degree approach

Alessandro Fonda [a,*], Rodica Toader [b]

[a] *Dipartimento di Matematica e Informatica, Università di Trieste, P.le Europa 1, I-34127 Trieste, Italy*
[b] *Dipartimento di Ingegneria Civile, Università di Udine, Via delle Scienze 208, I-33100 Udine, Italy*

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Abstract

We are concerned with non-autonomous radially symmetric systems with a singularity, which are T-periodic in time. By the use of topological degree theory, we prove the existence of large-amplitude periodic solutions whose minimal period is an integer multiple of T. Precise estimates are then given in the case of Keplerian-like systems, showing some resemblance between the orbits of those solutions and the circular orbits of the corresponding classical autonomous system.

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1. Introduction

In this paper, we are mainly interested in the study of periodic solutions to systems of the type

$$\ddot{x} + f\big(t, |x|\big)x = 0. \tag{1}$$

Here, f is a real function, T-periodic in t, which is defined on $\mathbb{R} \times]0, +\infty[$, so that 0 can be a singularity. We therefore look for solutions $x(t) \in \mathbb{R}^N$ which never attain the singularity, in the sense that

* Corresponding author.
E-mail addresses: a.fonda@units.it (A. Fonda), toader@uniud.it (R. Toader).

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$$x(t) \neq 0, \quad \text{for every } t \in \mathbb{R}. \tag{2}$$

We would like to deal, for example, with equations like

$$\ddot{x} + \frac{c(t)}{|x|^\gamma} x = e(t) \frac{x}{|x|}, \tag{3}$$

or

$$\ddot{x} + \frac{c(t)}{|x|^3} x + \frac{d(t)}{|x|^4} x = e(t) \frac{x}{|x|}. \tag{4}$$

In the above, $c(t)$, $d(t)$ and $e(t)$ are T-periodic functions, and γ is a positive constant. Notice that, if c, d, e are constant functions, with $e(t)$ identically zero, and $\gamma = 3$, then (3) is the equation of the Keplerian motion of a planet, while (4) is, in the simplest approximated form, the equation of its relativistic motion (cf. [24, p. 122]).

Since (1) is radially symmetric, its solutions have planar orbits. This fact simplifies considerably the analysis, and plays a crucial role in our approach.

More general systems, of the type

$$\ddot{x} + \nabla V(t, x) = 0, \tag{5}$$

were studied by many authors, mainly by the use of variational methods, assuming the function V to be T-periodic in t, differentiable in $x \neq 0$ with continuous gradient, and such that

$$\lim_{x \to 0} V(t, x) = -\infty. \tag{6}$$

In the planar case, Gordon [18] was able to prove a nice existence result for periodic solutions of (5) by the use of a variational method. The main difficulty of avoiding collision orbits (for which (2) is not satisfied) was overcome by introducing the so-called "strong force" assumption. His result, later improved by Capozzi, Greco and Salvatore [10], can be stated for our convenience as follows.

Theorem 1. *Let $N = 2$, assume* (6) *and that there exist $\nu \in [0, 2[$ and positive constants c_1, c_2 such that*

$$V(t, x) \leqslant c_1 |x|^\nu + c_2,$$

for every t and $x \neq 0$. If moreover

(SF) *there are a C^1-function $U : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, a neighborhood $\mathcal{N}$ of 0 and a positive constant c_3 such that $\lim_{x \to 0} U(x) = -\infty$ and*

$$-V(t, x) \geqslant \left|\nabla U(x)\right|^2 - c_3, \quad \textit{for every } x \in \mathcal{N} \setminus \{0\},$$

then, for every integer $k \geqslant 1$, system (5) *has a periodic solution with minimal period kT.*

The proof, originally given only for period T, consists in minimizing the associated functional over the orbits which turn around the origin exactly once in their period time. The strong force condition (SF) guarantees that the minimization procedure does not lead to a collision orbit. Theorem 1 applies to Eq. (3) provided that $\gamma \geqslant 4$, and does not apply to Eq. (4), since condition (SF) does not hold there.

After the pioneering paper by Gordon, much attention was given to systems in higher dimensions (see, e.g., [2,3,7,19–21,25,26]), and trying to avoid the strong force assumption (see, e.g., [12,13,29,31]). The possibility of obtaining collision orbits in situations when the strong force condition is not satisfied led Bahri and Rabinowitz [7] to the definition of some kind of "generalized solution." Many authors then studied this type of "solutions" (see [5] and the references therein). However, we will never consider such a situation, and for a solution we will always ask that (2) holds.

We will be mainly interested in situations which include the Keplerian case, where the strong force assumption does not hold. A remarkable result in this direction was obtained by Degiovanni, Giannoni and Marino [14]. In their paper, a lower semicontinuous function $\phi : [1, +\infty[\to [1, +\infty]$ is defined, with the following property:

$$\begin{aligned} &\phi(\alpha) = 1, && \text{if } \alpha = 1, \\ &\phi(\alpha) \in]1, +\infty[, && \text{if } 1 < \alpha < 2, \\ &\phi(\alpha) = +\infty, && \text{if } \alpha \geqslant 2. \end{aligned}$$

This function was recalled in [15], where Degiovanni and Giannoni, developing the ideas in [14], proposed the following result.

Theorem 2. *Let V be even in x and assume that, for some $\alpha \geqslant 1$,*

$$\frac{c}{|x|^\alpha} \leqslant -V(t,x) \leqslant \frac{c'}{|x|^\alpha}, \tag{7}$$

for every t and $x \neq 0$. If

$$0 < c' \leqslant 2^\alpha \phi(\alpha) c,$$

then, for every integer $k \geqslant 1$, system (5) *has a periodic solution $x_k(t)$ having period kT, whose orbit is symmetric with respect to the origin, and*

$$\lim_{k\to\infty} \bigl(\min|x_k|\bigr) = +\infty.$$

Theorem 2 holds in any dimension N and the function V is not required to be radially symmetric. The minimality of the period kT is not guaranteed, but there surely are infinitely many periodic solutions. The result applies to Eqs. (3) and (4) if $e(t)$ is identically zero and $\gamma = \alpha + 2 \geqslant 3$, with suitable bounds on $c(t)$ when $3 \leqslant \gamma < 4$.

Conditions similar to (7) were proposed by Ramos and Terracini [27] in the case $0 < \alpha < 1$, while considering autonomous equations. The Keplerian exponent $\alpha = 1$ proved to be particularly delicate when dealing with such methods, as shown by Capozzi, Solimini and Terracini [11].

A different approach was used by Ambrosetti and Coti Zelati in [4], where a perturbative situation of the type

$$V(t,x) = -\frac{c}{|x|^\alpha} + \varepsilon W(t,x)$$

was considered. Here $c > 0$ is a constant, α is positive, and ε is sufficiently small. They found several periodic solutions "bifurcating" from the circular orbits of the unperturbed system. Their results apply to Eq. (3), when $c(t)$ is constant and $e(t)$ is sufficiently small.

Another result which includes the Keplerian case can be found in [28], where Serra and Terracini considered C^2-potentials of the type

$$V(t,x) = -\frac{c}{|x|^\alpha} + \frac{\lambda}{2}|x|^2 + W(t,x),$$

where $c > 0$ is a constant, $\alpha \in \,]0,2[$ and W includes terms of lower order near 0 (see their assumptions (H1) and (H2)), thus proving the following.

Theorem 3. *Let $N \geqslant 3$ and assume V as above, with W such that*

$$W\left(t + \frac{T}{2}, -x\right) = W(t,x),$$

for every t, $x \neq 0$, and

$$\lim_{|x|\to\infty} \frac{\nabla W(t,x)}{|x|} = 0$$

uniformly in t. If $\lambda < (\frac{\pi}{T})^2$, then (5) *has a T-periodic solution, whose orbit is symmetric with respect to the origin.*

This result applies (with $\lambda = 0$) to Eq. (3) whenever $c(t)$ is constant, $2 < \gamma < 4$, and $e(t)$ is twice continuously differentiable. See also [1] where, under similar hypotheses, a different type of singularity is considered, and the existence of a T-periodic solution is proved.

We deal here with an equation like (1), which for convenience we write as

$$\ddot{x} = \big(-h\big(t, |x|\big) + e(t)\big)\frac{x}{|x|}. \tag{8}$$

We assume that $h : \mathbb{R} \times \,]0, +\infty[\, \to \mathbb{R}$ is L^1-Carathéodory and T-periodic with respect to its first variable, and that $e \in L^1_{\mathrm{loc}}(\mathbb{R})$ is T-periodic, with

(H0) $$\bar{e} := \frac{1}{T}\int_0^T e(t)\,dt \leqslant 0.$$

Let us state our main existence result.

Theorem 4. *Assume* (H0) *and the following hypotheses*:

(H1)
$$\lim_{r\to+\infty} r^3 h(t,r) = +\infty,$$

(H2)
$$\lim_{r\to+\infty} \frac{h(t,r)}{r} = 0$$

uniformly for almost every t. Then, there exists $k_1 \geqslant 1$ such that, for every integer $k \geqslant k_1$, Eq. (8) *has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover,*

$$\lim_{k\to\infty} \left(\min|x_k|\right) = +\infty$$

and

$$\lim_{k\to\infty} \frac{\min|x_k|}{\max|x_k|} = 1.$$

Notice that we have no assumption on h near the origin. The orbits of the solutions we find are close to be circular, in the sense that the quotient between the minimal and the maximal radius is close to 1, and stay far away from the origin. This fact can be used to deal with more general equations, like (5), when the potential V is assumed to be radially symmetric only outside a given compact set.

Our result directly applies to (3) and (4), as the following two corollaries show.

