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J. Differential Equations 214 (2005) 299–325

Journal of Differential Equations

www.elsevier.com/locate/jde

Periodic solutions of perturbed isochronous hamiltonian systems at resonance

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Received 6 May 2004; revised 6 May 2004

Available online 30 March 2005

Abstract

We look for periodic solutions of planar systems obtained by adding an asymptotically positively homogeneous nonlinear term to an isochronous hamiltonian system. Precise computations of the topological degree are obtained by elementary phase-plane analysis.

Keywords: Periodic solution; Hamiltonian systems; Topological degree

1. Introduction

We are interested in finding periodic solutions of the system

$$J\dot{u} = \nabla H(u) + f(t, u), \tag{1}$$

where the function $H : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 and $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and T-periodic in its first variable. Here, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix.

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0022-0396/$ - see front matter
doi:10.1016/j.jde.2005.02.003

Concerning the hamiltonian function, we assume it to be positively homogeneous of degree 2 and positive. In other terms, for every $u \in \mathbb{R}^2$ and $\lambda > 0$,

$$H(\lambda u) = \lambda^2 H(u), \tag{2}$$

and, moreover,

$$\min_{\|u\|=1} H(u) > 0. \tag{3}$$

In this situation, the origin is an isochronous center for the autonomous system

$$J\dot{u} = \nabla H(u); \tag{4}$$

all solutions of (4) are periodic with the same minimal period, which will be denoted by τ.

In this paper, we assume throughout that

T is an integer multiple of τ.

The autonomous system (4) thus has nontrivial T-periodic solutions. In this situation, we say that the system is “at resonance”.

In order to investigate the existence of T-periodic solutions for system (1), we are interested in computing the topological degree of $\mathcal{P} - \mathrm{Id}$ with respect to some large sets, where $\mathcal{P}$ is the associated Poincaré map for the period T.

In recent years, starting with the pioneering work of Dancer [3] and Fučík [10], there has been an intensive effort to describe the dynamics of “asymmetric” oscillators modeled by second-order scalar equations of the form

$$x'' + \mu x^+ - \nu x^- = g(t, x), \tag{5}$$

where μ and ν are positive real numbers and $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. In particular, the degree of $\mathcal{P} - \mathrm{Id}$ was computed in [5], assuming e.g. $g(t, x) = e(t)$, in the following manner. Denoting by $\phi(t)$ a nontrivial solution of $x'' + \mu x^+ - \nu x^- = 0$, with minimal period $\tau = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}$, the degree of $\mathcal{P} - \mathrm{Id}$ with respect to large disks centered at the origin is equal to $1 - \zeta$, where 2ζ is the number of zeros, in $[0, \tau[$, of the τ-periodic function

$$\Phi(\theta) = \int_0^T e(t)\phi(t + \theta)\, dt.$$

The above function Φ, first introduced by Dancer [3], plays a crucial role in the study of the dynamics of the asymmetric oscillator at resonance (see, for instance, [1,6,8,14], and the references therein). In [9], it was shown that most of the results obtained for the scalar equation (5) generalize to a system like (1), when the function f does not depend on u.

In the present paper, we generalize the above to system (1). In order to illustrate our results, assume, for the sake of simplicity, f to be positively homogeneous of degree $\beta \in [0, 1[$, in some sense to be precised in Section 3. Denoting by $\varphi(t)$ a nontrivial solution of the unperturbed system (4), we introduce the two functions

$$\Phi(\theta) = \int_0^T \langle f(t, \varphi(t+\theta)) \mid \varphi(t+\theta) \rangle \, dt,$$

$$\Psi(\theta) = \int_0^T \langle f(t, \varphi(t+\theta)) \mid \dot{\varphi}(t+\theta) \rangle \, dt.$$

Assuming all zeros of Φ to be simple, we can prove that the degree of $\mathcal{P} - \mathrm{Id}$ with respect to large disks centered at the origin is equal to $1 - \zeta_+ + \zeta_-$, where ζ_+ is the number of zeros of Φ with a positive slope on the set $\{t \in [0, \tau[: \Psi(t) > 0\}$, and ζ_- is the number of zeros of Φ with a negative slope, on the same set. When f does not depend on u, it is easy to see that $\Psi = \Phi'$; in that case, $\zeta_- = 0$, and the results proved in [5,9] are recovered as particular cases.