Corollary 1. *Assume* (H0), *let $c \in L^\infty(\mathbb{R})$ be such that*

(H3)
$$0 < c_1 \leqslant c(t) \leqslant c_2,$$

for some positive constants c_1 and c_2, and let $0 < \gamma < 4$. *Then, the conclusion of Theorem* 4 *holds for Eq.* (3).

Notice that, when $0 < \gamma < 1$, there is no singularity at all. However, the fact that γ has to be nonzero is crucial in order to avoid the possibility of linear resonance phenomena.

Corollary 2. *Assume* (H0), (H3), *and let $d \in L^\infty(\mathbb{R})$. Then, the conclusion of Theorem* 4 *holds for Eq.* (4).

The proof of Theorem 4 is given in the next section. It is based on a careful analysis of the behaviour of the solutions having large angular momentum. The radial component of our solutions is T-periodic and its oscillations are controlled in some way by its amplitude, while the angle component varies very slowly, and can be tuned so that the solution performs exactly one rotation in a period kT, for k large enough.

In Section 3 we will give more precise estimates on the solutions and their velocities for equations like (3) and (4). The resemblance with the circular orbits of the classical autonomous equations will then be even more evident.

In Section 4, we will discuss the possibility of obtaining multiplicity of solutions with the same minimal period, distinguishing them by the number of rotations they perform in their period time.

In Section 5 we show that the conclusion of Theorem 4 cannot be true without the assumption (H0). Indeed, we prove that, if $\bar{e} > 0$ and $h(t,r)$ tends to zero as r tends to infinity, there is a compact region with the property that all solutions exiting this region necessarily must have unbounded orbits.

In Appendix A we give an elementary proof of the fact that the solutions of a radially symmetric system like (1) have planar orbits. This is well known in the case $N = 3$, but it seems hard to find it proved, for higher dimensions, in standard textbooks.

2. Proof of the main result

This section is dedicated to the proof of Theorem 4. Let us first clarify our assumptions. Recall that L^1-Carathéodory means

- $h(\cdot, r)$ is measurable, for every $r > 0$;
- $h(t, \cdot)$ is continuous, for a.e. $t \in [0, T]$;
- for every compact interval $[a, b]$ in $]0, +\infty[$, there exists $\ell_{a,b} \in L^1(0, T)$ such that

$$r \in [a,b] \quad \Rightarrow \quad \left|h(t,r)\right| \leqslant \ell_{a,b}(t), \quad \text{for a.e. } t \in [0,T].$$

Since system (8) is radially symmetric, the orbit of a solution always lies on a plane containing the origin (see Appendix A). Moreover, since rotations leaving fixed the origin preserve the solutions, it is enough to look for those solutions whose orbits lie on a fixed plane. Therefore, from now on, we will assume, without loss of generality, that $N = 2$.

We may write the solutions in polar coordinates:

$$x(t) = \rho(t)\big(\cos\varphi(t), \sin\varphi(t)\big), \tag{9}$$

and (2) is satisfied if $\rho(t) > 0$, for every t. Eq. (8) is then equivalent to the system

$$\text{(S)} \qquad \begin{cases} \ddot{\rho} - \dfrac{\mu^2}{\rho^3} + h(t,\rho) = e(t), \\ \rho^2\dot{\varphi} = \mu, \end{cases}$$

where μ is the (scalar) angular momentum of $x(t)$, cf. [6]. Recall that μ is constant in time along any solution. In the following, when considering a solution of (S), we will always implicitly assume that $\rho > 0$.

Let us define the modified function $\tilde{h} : \mathbb{R} \times]0, +\infty[\to \mathbb{R}$ as follows:

$$\tilde{h}(t,r) = \begin{cases} h(t,r), & \text{if } r \geqslant 1, \\ (2r-1)h(t,r), & \text{if } \frac{1}{2} \leqslant r \leqslant 1, \\ 0, & \text{if } 0 < r \leqslant \frac{1}{2}. \end{cases}$$

This function still satisfies the L^1-Carathéodory conditions, as well as assumptions (H1) and (H2). Correspondingly, let us consider the modified equation

$$\ddot{x} = \big(-\tilde{h}\big(t, |x|\big) + e(t)\big)\frac{x}{|x|}, \tag{10}$$

which is equivalent to the system

$$(\tilde{\mathrm{S}}) \qquad \begin{cases} \ddot{\rho} - \dfrac{\mu^2}{\rho^3} + \tilde{h}(t,\rho) = e(t), \\ \rho^2\dot{\varphi} = \mu. \end{cases}$$

We will eventually find solutions of (10) which do not enter the circle of radius 1, so that along these solutions the functions h and $\tilde{h}$ coincide.

We begin by showing that, for μ large enough, the first equation in $(\tilde{\mathrm{S}})$ has a T-periodic solution. To this aim, we use degree theory. For $\lambda \in [0,1]$, we define

$$h_\lambda(t,r) = \lambda\tilde{h}(t,r) + \frac{1-\lambda}{r^2}, \qquad e_\lambda(t) = \lambda e(t) + (1-\lambda)\bar{e},$$

and consider the equation

$$\ddot{\rho} - \frac{\mu^2}{\rho^3} + h_\lambda(t,\rho) = e_\lambda(t). \tag{11}$$

Notice that the hypotheses (H1) and (H2) are also satisfied by h_λ and, moreover, the L^1-Carathéodory conditions hold true. In particular,

- for every compact interval $[a,b]$ in $]0,+\infty[$ there exists $\tilde{\ell}_{a,b} \in L^1(0,T)$ such that

$$r \in [a,b] \quad \Rightarrow \quad \big|h_\lambda(t,r)\big| \leqslant \tilde{\ell}_{a,b}(t), \quad \text{for a.e. } t \in [0,T] \text{ and every } \lambda \in [0,1].$$

Lemma 1. *For every $\varGamma > 0$ there exists $\mu(\varGamma) \geqslant 1$ such that, if $\mu \geqslant \mu(\varGamma)$, $\lambda \in [0,1]$, and ρ is a T-periodic solution of* (11)*, then $\|\rho\|_\infty \geqslant \varGamma$.*

Proof. Assume by contradiction that there are $\varGamma > 0$ and sequences $(\lambda_n)_n$, $(\mu_n)_n$, and $(\rho_n)_n$ such that $\lambda_n \in [0,1]$, $\lim_n \mu_n = +\infty$, and ρ_n is a T-periodic solution of (11) for $\lambda = \lambda_n$ and $\mu = \mu_n$, with $\|\rho_n\|_\infty < \varGamma$. Multiplying in (11) by ρ_n^3 and integrating gives

$$-3\int_0^T \dot{\rho}_n^2\rho_n^2 - \mu_n^2 T + \int_0^T h_{\lambda_n}(t,\rho_n)\rho_n^3 = \int_0^T e_{\lambda_n}\rho_n^3,$$

so that, since $\|e_{\lambda_n}\|_1 \leqslant \|e\|_1$,

$$\int_0^T h_{\lambda_n}(t,\rho_n)\rho_n^3 \geqslant \mu_n^2 T - \|e\|_1\varGamma^3.$$

On the other hand,

$$\int_0^T h_{\lambda_n}(t,\rho_n)\rho_n^3 = \int_{\{0<\rho_n<\frac{1}{2}\}} h_{\lambda_n}(t,\rho_n)\rho_n^3 + \int_{\{\frac{1}{2}\leqslant\rho_n<\Gamma\}} h_{\lambda_n}(t,\rho_n)\rho_n^3 \leqslant \frac{T}{2} + \|\tilde{\ell}_{\frac{1}{2},\Gamma}\|_1 \Gamma^3.$$

By the above, we deduce that

$$\mu_n^2 T \leqslant \frac{T}{2} + \big(\|\tilde{\ell}_{\frac{1}{2},\Gamma}\|_1 + \|e\|_1\big)\Gamma^3,$$

in contradiction with the fact that $\mu_n \to +\infty$. □

Lemma 2. *There exists a constant $\overline{C} \geqslant 0$ such that, if $\mu \geqslant 1$, $\lambda \in [0,1]$, and ρ is a T-periodic solution of* (11)*, then*

$$\min \rho \geqslant \frac{1}{2}\|\rho\|_\infty - \overline{C}.$$

Proof. By (H2), we can fix $\bar{r} > \frac{1}{2}$ such that

$$r \geqslant \bar{r} \quad \Rightarrow \quad \big|h_\lambda(t,r)\big| \leqslant \frac{r}{4T^2}, \quad \text{for a.e. } t \in [0,T] \text{ and every } \lambda \in [0,1].$$

Multiplying in (11) by ρ and integrating we get

$$\int_0^T \dot{\rho}^2 = \int_0^T h_\lambda(t,\rho)\rho - \int_0^T \frac{\mu^2}{\rho^2} - \int_0^T e_\lambda \rho.$$

Writing

$$\int_0^T h_\lambda(t,\rho)\rho = \Bigg(\int_{\{0<\rho<\frac{1}{2}\}} + \int_{\{\frac{1}{2}\leqslant\rho\leqslant\bar{r}\}} + \int_{\{\rho>\bar{r}\}}\Bigg) h_\lambda(t,\rho)\rho,$$

and estimating each term, we deduce that, if $\mu \geqslant 1$,

$$\begin{aligned}\|\dot{\rho}\|_2^2 &\leqslant \int_{\{0<\rho<\frac{1}{2}\}} \left(\frac{1}{\rho} - \frac{\mu^2}{\rho^2}\right) + \|\tilde{\ell}_{\frac{1}{2},\bar{r}}\|_1 \|\rho\|_\infty + \frac{1}{4T}\|\rho\|_\infty^2 + \|e_\lambda\|_1 \|\rho\|_\infty \\ &\leqslant \frac{1}{4T}\|\rho\|_\infty^2 + \big(\|\tilde{\ell}_{\frac{1}{2},\bar{r}}\|_1 + \|e\|_1\big)\|\rho\|_\infty,\end{aligned}$$

where we have used the fact that $\|e_\lambda\|_1 \leqslant \|e\|_1$, for every $\lambda \in [0,1]$.

Setting

$$\overline{C} := \big(\|\tilde{\ell}_{\frac{1}{2},\bar{r}}\|_1 + \|e\|_1\big)T,$$

we have

$$T\|\dot\rho\|_2^2 \leqslant \frac{1}{4}\big(\|\rho\|_\infty^2 + 4\overline{C}\|\rho\|_\infty\big) \leqslant \frac{1}{4}\big(\|\rho\|_\infty + 2\overline{C}\big)^2.$$

So,

$$\max\rho - \min\rho \leqslant \sqrt{T}\|\dot\rho\|_2 \leqslant \frac{1}{2}\|\rho\|_\infty + \overline{C},$$

thus proving the lemma. □

Let us now fix $\overline{\Gamma} > 2(1+\overline{C})$, where $\overline{C}$ is given by Lemma 2. Correspondingly, let $\overline{\mu} := \mu(\overline{\Gamma})$, with $\mu(\overline{\Gamma}) \geqslant 1$ as in Lemma 1.

Lemma 3. *Given A, B, with $\overline{\mu} \leqslant A \leqslant B$, there is a constant $C > 1$ such that, if $\mu \in [A, B]$, $\lambda \in [0, 1]$, and ρ is a T-periodic solution of* (11)*, then*

$$1 < \rho(t) < C, \qquad \big|\dot\rho(t)\big| < C,$$

for every $t \in [0, T]$.

Proof. By the choice of $\overline{\mu}$ and Lemma 1, it has to be $\|\rho\|_\infty \geqslant \overline{\Gamma}$, and by Lemma 2,

$$\min\rho \geqslant \frac{1}{2}\|\rho\|_\infty - \overline{C} \geqslant \frac{1}{2}\overline{\Gamma} - \overline{C} > 1.$$

Using (H1), let us choose $\tilde r > 0$ such that

$$r \geqslant \tilde r \quad\Rightarrow\quad r^3 h_\lambda(t, r) \geqslant 2B^2, \quad \text{for a.e. } t \in [0, T] \text{ and every } \lambda \in [0, 1].$$

Let us first show that $\min\rho < \tilde r$. Integrating (11), using (H0) we have

$$\int_0^T \frac{\mu^2}{\rho^3} = \int_0^T h_\lambda(t, \rho) - T\bar e \geqslant \int_0^T h_\lambda(t, \rho).$$

Arguing by contradiction, if $\min\rho \geqslant \tilde r$, the previous inequality would give

$$\int_0^T \frac{\mu^2}{\rho^3} \geqslant \int_0^T \frac{2B^2}{\rho^3},$$

which is impossible, since $\mu \in [A, B]$.

Having proved that $\min\rho < \tilde r$, by Lemma 2 we deduce that

$$\|\rho\|_\infty \leqslant 2(\min\rho + \overline{C}) < 2(\tilde r + \overline{C}).$$

Hence, setting $b = 2(\tilde{r} + \overline{C}\,)$, from Eq. (11) and the Carathéodory conditions, since $\dot{\rho}(t)$ vanishes somewhere,

$$\|\dot{\rho}\|_\infty \leqslant \|\ddot{\rho}\|_1 \leqslant \int_0^T \frac{\mu^2}{\rho^3} + \int_0^T \bigl|h_\lambda(t,\rho)\bigr| + \|e_\lambda\|_1 < B^2 T + \|\tilde{\ell}_{1,b}\|_1 + \|e\|_1 := b'.$$

Defining $C = \max\{b, b'\}$, the proof is completed. □

As a consequence of Lemma 3, if $\mu \geqslant \overline{\mu}$ and $\rho(t)$ is a T-periodic solution of

$$\ddot{\rho} - \frac{\mu^2}{\rho^3} + \tilde{h}(t,\rho) = e(t), \tag{12}$$

which is the first equation in ($\tilde{\mathrm{S}}$), then $\rho(t)$ also satisfies the first equation in (S), i.e.

$$\ddot{\rho} - \frac{\mu^2}{\rho^3} + h(t,\rho) = e(t). \tag{13}$$

The following lemma gives us an important information concerning this equation. Let us denote by C^1_T the set of T-periodic C^1-functions, with the usual norm of $C^1([0,T])$.

Lemma 4. *Given A, B, with $\overline{\mu} \leqslant A \leqslant B$, there is a continuum $\mathcal{C}_{A,B}$ in $[A,B] \times C^1_T$, connecting $\{A\} \times C^1_T$ with $\{B\} \times C^1_T$, whose elements (μ,ρ) are solutions of both Eqs.* (12) *and* (13)*.*

Proof. By the above remark, it is enough to prove the lemma for the modified equation (12). In order to apply degree theory, let us define the following operators:

$$L : D(L) \subset C^1\bigl([0,T]\bigr) \to L^1(0,T),$$
$$D(L) = \bigl\{\rho \in W^{2,1}(0,T):\ \rho(0) = \rho(T),\ \dot{\rho}(0) = \dot{\rho}(T)\bigr\},$$
$$L\rho = \ddot{\rho} - \rho,$$

and

$$N : [A,B] \times C^1\bigl([0,T]\bigr) \to L^1(0,T),$$
$$N(\mu,\rho)(t) = \frac{\mu^2}{\rho^3(t)} - \tilde{h}\bigl(t,\rho(t)\bigr) + e(t) - \rho(t).$$

The T-periodic problem for Eq. (12) is thus equivalent to

$$L\rho = N(\mu,\rho).$$

Since L is invertible, we can write equivalently

$$\rho - L^{-1}N(\mu,\rho) = 0. \tag{14}$$

Let $C > 0$ be the constant given by Lemma 3 and define G to be the following open subset of $C^1([0, T])$:

$$G = \{\rho \in C^1([0, T]):\ 1 < \rho(t) < C \text{ and } |\dot\rho(t)| < C,\ \text{for every } t \in [0, T]\}.$$

By Lemma 3, Eq. (14) has no solutions (μ, ρ) on $[A, B] \times \partial G$. Since $L^{-1}N(\mu, \cdot)$ is a compact operator, by the global continuation principle of Leray–Schauder (see, e.g., [32, Theorem 14.C]), the lemma will be proved if we show that the degree is nonzero for some $\mu \in [A, B]$.

In order to compute the degree, we consider Eq. (11). By Lemma 3, the degree has to be the same for every $\lambda \in [0, 1]$. Let us then take $\lambda = 0$, so that (11) becomes

$$\ddot\rho - \frac{\mu^2}{\rho^3} + \frac{1}{\rho^2} = \bar e.$$

Define the function

$$F : [1, C] \times [-C, C] \to \mathbb{R}^2,$$
$$F(u, v) = \left(v, \frac{\mu^2}{u^3} - \frac{1}{u^2} + \bar e\right).$$

By a result of Capietto, Mawhin and Zanolin [9, Theorem 1], one can compute the Leray–Schauder degree of $I - L^{-1}N(\mu, \cdot)$ as the Brouwer degree of F:

$$d_{\mathrm{LS}}(I - L^{-1}N(\mu, \cdot), G) = d_{\mathrm{B}}(F,]1, C[\times]{-C}, C[).$$

Since F has a unique zero (u_0, v_0), in the set $]1, C[\times]{-C}, C[$, and the Jacobian matrix $J_F(u_0, v_0)$ has a positive determinant, we conclude that the degree has to be equal to 1. □

We have thus showed that, for every $\mu \geqslant \overline{\mu}$, Eq. (13) has a T-periodic solution. Let us mention that, following [23], many authors have considered scalar equations with a singularity, like (13). A similar existence result was proved, e.g., in [8], via lower and upper solutions techniques. See also, among others, [16,17,22,30].

We can deduce from Lemma 4 that there is a connected set $\mathcal{C}$, contained in $[\overline{\mu}, +\infty[\times C^1_T$, which connects $\{\overline{\mu}\} \times C^1_T$ with $\{\mu^*\} \times C^1_T$, for every $\mu^* > \overline{\mu}$, whose elements (μ, ρ) are solutions of both Eqs. (12) and (13).

Lemma 5. *For every $\varepsilon > 0$, there exists $\mu_\varepsilon \geqslant \overline{\mu}$ such that, if $(\mu, \rho) \in \mathcal{C}$ with $\mu \geqslant \mu_\varepsilon$, then*

$$\int_0^T \frac{\mu}{\rho^2(t)}\, dt \leqslant \varepsilon.$$

Proof. Given $\varepsilon > 0$, set

$$\varepsilon' := \left(\frac{\varepsilon}{36T}\right)^2.$$

Let $\overline{C}$ be as in Lemma 2 and recall that $\bar{e} \leqslant 0$. By (H2), there exists $r' \geqslant \max\{-2\bar{e}/\varepsilon', 3\overline{C}\}$ such that

$$r \geqslant r' \quad \Rightarrow \quad \left|\tilde{h}(t,r)\right| \leqslant \frac{\varepsilon'}{2} r, \quad \text{for a.e. } t \in [0,T].$$

For $\Gamma := 2(r' + \overline{C})$, let $\mu(\Gamma)$ be as in Lemma 1. Set $\mu_\varepsilon := \max\{\mu(\Gamma), \overline{\mu}\}$.