However, we can also deal with the case when f is positively homogeneous of degree 1, provided that f itself is sufficiently small. As a particular interesting case, f could be a linear function. For example, if $T = \tau = 2\pi$, we show that the generalized Mathieu's equation

$$x'' + \mu x^+ - \nu x^- + \varepsilon \cos(nt) x = 0,$$

with $\mu \neq \nu$, $\sqrt{\mu}$ and $\sqrt{\nu}$ being irrational, only has the trivial T-periodic solution, for ε small enough, with associated degree $1 - n$, where n is any positive integer. Recall that, in the linear case $\mu = \nu$, the degree is either 1 or -1.

The precise knowledge of degree values can be useful to prove bifurcation results for equations with varying parameters. Those type of applications will appear elsewhere. See also [7] for related results concerning the dynamics of the solutions.

The paper is organized as follows. In Section 2, a preliminary result, about degree computations in the plane, is obtained. Roughly speaking, the basic idea is to reduce the degree of a map $Q : \mathbb{R}^2 \to \mathbb{R}^2$, with respect to a star-shaped open set $\Omega \subset \mathbb{R}^2$, to the degree of an angular coordinate of $Q + \mathrm{Id}$ with respect to some appropriate subset of the boundary of Ω.

Using that result, we compute in Section 3 the degree of $\mathcal{P} - \mathrm{Id}$, with respect to large disks centered at the origin, assuming f to be asymptotically positively homogeneous of degree $\beta \in [0, 1[$.

In Section 4, we provide some examples where our theory applies.

In Section 5, we treat the case where f is asymptotically positively homogeneous of degree 1, presenting some examples of applications.

The results of Sections 3 and 5 require that Φ and Ψ do not vanish simultaneously. In Section 6, considering only the case where $f(t, u)$ is independent of u, we are able to get rid of that hypothesis and generalize our main result. This generalization relies on a refined development of the Poincaré map with respect to a parameter determining the size of the set on which the degree is computed.

Notation. For a real-valued function $\mathcal{G} : E \to \mathbb{R}$ and a real number c, we denote by $\{\mathcal{G} > c\}$ the set $\{x \in E : \mathcal{G}(x) > c\}$. Analogously if $>$ is replaced by $\geqslant$, $<$, or $\leqslant$.

2. On the computation of the degree

We first introduce some kind of generalized polar coordinates. Let $\varphi : \mathbb{R} \to \mathbb{R}^2$ be a periodic C^1-function, with minimal period $\tau > 0$, such that

$$\forall t \in \mathbb{R}, \quad \langle \dot{\varphi}(t) \mid J\varphi(t) \rangle < 0.$$

The orbit of φ parametrizes in clockwise direction the border of a domain Ω which is strictly star-shaped with respect to the origin: precisely, we have

$$\Omega = \{\rho\varphi(\theta) : \theta \in [0, \tau[, \rho \in [0, 1[\,\},$$

and every ray emanating from the origin crosses the orbit of φ at precisely one point.

Let $P : \bar{\Omega} \to \mathbb{R}^2$ be a continuous function such that,

$$\forall t \in \mathbb{R}, \quad P(\varphi(t)) \neq \varphi(t). \tag{6}$$

We want to compute the degree of the function $Q = P - \mathrm{Id}$ with respect to Ω, i.e. the number of rotations around the origin of $Q(\varphi(t))$, in clockwise direction, when t varies from 0 to τ. We will compare the direction of $Q(\varphi(t))$ to that of $\varphi(t)$. To this aim, let us introduce the following two functions:

$$h(t) = \langle Q(\varphi(t)) \mid \varphi(t) \rangle, \qquad \eta(t) = \langle Q(\varphi(t)) \mid J\varphi(t) \rangle.$$

If there is a “missing direction” for $Q(\varphi(t))$ with respect to $\varphi(t)$, the degree is 1 (see e.g. [11]). Otherwise, the vector $\xi(t) = (h(t), \eta(t))$, which never vanishes, will cover every direction as t varies from 0 to τ. In this case, we construct

$$t_1^- < t_1^+ \leqslant t_2^- < t_2^+ \leqslant \cdots \leqslant t_n^- < t_n^+ \leqslant t_{n+1}^- = t_1^- + \tau$$

in the following way:

(a) $h(t_k^-) = 0 = h(t_k^+)$;
(b) if $t \in]t_k^-, t_k^+[$, then $h(t) > 0$;
(c) $]t_k^-, t_k^+[\cap \eta^{-1}(0) \neq \emptyset$;
(d) if $t \in]t_k^+, t_{k+1}^-[$, then either $\eta(t) \neq 0$, or $h(t) < 0$.

The number of such points is finite because of the uniform continuity of η and the fact that there is a constant $c > 0$ for which

$$|h(t)| + |\eta(t)| \geqslant c,$$

for every $t \in \mathbb{R}$. In this situation, set

$$V =]t_1^-, t_1^+[\,\cup\,]t_2^-, t_2^+[\,\cup\, \cdots \,\cup\,]t_n^-, t_n^+[.$$

When t goes from t_k^- to t_k^+, two situations can occur: either $Q(\varphi(t))$ comes back to the same direction with respect to $\varphi(t)$, in which case $\eta(t_k^-)$ and $\eta(t_k^+)$ have the same sign, or it makes half a turn with respect to $\varphi(t)$, the sense of rotation depending on the signs of $\eta(t_k^-)$ and $\eta(t_k^+)$. Therefore, counting the number of crossings with the half-line $\{\eta = 0, h > 0\}$, we have

$$\deg(Q, \Omega) = 1 + \frac{1}{2} \sum_{k=1}^{n} \Big(\operatorname{sign}(\eta(t_k^-)) - \operatorname{sign}(\eta(t_k^+)) \Big) = 1 - \deg(\eta, V).$$

Assume moreover that

$$\forall t \in \mathbb{R}, \qquad P(\varphi(t)) \neq (0, 0). \tag{7}$$

In that case, we can define two continuous functions $R, \Theta : \mathbb{R} \to \mathbb{R}$ by

$$P(\varphi(t)) = R(t)\varphi(t + \Theta(t)),$$

with $R(t) > 0$ for all $t \in \mathbb{R}$. We want to modify the function Θ to make it describe the degree. Define $\tau_{\mathrm{opp}}(t)$ to be such that

$$\tau_{\mathrm{opp}}(t) \in]0, \tau[\quad \text{and} \quad \frac{\varphi(t)}{\|\varphi(t)\|} = -\frac{\varphi(t + \tau_{\mathrm{opp}}(t))}{\|\varphi(t + \tau_{\mathrm{opp}}(t))\|}. \tag{8}$$

Consider the open set

$$\hat{U} = \left\{ t \in \mathbb{R} : \Theta(t) \notin \tau_{\mathrm{opp}}(t) + \tau\mathbb{Z} \right\},$$

and define $\hat{\Theta} : \hat{U} \to]-\tau, \tau[$ such that, for every $t \in \hat{U}$,

$$\begin{cases} \hat{\Theta}(t) \in \Theta(t) + \tau\mathbb{Z}, \\ \tau_{\text{opp}}(t) - \tau < \hat{\Theta}(t) < \tau_{\text{opp}}(t). \end{cases}$$