Let (μ,ρ) be an element of $\mathcal{C}$, with $\mu \geqslant \mu_\varepsilon$. By Lemma 1, $\|\rho\|_\infty \geqslant \Gamma$, and by Lemma 2, $\rho(t) \geqslant \frac{1}{2}\|\rho\|_\infty - \overline{C} \geqslant r'$, for every $t \in \mathbb{R}$. Hence,

$$\frac{1}{T}\int_0^T \tilde{h}(t,\rho) \leqslant \frac{1}{T}\int_0^T \frac{\varepsilon'}{2}\rho \leqslant \frac{\varepsilon'}{2}\|\rho\|_\infty.$$

Integrating in (12), we have

$$\frac{1}{T}\int_0^T \frac{\mu^2}{\rho^3} = \frac{1}{T}\int_0^T \tilde{h}(t,\rho) - \bar{e} \leqslant \frac{\varepsilon'}{2}\|\rho\|_\infty - \bar{e}.$$

On the other hand,

$$\frac{1}{T}\int_0^T \frac{\mu^2}{\rho^3} \geqslant \frac{\mu^2}{\|\rho\|_\infty^3},$$

so that

$$\frac{\mu^2}{\|\rho\|_\infty^4} \leqslant \frac{\varepsilon'}{2} - \frac{\bar{e}}{\|\rho\|_\infty} \leqslant \frac{\varepsilon'}{2} - \frac{\bar{e}}{r'} \leqslant \varepsilon'.$$

Then, using again Lemma 2,

$$\begin{aligned}\int_0^T \frac{\mu}{\rho^2(t)}\,dt &\leqslant T\frac{\mu}{(\min\rho)^2} \leqslant T\frac{4\mu}{(\|\rho\|_\infty - 2\overline{C})^2} \\ &= T\frac{\mu}{\|\rho\|_\infty^2}\left(\frac{2\|\rho\|_\infty}{\|\rho\|_\infty - 2\overline{C}}\right)^2 \leqslant T\sqrt{\varepsilon'}\left(\frac{2r'}{r' - 2\overline{C}}\right)^2 \\ &\leqslant T\sqrt{\varepsilon'}\left(\frac{6\overline{C}}{3\overline{C} - 2\overline{C}}\right)^2 = 36T\sqrt{\varepsilon'} = \varepsilon,\end{aligned}$$

thus proving the lemma. □

Since the function

$$(\mu,\rho)\mapsto\int_0^T\frac{\mu}{\rho^2(t)}\,dt$$

is continuous from $\mathcal{C}$ to $\mathbb{R}$, and $\mathcal{C}$ is connected, its image is an interval. By Lemmas 4 and 5 this interval is of the type $]0,\overline{\theta}]$ for some $\overline{\theta}>0$.

Lemma 6. *For every $\theta\in\,]0,\overline{\theta}]$, there are (μ,ρ,φ), verifying system* (S)*, for which $(\mu,\rho)\in\mathcal{C}$, and*

$$\rho(t+T)=\rho(t),\qquad \varphi(t+T)=\varphi(t)+\theta,$$

for every $t\in\mathbb{R}$.

Proof. Given $\theta\in\,]0,\overline{\theta}]$, there are $(\mu,\rho)\in\mathcal{C}$ such that

$$\int_0^T\frac{\mu}{\rho^2(t)}\,dt=\theta.$$

In particular, the first equation in (S) is satisfied and ρ is T-periodic. Moreover, defining

$$\varphi(t)=\int_0^t\frac{\mu}{\rho^2(s)}\,ds,$$

the second equation in (S) is also satisfied and

$$\varphi(t+T)-\varphi(t)=\int_t^{t+T}\frac{\mu}{\rho^2(s)}\,ds=\int_0^T\frac{\mu}{\rho^2(s)}\,ds=\theta.\qquad\square$$

We are going to complete now the proof of Theorem 4. For every $\theta\in\,]0,\overline{\theta}]$, the solution of system (S) found in Lemma 6 provides, through (9), a solution to Eq. (8) such that

$$x(t+T)=e^{i\theta}x(t),$$

for every $t\in\mathbb{R}$.

In particular, if $\theta=\frac{2\pi}{k}$ for some integer $k\geqslant 1$, then $x(t)$ is periodic with minimal period kT, and rotates exactly once around the origin in the period time kT. Hence, for every integer $k\geqslant 2\pi/\overline{\theta}$, we have such a kT-periodic solution, which we denote by $x_k(t)$. Let $(\rho_k(t),\varphi_k(t))$ be its polar coordinates, and μ_k be its angular momentum. By the above construction, $(\mu_k, \rho_k, \varphi_k)$ verify system (S), $(\mu_k, \rho_k) \in \mathcal{C}$, and

$$\int_0^T \frac{\mu_k}{\rho_k^2(t)}\, dt = \frac{2\pi}{k}. \tag{15}$$

We claim that

$$\lim_{k\to\infty} \mu_k = +\infty. \tag{16}$$

Indeed, if $(\mu_{k_j})_j$ were a bounded subsequence, with $\mu_{k_j} \in [\overline{\mu}, B]$ for some B, using Lemma 3 with $\lambda = 1$, there would be a constant $C > 0$ such that $\|\rho_{k_j}\|_\infty < C$, and hence

$$\int_0^T \frac{\mu_{k_j}}{\rho_{k_j}^2(t)}\, dt > \frac{\overline{\mu} T}{C^2},$$

for every j, in contradiction with (15).

By (16) and Lemma 1, with $\lambda = 1$, we have

$$\lim_{k\to\infty} \|\rho_k\|_\infty = +\infty.$$

By Lemma 2, again with $\lambda = 1$,

$$\lim_{k\to\infty} (\min \rho_k) = +\infty. \tag{17}$$

Using (H2), for any fixed $\varepsilon > 0$ there is $r_\varepsilon > 0$ such that

$$r \geqslant r_\varepsilon \quad \Rightarrow \quad \big|h(t,r)\big| \leqslant \varepsilon r, \quad \text{for a.e. } t \in [0,T].$$

By (17), there is $k_\varepsilon > 0$ such that

$$k \geqslant k_\varepsilon \quad \Rightarrow \quad \rho_k(t) \geqslant \max\left\{r_\varepsilon, \frac{\|e\|_1}{\varepsilon T}\right\},$$

for every t, so that, from the first equation in (S),

$$\begin{aligned}
\|\dot\rho_k\|_2^2 &= \int_0^T \left(-\frac{\mu^2}{\rho_k^2} + h(t,\rho_k)\rho_k\right) - \int_0^T e\rho_k \\
&\leqslant \int_0^T h(t,\rho_k)\rho_k - \int_0^T e\rho_k \\
&\leqslant \varepsilon T \|\rho_k\|_\infty^2 + \|e\|_1 \|\rho_k\|_\infty \\
&\leqslant 2\varepsilon T \|\rho_k\|_\infty^2.
\end{aligned}$$

Hence,

$$\frac{\max \rho_k - \min \rho_k}{\max \rho_k} \leqslant \frac{\sqrt{T}\,\|\dot{\rho}_k\|_2}{\|\rho_k\|_\infty} \leqslant \sqrt{2\varepsilon}T.$$

The above proves that

$$\lim_{k\to\infty} \frac{\max \rho_k - \min \rho_k}{\max \rho_k} = 0, \tag{18}$$

thus concluding the proof of Theorem 4.

3. Applications to Keplerian-like systems

In this section we study more in detail equations like (3) and (4). To this aim, let us state the following direct consequence of Theorem 4.

Corollary 3. *Assume* (H0)*, and that there is* $\gamma \in \,]0,4[$ *such that*

(H4) $$0 < c_1 \leqslant \liminf_{r\to+\infty} r^{\gamma-1}h(t,r) \leqslant \limsup_{r\to+\infty} r^{\gamma-1}h(t,r) \leqslant c_2$$

uniformly for almost every t. Then, there exists $k_1 \geqslant 1$ such that, for every integer $k \geqslant k_1$, Eq. (8) *has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover,*

$$\lim_{k\to\infty} \big(\min|x_k|\big) = +\infty$$

and

$$\lim_{k\to\infty} \frac{\min|x_k|}{\max|x_k|} = 1.$$

Notice that Corollaries 1 and 2 follow directly from Corollary 3.

Our aim will be now to give more precise estimates on the amplitude of the solutions having large minimal period, and on their radial and tangential speed. The radial and tangential velocities of $x(t)$ are, as well known, defined by

$$\dot{x}^{\mathrm{rad}} = \frac{\dot{x}\cdot x}{|x|^2}x, \qquad \dot{x}^{\tan} = \dot{x} - \dot{x}^{\mathrm{rad}}.$$

Recall that, passing to polar coordinates, Eq. (8) is equivalent to (S).