The function $\hat{\Theta}$ is continuous on $\hat{U}$. Because of (b), each interval $[t_k^-, t_k^+]$ is contained in $\hat{U}$, so that $V \subseteq \hat{U}$. Moreover, the functions $\hat{\Theta}$ and η have opposite sign in V. Hence,

$$\deg(Q, \Omega) = 1 - \deg(\eta, V) = 1 + \deg(\hat{\Theta}, V).$$

If $\hat{\Theta}$ vanishes at some point $t \in V$, $P(\varphi(t))$ and $\varphi(t)$ will have the same direction. But, on V, we have $h(t) > 0$, so that $Q(\varphi(t))$ and $\varphi(t)$ will also have the same direction, which implies that $R(t) > 1$. Consequently,

$$\deg(\hat{\Theta}, V) = \deg(\hat{\Theta}, V \cap \{R > 1\}),$$

and we can write

$$\deg(Q, \Omega) = 1 + \deg(\hat{\Theta}, V \cap \{R > 1\}).$$

Let us see what happens outside the intervals $[t_k^-, t_k^+]$, i.e. outside V. Consider the interval $]t_k^+, t_{k+1}^-[$. If $\hat{\Theta}$ vanishes at some point $t \in]t_k^+, t_{k+1}^-[\cap \hat{U}$, we have $\eta(t) = 0$ and therefore, because of (d), $h(t) < 0$. Since $P(\varphi(t))$ and $\varphi(t)$ have the same direction, this implies that $R(t) < 1$, so that $t \notin \{R > 1\}$. Therefore, we have

$$\deg(\hat{\Theta}, V \cap \{R > 1\}) = \deg(\hat{\Theta},]t_1^-, t_1^- + \tau[\cap \hat{U} \cap \{R > 1\}).$$

Summing up these considerations, and using the periodicity, this leads to the following.

Lemma 1. *In the above situation, we have*

$$\deg(P - \mathrm{Id}, \Omega) = 1 + \deg(\hat{\Theta},]a, a + \tau[\cap \hat{U} \cap \{R > 1\}),$$

where a is chosen so that $\hat{\Theta}(a) \neq 0$, i.e. $\Theta(a) \notin \tau\mathbb{Z}$.

In the case when $\hat{\Theta}$ coincides with Θ, we have the following.

Theorem 1. *Let* $P : \bar{\Omega} \to \mathbb{R}^2$ *be continuous and such that*

$$P(\varphi(t)) = R(t)\varphi(t + \Theta(t)),$$

where $R, \Theta : \mathbb{R} \to \mathbb{R}$ *are continuous and, for every* $t \in \mathbb{R}$,

$$R(t) > 0, \tag{9}$$

$$\tau_{\text{opp}}(t) - \tau < \Theta(t) < \tau_{\text{opp}}(t) \tag{10}$$

and

$$|R(t) - 1| + |\Theta(t)| \neq 0, \tag{11}$$

where τ_{opp} *is defined in* (8). *Then,*

$$\begin{aligned} \deg(P - \text{Id}, \Omega) &= 1 + \deg(\Theta, \ ]a, a + \tau[\cap \{R > 1\}) \\ &= 1 - \deg(\Theta, \ ]a, a + \tau[\cap \{R < 1\}), \end{aligned}$$

where a *is chosen so that* $\Theta(a) \neq 0$.

Proof. Just notice that, because of (9)–(11), both (6) and (7) are verified, and $\hat{\Theta} = \Theta$. So, Lemma 1 applies to give the first equality for the degree. The second equality follows by excision. □

3. Asymptotically positively homogeneous nonlinearities

In this section we consider system (1), i.e.

$$J\dot{u} = \nabla H(u) + f(t, u), \tag{12}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function, with locally Lipschitz continuous gradient, satisfying (2) and (3), and $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, T-periodic in its first variable, and locally Lipschitz continuous in its second variable. Consider a finite number of directions

$$\alpha_1 < \alpha_2 < \cdots < \alpha_m < \alpha_{m+1} = \alpha_1 + 2\pi,$$

and define the set $\Sigma = \{\rho e^{i\alpha_k} : \rho \geqslant 0, \ k = 1, 2, \ldots, m\}$, which is made of m rays starting from the origin. We assume that, for some $\alpha > 0$ and $\beta \in [0, 1[$,

$$\|f(t, u)\| \leqslant \alpha(\|u\|^\beta + 1), \tag{13}$$

for every $t\in\mathbb{R}$ and $u\in\mathbb{R}^2$, and that there exists a continuous function $F:\mathbb{R}\times(\mathbb{R}^2\setminus\Sigma)\to\mathbb{R}^2$ such that

$$F(t,u)=\lim_{\lambda\to+\infty}\frac{f(t,\lambda u)}{\lambda^\beta}, \tag{14}$$

the above limit being uniform with respect to (t,u) when u varies in compact subsets of $\mathbb{R}^2\setminus\Sigma$. It is easy to see that

$$\|F(t,u)\|\leqslant\alpha\|u\|^\beta, \qquad F(t,\lambda u)=\lambda^\beta F(t,u),$$

for every $t\in\mathbb{R}$, $u\in\mathbb{R}^2\setminus\Sigma$, and $\lambda>0$.

It is useful to fix a solution of the autonomous system (4). Let $\varphi:\mathbb{R}\to\mathbb{R}^2$ be such that

$$J\dot\varphi(t)=\nabla H(\varphi(t)) \quad \text{and} \quad H(\varphi(t))=\tfrac{1}{2},$$

for every $t\in\mathbb{R}$. Recall that φ is periodic with minimal period τ. Define, for $\delta>0$, the set

$$\Omega_\delta=\left\{\frac{1}{\delta}\rho\varphi(\theta):\theta\in[0,\tau[,\ \rho\in[0,1[\right\}.$$

Notice that, with the notations of Section 2, we have $\Omega_1=\Omega$.

Denote by $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ the Poincaré map for the period T associated to (12). It is well defined since the right-hand side of Eq. (12) grows at most linearly. We would like to compute the degree of $\mathcal{P}-\mathrm{Id}$ with respect to the set Ω_δ, for a sufficiently small δ.

By the change of variable $v=\delta u$, for some $\delta>0$, Eq. (12) becomes

$$J\dot v=\nabla H(v)+\delta f\left(t,\frac{v}{\delta}\right). \tag{15}$$

Let us denote by $\tilde{\mathcal{P}}_\delta:\mathbb{R}^2\to\mathbb{R}^2$ the Poincaré map associated to (15). We have $\tilde{\mathcal{P}}_\delta(v)=\delta\mathcal{P}(\frac{v}{\delta})$, and

$$\deg(\mathcal{P}-\mathrm{Id},\Omega_\delta)=\deg(\tilde{\mathcal{P}}_\delta-\mathrm{Id},\Omega).$$

For any $\theta_0\in[0,\tau]$, we write

$$\tilde{\mathcal{P}}_\delta(\varphi(\theta_0))=r_1\varphi(\theta_1);$$

in the sequel, we will try to evaluate θ_1 and r_1.

If $v(t)$ is a solution of (15) with starting point on $\partial\Omega$, we can write

$$v(t) = r(t)\varphi(t + \theta(t)),$$

with $r(0) = 1$. As long as $r(t) > 0$, the functions $\theta(t)$ and $r(t)$ are of class C^1 and satisfy (see [9])

$$\begin{cases} \theta' = \dfrac{\delta}{r} \left\langle f\left(t, \dfrac{r}{\delta}\,\varphi(t+\theta)\right) \Big| \varphi(t+\theta)\right\rangle, \\ r' = -\delta \left\langle f\left(t, \dfrac{r}{\delta}\,\varphi(t+\theta)\right) \Big| \dot\varphi(t+\theta)\right\rangle. \end{cases} \tag{16}$$

Denote by $(\theta(t;\theta_0;\delta), r(t;\theta_0;\delta))$ the solution of (16) with starting point

$$\theta(0;\theta_0;\delta) = \theta_0 \in [0,\tau], \qquad r(0;\theta_0;\delta) = 1.$$