Lemma 7. *Assume* (H0) *and* (H4)*, with $0 < \gamma < 4$. For every $\varepsilon \in \,]0,1[$ there is $\mu_\varepsilon > 0$ such that, if $\mu \geqslant \mu_\varepsilon$ and ρ is a T-periodic solution of the first equation in* (S)*, then, for every t,*

- *when, either* $\bar{e} < 0$ *and* $0 < \gamma < 1$*, or* $\bar{e} = 0$*, one has*

$$(1-\varepsilon)\left(\frac{\mu^2}{c_2}\right)^{\frac{1}{4-\gamma}} \leqslant \rho(t) \leqslant (1+\varepsilon)\left(\frac{\mu^2}{c_1}\right)^{\frac{1}{4-\gamma}};$$

- *when* $\bar{e} \leqslant 0$ *and* $\gamma = 1$*, it is*

$$(1-\varepsilon)\left(\frac{\mu^2}{c_2-\bar{e}}\right)^{\frac{1}{3}} \leqslant \rho(t) \leqslant (1+\varepsilon)\left(\frac{\mu^2}{c_1-\bar{e}}\right)^{\frac{1}{3}};$$

- *when* $\bar{e} < 0$ *and* $1 < \gamma < 4$*,*

$$(1-\varepsilon)\left(\frac{\mu^2}{-\bar{e}}\right)^{\frac{1}{3}} \leqslant \rho(t) \leqslant (1+\varepsilon)\left(\frac{\mu^2}{-\bar{e}}\right)^{\frac{1}{3}};$$

- *finally, when* $\bar{e} \leqslant 0$ *and* $1 \leqslant \gamma < 4$*, we have*

$$\left|\dot{\rho}(t)\right| \leqslant \|e\|_1 + \varepsilon. \tag{19}$$

Proof. Let c_1' and c_2' be such that

$$0 < c_1' < c_1 \leqslant c_2 < c_2'. \tag{20}$$

By (H4), there is r_* such that

$$r \geqslant r_* \quad \Rightarrow \quad \frac{c_1'}{r^{\gamma-1}} \leqslant h(t,r) \leqslant \frac{c_2'}{r^{\gamma-1}}, \quad \text{for a.e. } t \in [0,T]. \tag{21}$$

From Lemmas 1 and 2 we know that, for sufficiently large μ, it has to be $\rho(t) \geqslant r_*$, for every t. Let us consider the function $g_{\mu,c} :]0,+\infty[\to \mathbb{R}$ defined by

$$g_{\mu,c}(r) = -\frac{\mu^2}{r^3} + \frac{c}{r^{\gamma-1}}. \tag{22}$$

Note that the equation $g_{\mu,c}(r) = \bar{e}$ has a unique solution, which we denote by $\bar{r}_{\mu,c}$:

$$g_{\mu,c}(\bar{r}_{\mu,c}) = \bar{e}.$$

Integrating in (13) and using (21), we get

$$\frac{1}{T}\int_0^T \left(-\frac{\mu^2}{\rho^3} + \frac{c_1'}{\rho^{\gamma-1}}\right) \leqslant \bar{e} \leqslant \frac{1}{T}\int_0^T \left(-\frac{\mu^2}{\rho^3} + \frac{c_2'}{\rho^{\gamma-1}}\right).$$

Hence, there exist $\bar{t}_1, \bar{t}_2 \in [0,T]$ such that $\rho(\bar{t}_1) \leqslant \bar{r}_{\mu,c_1'}$ and $\rho(\bar{t}_2) \geqslant \bar{r}_{\mu,c_2'}$. As $\bar{r}_{\mu,c_2'} \leqslant \bar{r}_{\mu,c_1'}$, we have that there exists $\bar{t} \in [0,T]$ such that

$$\rho(\bar{t}) \in [\bar{r}_{\mu,c_2'}, \bar{r}_{\mu,c_1'}]. \tag{23}$$

Let us now give some estimates on $\bar{r}_{\mu,c}$. It is easy to see that

$$\lim_{\mu\to+\infty} \bar{r}_{\mu,c} = +\infty. \tag{24}$$

If $\bar{e} = 0$, we immediately have

$$\bar{r}_{\mu,c} = \left(\frac{\mu^2}{c}\right)^{\frac{1}{4-\gamma}}.$$

If $\bar{e} < 0$ and $0 < \gamma < 1$, we do not have the equality, but nevertheless, using (24),

$$\lim_{\mu\to+\infty} \frac{\bar{r}_{\mu,c}}{(\frac{\mu^2}{c})^{\frac{1}{4-\gamma}}} = 1.$$

If $\bar{e} \leqslant 0$ and $\gamma = 1$, it is easy to compute

$$\bar{r}_{\mu,c} = \left(\frac{\mu^2}{c-\bar{e}}\right)^{\frac{1}{3}}.$$

Finally, if $\bar{e} < 0$ and $1 < \gamma < 4$, let us fix $\varepsilon' \in]0,1[$. Since

$$g_{\mu,c}\left((1-\varepsilon')\left(\frac{\mu^2}{-\bar{e}}\right)^{\frac{1}{3}}\right) = \frac{\bar{e}}{(1-\varepsilon')^3} + \frac{c(-\bar{e})^{\frac{\gamma-1}{3}}}{(1-\varepsilon')^{\gamma-1}\mu^{\frac{2(\gamma-1)}{3}}},$$

$$g_{\mu,c}\left((1+\varepsilon')\left(\frac{\mu^2}{-\bar{e}}\right)^{\frac{1}{3}}\right) = \frac{\bar{e}}{(1+\varepsilon')^3} + \frac{c(-\bar{e})^{\frac{\gamma-1}{3}}}{(1+\varepsilon')^{\gamma-1}\mu^{\frac{2(\gamma-1)}{3}}},$$

for μ sufficiently large we have

$$g_{\mu,c}\left((1-\varepsilon')\left(\frac{\mu^2}{-\bar{e}}\right)^{\frac{1}{3}}\right) < \bar{e} < g_{\mu,c}\left((1+\varepsilon')\left(\frac{\mu^2}{-\bar{e}}\right)^{\frac{1}{3}}\right),$$

hence

$$(1-\varepsilon')\left(\frac{\mu^2}{-\bar{e}}\right)^{\frac{1}{3}} < \bar{r}_{\mu,c} < (1+\varepsilon')\left(\frac{\mu^2}{-\bar{e}}\right)^{\frac{1}{3}}.$$

The above proves that, in this case,

$$\lim_{\mu\to+\infty} \frac{\bar{r}_{\mu,c}}{(\frac{\mu^2}{-\bar{e}})^{\frac{1}{3}}} = 1.$$

As already seen in the previous section, if μ is sufficiently large, we have

$$\left|\frac{\min\rho}{\max\rho} - 1\right| < \varepsilon.$$

Hence,

$$(1-\varepsilon)\rho(\bar{t}) \leqslant \rho(t) \leqslant (1+\varepsilon)\rho(\bar{t}),$$

for every t, and the above estimates on $\rho(\bar{t})$, together with the arbitrary choice of c_1', c_2' satisfying (20), lead to all the claimed estimates for $\rho(t)$.

Notice that, if $1 \leqslant \gamma < 4$, the function $g_{\mu,c}$ defined in the proof of Lemma 7 is bounded above:

$$g_{\mu,c}(r) \leqslant (4-\gamma)\left(\frac{c}{3}\right)^{\frac{3}{4-\gamma}}\left(\frac{\gamma-1}{\mu^2}\right)^{\frac{\gamma-1}{4-\gamma}} := M_{\mu,c},$$

and

$$\lim_{\mu\to+\infty} M_{\mu,c} = 0.$$

Hence, for μ sufficiently large, we have

$$\ddot{\rho}(t) \geqslant -g_{\mu,c_2'}\big(\rho(t)\big) + e(t) \geqslant -M_{\mu,c_2'} + e(t) \geqslant -\frac{1}{T}\varepsilon + e(t).$$

So, for every t_1, t_2 in $[0, T]$,

$$\dot{\rho}(t_2) - \dot{\rho}(t_1) = \int_{t_1}^{t_2} \ddot{\rho}(t)\,dt \geqslant -\varepsilon - \|e\|_1,$$

from which we deduce, since $\dot{\rho}$ vanishes somewhere, that

$$\big|\dot{\rho}(t)\big| \leqslant \varepsilon + \|e\|_1,$$

for every t. □

We will now concentrate on the case $\bar{e} = 0$.

Theorem 5. *Assume* (H4)*, with* $0 < \gamma < 4$*. If* $\bar{e} = 0$*, then the solutions* $x_k(t)$ *of Eq.* (8) *obtained in Corollary* 3 *satisfy the following estimates*:

$$\big(c_1^{\frac{4}{\gamma}} c_2^{-1}\big)^{\frac{1}{4-\gamma}} \leqslant \liminf_{k\to\infty} \frac{|x_k(t)|}{(\frac{kT}{2\pi})^{\frac{2}{\gamma}}} \leqslant \limsup_{k\to\infty} \frac{|x_k(t)|}{(\frac{kT}{2\pi})^{\frac{2}{\gamma}}} \leqslant \big(c_2^{\frac{4}{\gamma}} c_1^{-1}\big)^{\frac{1}{4-\gamma}}, \tag{25}$$

$$\big(c_1^{8-\gamma} c_2^{-4}\big)^{\frac{1}{\gamma(4-\gamma)}} \leqslant \liminf_{k\to\infty} \frac{|\dot{x}_k^{\tan}(t)|}{(\frac{kT}{2\pi})^{\frac{2-\gamma}{\gamma}}} \leqslant \limsup_{k\to\infty} \frac{|\dot{x}_k^{\tan}(t)|}{(\frac{kT}{2\pi})^{\frac{2-\gamma}{\gamma}}} \leqslant \big(c_2^{8-\gamma} c_1^{-4}\big)^{\frac{1}{\gamma(4-\gamma)}}, \tag{26}$$

and moreover, if $1 \leqslant \gamma < 4$*,*

$$\limsup_{k\to\infty}\big|\dot{x}_k^{\mathrm{rad}}(t)\big| \leqslant \|e\|_1 \tag{27}$$

uniformly in t.

Proof. Let μ_k, $\rho_k(t)$ and $\varphi_k(t)$ be the angular momentum and the polar coordinates corresponding to $x_k(t)$, respectively. Fix $\varepsilon \in]0,1[$. Using Lemma 7 and (16), for k sufficiently large we can write

$$(1-\varepsilon)\left(\frac{\mu_k^2}{c_2}\right)^{\frac{1}{4-\gamma}} \leqslant \rho_k(t) \leqslant (1+\varepsilon)\left(\frac{\mu_k^2}{c_1}\right)^{\frac{1}{4-\gamma}}, \tag{28}$$

for every t. As

$$\frac{1}{T}\int_0^T \frac{\mu_k}{\rho_k^2(t)}\,dt = \frac{2\pi}{kT},$$

for k sufficiently large, we have

$$(1+\varepsilon)^{-2}\mu_k\left(\frac{\mu_k^2}{c_1}\right)^{\frac{-2}{4-\gamma}} \leqslant \frac{2\pi}{kT} \leqslant (1-\varepsilon)^{-2}\mu_k\left(\frac{\mu_k^2}{c_2}\right)^{\frac{-2}{4-\gamma}},$$

and hence,

$$c_1^{\frac{2}{\gamma}}\left(\frac{kT}{2\pi}(1+\varepsilon)^{-2}\right)^{\frac{4-\gamma}{\gamma}} \leqslant \mu_k \leqslant c_2^{\frac{2}{\gamma}}\left(\frac{kT}{2\pi}(1-\varepsilon)^{-2}\right)^{\frac{4-\gamma}{\gamma}}.$$

This estimate together with (28) leads to

$$\frac{1-\varepsilon}{(1+\varepsilon)^{\frac{4}{\gamma}}}\left(c_1^{\frac{4}{\gamma}}c_2^{-1}\right)^{\frac{1}{4-\gamma}}\left(\frac{kT}{2\pi}\right)^{\frac{2}{\gamma}} \leqslant \rho_k(t) \leqslant \frac{1+\varepsilon}{(1-\varepsilon)^{\frac{4}{\gamma}}}\left(c_2^{\frac{4}{\gamma}}c_1^{-1}\right)^{\frac{1}{4-\gamma}}\left(\frac{kT}{2\pi}\right)^{\frac{2}{\gamma}}.$$

Since ε is arbitrary, the proof of (25) is thus concluded.

To prove (26) note that

$$\left|\dot{x}_k^{\tan}(t)\right| = \rho_k(t)\dot{\varphi}_k(t) = \frac{\mu_k}{\rho_k(t)}.$$

Using the above estimates, we obtain

$$\frac{(1-\varepsilon)^{\frac{4}{\gamma}}}{(1+\varepsilon)^{\frac{8-\gamma}{\gamma}}}\left(\frac{c_1^{8-\gamma}}{c_2^4}\right)^{\frac{1}{\gamma(4-\gamma)}}\left(\frac{kT}{2\pi}\right)^{\frac{2-\gamma}{\gamma}} \leqslant \frac{\mu_k}{\rho_k(t)} \leqslant \frac{(1+\varepsilon)^{\frac{4}{\gamma}}}{(1-\varepsilon)^{\frac{8-\gamma}{\gamma}}}\left(\frac{c_2^{8-\gamma}}{c_1^4}\right)^{\frac{1}{\gamma(4-\gamma)}}\left(\frac{kT}{2\pi}\right)^{\frac{2-\gamma}{\gamma}}.$$

Using again the arbitrariness of ε, (26) is proved, as well.

The proof of (27) follows directly from (19). □

Let us state the corresponding results for Eqs. (3) and (4).

Corollary 4. *Assume* (H3)*,* $0<\gamma<4$*, and* $\bar{e}=0$*. Then, the solutions* $x_k(t)$ *of Eq.* (3) *obtained in Corollary* 1 *satisfy the same estimates as in Theorem* 5.

Corollary 5. *Assume* (H3), $d \in L^\infty(\mathbb{R})$ *and* $\bar{e} = 0$. *Then, the solutions* $x_k(t)$ *of Eq.* (4) *obtained in Corollary* 2 *satisfy the same estimates as in Theorem* 5, *with* $\gamma = 3$.

Remark 1. Notice that, when $\bar{e} = 0$, the estimates in Theorem 5 imply that

$$\lim_{k\to\infty} \left|\dot{x}_k^{\tan}(t)\right| = +\infty, \quad \text{if } 0 < \gamma < 2,$$

$$\lim_{k\to\infty} \left|\dot{x}_k^{\tan}(t)\right| = 0, \quad \text{if } 2 < \gamma < 4,$$

while $|\dot{x}_k^{\tan}(t)|$ remains bounded, for large k, if $\gamma = 2$.

It is also interesting to write the following particular case.

Corollary 6. *If* $\bar{e} = 0$ *and there are* $\gamma \in\,]0, 4[$ *and a constant* $c > 0$ *such that*

$$\text{(H5)} \qquad \lim_{r\to+\infty} r^{\gamma-1} h(t, r) = c$$

uniformly for almost every t, *then the solutions* $x_k(t)$ *of Eq.* (8) *obtained in Corollary* 3 *satisfy the following estimates*:

$$\lim_{k\to\infty} \frac{|x_k(t)|}{(\frac{kT}{2\pi})^{\frac{2}{\gamma}}} = c^{\frac{1}{\gamma}},$$

$$\lim_{k\to\infty} \frac{|\dot{x}_k^{\tan}(t)|}{(\frac{kT}{2\pi})^{\frac{2-\gamma}{\gamma}}} = c^{\frac{1}{\gamma}},$$

and moreover, if $1 \leqslant \gamma < 4$,

$$\limsup_{k\to\infty} \left|\dot{x}_k^{\mathrm{rad}}(t)\right| \leqslant \|e\|_1$$

uniformly in t.

Proof. It is sufficient to notice that (H5) corresponds to (H4), with $c_1 = c_2$, and to apply Theorem 5. □

As an immediate consequence, we have the following.

Corollary 7. *Assume* $\bar{e} = 0$, $\gamma \in\,]0, 4[$ *and that* $c(t)$ *is constant with a positive value* c. *Then, the solutions* $x_k(t)$ *of Eq.* (3) *obtained in Corollary* 1 *satisfy the same estimates as in Corollary* 6.

Remark 2. When the function c is a positive constant and e is identically 0, the circular solutions of (3) with radius $\rho > 0$ have minimal period

$$\tau_\rho = \frac{2\pi}{\sqrt{c}} \rho^{\frac{\gamma}{2}}$$

and tangential speed

$$v_\rho = \sqrt{c}\rho^{\frac{2-\gamma}{2}}.$$

This situation is reflected by our estimates in Corollary 6. Indeed, the orbits of the large-amplitude solutions we have found are, in a sense, close to be circular, since the quotient between the maximal and minimal radius is close to 1.

We have a similar situation for Eq. (4).

Corollary 8. *Assume $\bar{e} = 0$, $d \in L^\infty(\mathbb{R})$ and that $c(t)$ is constant with a positive value c. Then, the solutions $x_k(t)$ of Eq.* (4) *obtained in Corollary* 2 *satisfy the same estimates as in Corollary* 6*, with $\gamma = 3$.*

Analogous type of results can be deduced from Lemma 7 when $\bar{e} < 0$, distinguishing the three cases $0 < \gamma < 1$, $\gamma = 1$ and $1 < \gamma < 4$. For briefness, let us only state the following.

Theorem 6. *Assume* (H4)*, with $1 < \gamma < 4$. If $\bar{e} < 0$, then the solutions $x_k(t)$ of Eq.* (8) *obtained in Corollary* 3 *satisfy the following estimates*:

$$\lim_{k\to\infty} \frac{|x_k(t)|}{k^2} = -\left(\frac{T}{2\pi}\right)^2 \bar{e},$$

$$\lim_{k\to\infty} \frac{|\dot{x}_k^{\tan}(t)|}{k} = -\frac{T}{2\pi}\bar{e},$$

$$\limsup_{k\to\infty} \left|\dot{x}_k^{\mathrm{rad}}(t)\right| \leqslant \|e\|_1$$

uniformly in t.

Remark 3. As in Remark 2, we can compute explicitly the minimal period and the tangential speed of the circular solutions of (3) with radius $\rho > 0$, when the function c is a positive constant and the function e is constant with value $\bar{e} < 0$, thus obtaining

$$\tau_\rho = \frac{2\pi\rho^2}{\sqrt{c\rho^{4-\gamma} - \bar{e}\rho^3}}, \qquad v_\rho = \frac{\sqrt{c\rho^{4-\gamma} - \bar{e}\rho^3}}{\rho}.$$

This situation is well reflected by Theorem 6.

4. Multiplicity results

In this section, we will prove the existence of solutions making a given number of revolutions around the origin. We first consider the general equation (8), and state the following.

Theorem 7. *Let $e(t)$ be continuous, satisfying* (H0)*, with minimal period T. Let $h(t,r)$ be continuous, and assume* (H1) *and*

(H6) $$\lim_{r\to+\infty} h(t,r) = 0$$

uniformly for every t. Then, for every integer $m \geqslant 1$, there exists $k_m \geqslant 1$ such that, for every integer $k \geqslant k_m$, if k and m are relatively prime, Eq. (8) *has a periodic solution $x_{k,m}(t)$ with minimal period kT, which makes exactly m revolutions around the origin in the period time kT. Moreover,*

$$\lim_{k\to\infty}\bigl(\min|x_{k,m}|\bigr) = +\infty$$

and

$$\lim_{k\to\infty}\frac{\min|x_{k,m}|}{\max|x_{k,m}|} = 1.$$

Proof. Following the proof of Theorem 4, if $k \geqslant 2\pi m/\overline{\theta}$, there is a solution $x(t)$ of (8) such that

$$x(t+T) = e^{\frac{2\pi m}{k}i}x(t),$$

for every t. We denote by $x_{k,m}(t)$ such a solution, and by $\rho_{k,m}(t)$, $\varphi_{k,m}(t)$ and $\mu_{k,m}$ its polar coordinates and angular momentum, respectively. Such a solution is kT-periodic and rotates exactly m times around the origin when t varies from 0 to kT. Recall that $\rho_{k,m}(t)$ is T-periodic and, as in (16),

$$\lim_{k\to\infty}\mu_{k,m} = +\infty.$$

Moreover, as in (17) and (18),

$$\lim_{k\to\infty}(\min\rho_{k,m}) = +\infty, \qquad \lim_{k\to\infty}\frac{\min\rho_{k,m}}{\max\rho_{k,m}} = 1.$$