Writing briefly $r(t)$ for $r(t;\theta_0;\delta)$ and $\theta(t)$ for $\theta(t;\theta_0;\delta)$, by (13) we have, for some constants c_1, c_2, c_3 depending only on φ,

$$\begin{aligned} |r(t) - 1| &= \left| \delta \int_0^t \left\langle f\left(s, \frac{r(s)}{\delta}\,\varphi(s+\theta(s))\right) \Big| \dot\varphi(s+\theta(s)) \right\rangle ds \right| \\ &\leqslant \delta \int_0^t \alpha \left(\left\| \frac{1}{\delta} r(s)\varphi(s+\theta(s)) \right\|^{\beta} + 1 \right) \|\dot\varphi(s+\theta(s))\|\, ds \\ &\leqslant \alpha c_1 \delta^{1-\beta} \int_0^t |r(s) - 1|\, ds + \alpha(c_2\delta^{1-\beta} + c_3\delta), \end{aligned}$$

so that, by Gronwall inequality,

$$|r(t) - 1| \leqslant \alpha(c_2\delta^{1-\beta} + c_3\delta)\exp(\alpha c_1 \delta^{1-\beta} t). \tag{17}$$

Hence,

$$\lim_{\delta\to 0^+} r(t;\theta_0;\delta) = 1, \tag{18}$$

uniformly with respect to $t \in [0,T]$ and $\theta_0 \in [0,\tau]$. In particular, for δ small enough, we have $r(t) > 0$ for every $t \in [0,T]$. Analogously, one shows that

$$\lim_{\delta\to 0^+} \theta(t;\theta_0;\delta) = \theta_0, \tag{19}$$

with the same type of uniformity.

Therefore, for δ small enough, we can write

$$\begin{cases} \theta_1 = \theta_0 + \delta \displaystyle\int_0^T \frac{1}{r(t)} \left\langle f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right) \middle| \varphi(t+\theta(t))\right\rangle dt, \\ r_1 = 1 - \delta \displaystyle\int_0^T \left\langle f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right) \middle| \dot\varphi(t+\theta(t))\right\rangle dt. \end{cases}$$

Define the two functions

$$\Phi(\theta) = \int_0^T \langle F(t, \varphi(t+\theta)) \mid \varphi(t+\theta)\rangle\, dt,$$

$$\Psi(\theta) = \int_0^T \langle F(t, \varphi(t+\theta)) \mid \dot\varphi(t+\theta)\rangle\, dt.$$

Because of the properties of φ, the set $\{t \in [0, \tau] : \varphi(t) \in \Sigma\}$ is finite, so that Φ and Ψ are well defined and they are continuous and τ-periodic. We now need the following.

Lemma 2. *We have*

$$\begin{cases} \theta_1 = \theta_0 + \delta^{1-\beta}[\Phi(\theta_0) + R_1(\theta_0, \delta)], \\ r_1 = 1 - \delta^{1-\beta}[\Psi(\theta_0) + R_2(\theta_0, \delta)], \end{cases}$$

where R_1 *and* R_2 *are such that*

$$\lim_{\delta\to 0^+} R_1(\theta_0, \delta) = \lim_{\delta\to 0^+} R_2(\theta_0, \delta) = 0,$$

uniformly for $\theta_0 \in [0, \tau]$.

Proof. We have to show that

$$\lim_{\delta\to 0^+} \delta^\beta \int_0^T \frac{1}{r(t)} \left\langle f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right) \middle| \varphi(t+\theta(t))\right\rangle dt = \Phi(\theta_0)$$

and

$$\lim_{\delta\to 0^+} \delta^\beta \int_0^T \left\langle f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right) \middle| \dot\varphi(t+\theta(t))\right\rangle dt = \Psi(\theta_0),$$

uniformly with respect to $\theta_0 \in [0, \tau]$. We prove the second one, the first being similar. Notice that we can write

$$\delta^\beta \int_0^T \left\langle f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right) \middle| \dot\varphi(t+\theta(t))\right\rangle dt$$

$$= \int_0^T r(t)^\beta \left\langle \frac{f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right)}{\left(\frac{r(t)}{\delta}\right)^\beta} \middle| \dot\varphi(t+\theta(t))\right\rangle dt.$$