Let us show that, if k is large enough and relatively prime with m, then kT is the minimal period of $x_{k,m}(t)$. If not, $x_{k,m}(t)$ has minimal period $\frac{kT}{m'}$, for some integer $m' \geqslant 2$ which divides m, thus making $\frac{m}{m'}$ revolutions in the period time $\frac{kT}{m'}$. Hence, also $\rho_{k,m}(t)$ has period $\frac{kT}{m'}$. Let

$$k = pm' + q,$$

for some integers p, q with $q < m'$. Then,

$$\rho_{k,m}(t) = \rho_{k,m}\left(t + \frac{kT}{m'}\right) = \rho_{k,m}\left(t + pT + \frac{qT}{m'}\right) = \rho_{k,m}\left(t + \frac{qT}{m'}\right),$$

for every t, thus showing that $\frac{qT}{m'}$ is a period of $\rho_{k,m}(t)$. Since $\frac{qT}{m'} < T$ and $e(t)$ has minimal period T, there is $\tilde{t} \in [0,T]$ such that

$$e\left(\tilde{t} + \frac{qT}{m'}\right) \neq e(\tilde{t}).$$

By (H6), there is $\hat{r} > 0$ such that

$$r \geqslant \hat{r} \quad \Rightarrow \quad \big|h(t,r)\big| \leqslant \frac{1}{3}\left|e\left(\tilde{t} + \frac{qT}{m'}\right) - e(\tilde{t})\right|, \quad \text{for every } t \in [0,T].$$

By the above, if k is large, then $\rho_{k,m}(t) \geqslant \hat{r}$, for every t. From the first equation in (S) we deduce that

$$\begin{aligned} e\left(\tilde{t} + \frac{qT}{m'}\right) - h\left(\tilde{t} + \frac{qT}{m'}, \rho_{k,m}\left(\tilde{t} + \frac{qT}{m'}\right)\right) &= \ddot{\rho}_{k,m}\left(\tilde{t} + \frac{qT}{m'}\right) - \frac{\mu^2}{\rho_{k,m}^3(\tilde{t} + \frac{qT}{m'})} \\ &= \ddot{\rho}_{k,m}(\tilde{t}) - \frac{\mu^2}{\rho_{k,m}^3(\tilde{t})} \\ &= e(\tilde{t}) - h\big(\tilde{t}, \rho_{k,m}(\tilde{t})\big). \end{aligned}$$

Consequently,

$$\begin{aligned} \left|e\left(\tilde{t} + \frac{qT}{m'}\right) - e(\tilde{t})\right| &= \left|h\left(\tilde{t} + \frac{qT}{m'}, \rho_{k,m}\left(\tilde{t} + \frac{qT}{m'}\right)\right) - h\big(\tilde{t}, \rho_{k,m}(\tilde{t})\big)\right| \\ &\leqslant \left|h\left(\tilde{t} + \frac{qT}{m'}, \rho_{k,m}\left(\tilde{t} + \frac{qT}{m'}\right)\right)\right| + \big|h\big(\tilde{t}, \rho_{k,m}(\tilde{t})\big)\big| \\ &\leqslant \frac{2}{3}\left|e\left(\tilde{t} + \frac{qT}{m'}\right) - e(\tilde{t})\right|, \end{aligned}$$

and we have got a contradiction. Hence, $x_{k,m}(t)$ must have minimal period kT. □

Remark 4. The estimates on the solutions made in Section 3, in the case when (H4) is satisfied, can be carried out in the same way for the solutions $x_{k,m}(t)$. Since these solutions make m revolutions around the origin in the time kT, the analogous of Theorems 5, 6 and their corollaries hold, simply replacing 2π by $2m\pi$. For instance, the analogous of (25) is

$$\big(c_1^{\frac{4}{\gamma}} c_2^{-1}\big)^{\frac{1}{4-\gamma}} \leqslant \liminf_{k\to\infty} \frac{|x_{k,m}(t)|}{(\frac{kT}{2m\pi})^{\frac{2}{\gamma}}} \leqslant \limsup_{k\to\infty} \frac{|x_{k,m}(t)|}{(\frac{kT}{2m\pi})^{\frac{2}{\gamma}}} \leqslant \big(c_2^{\frac{4}{\gamma}} c_1^{-1}\big)^{\frac{1}{4-\gamma}}.$$

Let us state now our multiplicity result.

Theorem 8. *Let $e(t)$ be continuous, satisfying* (H0)*, with minimal period T. Let $h(t,r)$ be continuous, and assume* (H1) *and* (H6)*. Then, for every $n \in \mathbb{N}$, there is $k(n) \geqslant 1$ such that, for every prime integer $k \geqslant k(n)$, Eq.* (8) *has at least n geometrically distinct periodic solutions with minimal period kT.*

Proof. Given $n \geqslant 1$, let $p_1, p_2, \ldots, p_n$ be the first n prime numbers. Correspondingly, let $k_{p_1}, k_{p_2}, \ldots, k_{p_n}$ be as in Theorem 7. Define

$$k(n) = \max\{k_{p_1}, k_{p_2}, \ldots, k_{p_n}\}.$$

By Theorem 7, for every prime number $k \geqslant k(n)$, Eq. (8) has periodic solutions $x_{k,p_1}, x_{k,p_2}, \dots, x_{k,p_n}$, with minimal period kT, which make exactly $p_1, p_2, \dots, p_n$ rotations around the origin, respectively, in the period time kT. □

The following corollaries are immediate consequences of the above theorems.

Corollary 9. *Let $c(t)$ and $e(t)$ be continuous, satisfying* (H0), (H3), *and assume* $1 < \gamma < 4$. *If $e(t)$ has minimal period T, then the conclusions of Theorems* 7 *and* 8 *hold for Eq.* (3).

Corollary 10. *Let $c(t)$, $d(t)$ and $e(t)$ be continuous, assume* (H0) *and* (H3). *If $e(t)$ has minimal period T, then the conclusions of Theorems* 7 *and* 8 *hold for Eq.* (4).

Let us concentrate now on Eq. (3). We have considered above the case when $e(t)$ has minimal period T. We consider now the case when $e(t)$ is constant but $c(t)$ has minimal period T.

Theorem 9. *Assume $e(t)$ is constant, with $e(t) \equiv \bar{e} \leqslant 0$, and* $1 < \gamma < 4$. *If $c(t)$ satisfies* (H3) *and has minimal period T, then the same conclusions of Theorems* 7 *and* 8 *hold for Eq.* (3).

Proof. Repeating the proof of Theorem 7, by contradiction we see that $\rho_{k,m}(t)$ has to be $\frac{qT}{m'}$-periodic, and $\frac{qT}{m'} < T$. From the first equation in (S),

$$c(t) = -\rho_{k,m}^{\gamma-1}(t)\ddot{\rho}_{k,m}(t) + \rho_{k,m}^{\gamma-4}(t)\mu^2 - \rho_{k,m}^{\gamma-1}(t)\bar{e},$$

so that $c(t)$ has to be $\frac{qT}{m'}$-periodic, as well, which is a contradiction. □

5. The case $\bar{e} > 0$

In this section we will show that assumption (H0) is essential in order to have large amplitude periodic solutions to Eq. (8). This fact can easily be seen for the circular solutions of Eq. (3), when the functions $c(t)$ and $e(t)$ are constant, with c positive, and $\gamma > 1$. In that case, if $\bar{e} > 0$, the circular solutions are confined in a compact region and their periods tend to infinity as they approach the boundary of that region.

In the general case, we will show that, if $\bar{e} > 0$ and the function $h(t,r)$ tends to zero when r goes to infinity, periodic solutions of arbitrary large amplitude cannot exist.

Theorem 10. *Assume $\bar{e} > 0$, and that* (H6) *holds. Then, every solution of* (8) *which attains a large enough distance from the origin is unbounded. More precisely, there is $R > 0$ such that, if $x(t)$ is a solution of* (8) *with $|x(t_0)| \geqslant R$ for some $t_0 \in \mathbb{R}$, then the orbit of $x(t)$ is unbounded.*

Proof. By (H6), there is $r_1 > 0$ such that

$$r \geqslant r_1 \quad \Rightarrow \quad \bigl|h(t,r)\bigr| \leqslant \frac{\bar{e}}{2}, \quad \text{for a.e. } t \in \mathbb{R}.$$

Choose R such that

$$R > r_1 + \frac{\|e\|_1^2}{\bar{e}} + \frac{5}{2}T\|e\|_1 + \frac{17}{16}T^2\bar{e}. \tag{29}$$

Consider the equivalent system (S), and let $\rho(t_0) = |x(t_0)| \geqslant R$. We first assume $\dot\rho(t_0) \geqslant 0$, and prove by induction the following proposition:

(P_n) For every $t \in [t_0 + nT, t_0 + (n+1)T]$,

$$\dot\rho(t) \geqslant \frac{n-1}{2} T\bar e - \|e\|_1,$$

$$\rho(t) \geqslant R - (n+1)T\|e\|_1 + \frac{n^2 - 3n - 2}{4} T^2 \bar e,$$

and, moreover,

$$\dot\rho\big(t_0 + (n+1)T\big) \geqslant \frac{n+1}{2} T\bar e,$$

$$\rho\big(t_0 + (n+1)T\big) \geqslant R - (n+1)T\|e\|_1 + \frac{(n-2)(n+1)}{4} T^2 \bar e.$$

We check that (P_0) is true. First we show that

$$\rho(t) > r_1, \quad \text{for every } t \in [t_0, t_0 + T]. \tag{30}$$

By contradiction, assume that there is $t_1 \in [t_0, t_0 + T]$ such that $\rho(t_1) = r_1$ and $\rho(t) > r_1$ for every $t \in [t_0, t_1[$. Then, for $t \in [t_0, t_1[$,

$$\ddot\rho(t) \geqslant e(t) - \frac{\bar e}{2},$$

so that

$$\dot\rho(t) = \dot\rho(t_0) + \int_{t_0}^{t} \ddot\rho(s)\,ds \geqslant -\|e\|_1 - \frac{T\bar e}{2},$$

and hence

$$\rho(t) = \rho(t_0) + \int_{t_0}^{t} \dot\rho(s)\,ds \geqslant R - T\left(\|e\|_1 + \frac{T\bar e}{2}\right).$$

On the other hand, by the choice of R,

$$R - T\left(\|e\|_1 + \frac{T\bar e}{2}\right) \geqslant r_1 + \frac{\|e\|_1^2}{\bar e} + \frac{3}{2} T\|e\|_1 + \frac{9}{16} T^2 \bar e,$$

thus contradicting the fact that $\rho(t_1) = r_1$.

Having proved that (30) holds, we have that all the above inequalities hold for every $t \in [t_0, t_0 + T]$, and hence the first two inequalities in (P_0) hold true. Moreover,

$$\dot\rho(t_0+T) = \dot\rho(t_0) + \int_{t_0}^{t_0+T} \ddot\rho(s)\,ds \geqslant \int_{t_0}^{t_0+T} \left(e(s) - \frac{\bar e}{2}\right) ds = \frac{T\bar e}{2}$$

and

$$\rho(t_0+T) = \rho(t_0) + \int_{t_0}^{t_0+T} \dot\rho(s)\,ds \geqslant R - T\left(\|e\|_1 + \frac{T\bar e}{2}\right).$$

Hence, (P_0) has been proved. Now assume that (P_n) holds, for some $n \in \mathbb{N}$. First we show that

$$\rho(t) > r_1, \quad \text{for every } t \in \big[t_0 + (n+1)T, t_0 + (n+2)T\big]. \tag{31}$$

By contradiction, assume that there is $t_1 \in [t_0 + (n+1)T, t_0 + (n+2)T]$ such that $\rho(t_1) = r_1$ and $\rho(t) > r_1$ for every $t \in [t_0 + (n+1)T, t_1[$. Then, for $t \in [t_0 + (n+1)T, t_1[$,

$$\ddot\rho(t) \geqslant e(t) - \frac{\bar e}{2},$$

so that

$$\begin{aligned}
\dot\rho(t) &= \dot\rho\big(t_0 + (n+1)T\big) + \int_{t_0+(n+1)T}^{t} \ddot\rho(s)\,ds \\
&\geqslant \frac{n+1}{2}T\bar e - \|e\|_1 - \frac{T\bar e}{2} = \frac{n}{2}T\bar e - \|e\|_1,
\end{aligned}$$

and hence

$$\begin{aligned}
\rho(t) &= \rho\big(t_0 + (n+1)T\big) + \int_{t_0+(n+1)T}^{t} \dot\rho(s)\,ds \\
&\geqslant R - (n+1)T\|e\|_1 + \frac{(n-2)(n+1)}{4}T^2\bar e + \\
&\quad + \big(t - \big(t_0 + (n+1)T\big)\big)\left(\frac{n}{2}T\bar e - \|e\|_1\right) \\
&> R - (n+1)T\|e\|_1 + \frac{(n-2)(n+1)}{4}T^2\bar e - T\left(\|e\|_1 + \frac{T\bar e}{2}\right) \\
&= R - (n+2)T\|e\|_1 + \frac{(n+1)^2 - 3(n+1) - 2}{4}T^2\bar e.
\end{aligned}$$

The function

$$x \mapsto R - (x+2)T\|e\|_1 + \frac{(x+1)^2 - 3(x+1) - 2}{4} T^2 \bar{e}$$

has as minimum value

$$R - \frac{\|e\|_1^2}{\bar{e}} - \frac{5}{2} T\|e\|_1 - \frac{17}{16} T^2 \bar{e},$$

and this, together with (29), contradicts the fact that $\rho(t_1) = r_1$.

Having proved that (31) holds, we have that all the above inequalities hold for every $t \in [t_0 + (n+1)T, t_0 + (n+2)T]$, and hence the first two inequalities in (P_{n+1}) hold true. Moreover,

$$\begin{aligned}
\dot{\rho}\big(t_0 + (n+2)T\big) &= \dot{\rho}\big(t_0 + (n+1)T\big) + \int_{t_0+(n+1)T}^{t_0+(n+2)T} \ddot{\rho}(s)\, ds \\
&\geqslant \frac{n+1}{2} T\bar{e} + \int_{t_0+(n+1)T}^{t_0+(n+2)T} \left(e(s) - \frac{\bar{e}}{2} \right) ds \\
&= \frac{n+2}{2} T\bar{e}
\end{aligned}$$

and

$$\begin{aligned}
\rho\big(t_0 + (n+2)T\big) &= \rho\big(t_0 + (n+1)T\big) + \int_{t_0+(n+1)T}^{t_0+(n+2)T} \dot{\rho}(s)\, ds \\
&\geqslant R - (n+1)T\|e\|_1 + \frac{(n-2)(n+1)}{4} T^2 \bar{e} + T\left(\frac{n}{2} T\bar{e} - \|e\|_1 \right) \\
&= R - (n+2)T\|e\|_1 + \frac{(n-1)(n+2)}{4} T^2 \bar{e}.
\end{aligned}$$

Hence, (P_{n+1}) has been proved.

Having proved that (P_n) holds for every $n \in \mathbb{N}$, as an immediate consequence we have that

$$\lim_{t \to +\infty} \rho(t) = \lim_{t \to +\infty} \dot{\rho}(t) = +\infty.$$

Assume now that $\dot{\rho}(t_0) < 0$. Setting $\eta(t) = \rho(-t)$, we have that η verifies

$$\ddot{\eta} - \frac{\mu^2}{\eta^3} + h(-t, \eta) = e(-t).$$

Moreover, $\eta(-t_0) > R$ and $\dot{\eta}(-t_0) > 0$. So, the same arguments used above for ρ can now be used for η, thus leading to

$$\lim_{t\to+\infty} \eta(t) = \lim_{t\to+\infty} \dot{\eta}(t) = +\infty,$$

which means

$$\lim_{t\to-\infty} \rho(t) = +\infty, \qquad \lim_{t\to-\infty} \dot{\rho}(t) = -\infty.$$

The proof is thus completed. □

As a direct consequence, we get the following two corollaries concerning Eqs. (3) and (4).

Corollary 11. *Assume* $\bar{e} > 0$, $c \in L^\infty(\mathbb{R})$, *and* $\gamma > 1$. *Then, the same conclusion of Theorem* 10 *holds for Eq.* (3).

Corollary 12. *Assume* $\bar{e} > 0$ *and* $c, d \in L^\infty(\mathbb{R})$. *Then, the same conclusion of Theorem* 10 *holds for Eq.* (4).

Remark 5. Notice that in the proof of Theorem 10 we do not need the function $h(t,r)$ to be periodic in the first variable t. In particular, in the above corollaries, $c(t)$ and $d(t)$ could be arbitrary bounded functions.

Appendix A. Planar orbits

The orbits of the solutions to a general system like (1) are planar. For $N = 3$, this fact is well known and can be found, e.g., in [6]. For the reader's convenience, we give here a simple proof valid for any dimension $N \geqslant 3$. Let $x(t)$ be such a solution, defined on an interval I, with

$$x(t) = \big(x_1(t), \ldots, x_N(t)\big) \in \mathbb{R}^N.$$

We claim that the functions

$$t \mapsto x_i(t)\dot{x}_j(t) - x_j(t)\dot{x}_i(t),$$

with $i, j \in \{1, \ldots, N\}$, are constant on I. Indeed,

$$\frac{d}{dt}(x_i\dot{x}_j - x_j\dot{x}_i) = \dot{x}_i\dot{x}_j + x_i\ddot{x}_j - \dot{x}_j\dot{x}_i - x_j\ddot{x}_i = x_i f\big(t, |x|\big)x_j - x_j f\big(t, |x|\big)x_i = 0.$$

Assume that $x(t_0)$ and $\dot{x}(t_0)$ are linearly independent, for some $t_0 \in I$, and let $\mathcal{P}(t_0)$ be the plane generated by these two vectors. Up to a permutation of the coordinates, we may assume, for instance, that

$$x_1(t_0)\dot{x}_2(t_0) - x_2(t_0)\dot{x}_1(t_0) \neq 0.$$

Then, $x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t) \neq 0$ for every $t \in I$. In particular, $x(t)$ and $\dot{x}(t)$ are linearly independent and generate a plane, which we denote by $\mathcal{P}(t)$. Writing the equations of this plane, we have that a vector $v = (v_1, \dots, v_N)$ belongs to $\mathcal{P}(t)$ if and only if

$$\begin{cases} v_3 = \dfrac{x_3(t)\dot{x}_2(t) - x_2(t)\dot{x}_3(t)}{x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)} v_1 + \dfrac{x_1(t)\dot{x}_3(t) - x_3(t)\dot{x}_1(t)}{x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)} v_2, \\ \vdots \\ v_N = \dfrac{x_N(t)\dot{x}_2(t) - x_2(t)\dot{x}_N(t)}{x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)} v_1 + \dfrac{x_1(t)\dot{x}_N(t) - x_N(t)\dot{x}_1(t)}{x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)} v_2. \end{cases}$$

Since all the coefficients are constant in t, we deduce that $\mathcal{P}(t)$ does not depend on t. Hence, the whole orbit of $x(t)$ is contained in the plane $\mathcal{P}(t_0)$.

Assume now that $x(t_0)$ and $\dot{x}(t_0)$ are linearly dependent, for some $t_0 \in I$. In this case the orbit must be contained in the straight line joining $x(t_0)$ with the origin. Otherwise, for some $t_1 \in I$, the vectors $x(t_1)$ and $\dot{x}(t_1)$ would have to be linearly independent and, by the previous arguments, this would lead to a contradiction.

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