Fix $\varepsilon > 0$. Corresponding to each direction α_k, we consider a small cone determined by $[\alpha_k - \eta, \alpha_k + \eta]$, for some $\eta > 0$. Let Σ_η be the union of these cones, and define

$$A_\eta(\theta_0) = \{t \in [0, T] : \varphi(t + \theta_0) \in \Sigma_\eta\}.$$

Writing the above integral and the one defining $\Phi(\theta_0)$ as

$$\int_0^T \cdots = \int_{A_\eta(\theta_0)} \cdots + \int_{[0,T]\setminus A_\eta(\theta_0)} \cdots ,$$

we have that, taking η small enough,

$$\left| \int_{A_\eta(\theta_0)} \langle F(t, \varphi(t+\theta_0)) \mid \dot\varphi(t+\theta_0)\rangle\, dt \right| \leqslant \int_{A_\eta(\theta_0)} \alpha \|\varphi(t+\theta_0)\|^\beta \|\dot\varphi(t+\theta_0)\|\, dt \leqslant \frac{\varepsilon}{4},$$

and, for δ also small enough, by (13) and (18),

$$\left| \delta^\beta \int_{A_\eta(\theta_0)} \left\langle f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right) \middle| \dot\varphi(t+\theta(t))\right\rangle dt \right|$$

$$\leqslant \int_{A_\eta(\theta_0)} \alpha \left(2\|\varphi(t+\theta(t))\|^\beta + 1\right) \|\dot\varphi(t+\theta(t))\|\, dt \leqslant \frac{\varepsilon}{4}.$$

On the other hand, for $t \in [0, T] \setminus A_\eta(\theta_0)$, by (18) and (19),

$$\lim_{\delta\to 0^+} r(t)^\beta \left\langle \frac{f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right)}{\left(\frac{r(t)}{\delta}\right)^\beta} \middle| \dot\varphi(t+\theta(t))\right\rangle = \langle F(t, \varphi(t+\theta_0)) \mid \dot\varphi(t+\theta_0)\rangle,$$

uniformly in $t \in [0,T] \setminus A_\eta(\theta_0)$ and $\theta_0 \in [0,\tau]$, so that, for δ small enough,

$$\left| \int_{[0,T]\setminus A_\eta(\theta_0)} \left[r(t)^\beta \left\langle \frac{f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right)}{\left(\frac{r(t)}{\delta}\right)^\beta} \middle| \dot\varphi(t+\theta(t)) \right\rangle \right.\right.$$
$$\left.\left. -\langle F(t,\varphi(t+\theta_0)) \mid \dot\varphi(t+\theta_0)\rangle \right] dt \right| \leqslant \frac{\varepsilon}{2}.$$

So, taking η and δ small enough, for every $\theta_0 \in [0,\tau]$ we have

$$\left| \delta^\beta \int_0^T \left\langle f\left(t, \frac{r(t)}{\delta}\,\varphi(t+\theta(t))\right) \middle| \dot\varphi(t+\theta(t)) \right\rangle dt - \Psi(\theta_0) \right| \leqslant \varepsilon,$$

and the lemma is thus proved. □

We are now ready to prove the following.

Theorem 2. *Assume that*

$$\forall \theta \in \mathbb{R}, \quad |\Phi(\theta)| + |\Psi(\theta)| \neq 0.$$

Then, for every sufficiently small δ,

$$\begin{aligned} \deg(\mathcal{P} - \mathrm{Id}, \Omega_\delta) &= 1 - \deg(\Phi, \,]a, a+\tau[\, \cap \{\Psi > 0\}) \\ &= 1 + \deg(\Phi, \,]a, a+\tau[\, \cap \{\Psi < 0\}), \end{aligned}$$

where a *is chosen so that* $\Phi(a) \neq 0$.

Proof. We compute the rotation number of $\tilde{\mathcal{P}}_\delta - \mathrm{Id}$ on the curve $\partial\Omega$, which we denote by $\mathrm{rot}(\tilde{\mathcal{P}}_\delta - \mathrm{Id}, \partial\Omega)$. Of course,

$$\deg(\tilde{\mathcal{P}}_\delta - \mathrm{Id}, \Omega) = \mathrm{rot}(\tilde{\mathcal{P}}_\delta - \mathrm{Id}, \partial\Omega).$$

Let $c > 0$ be such that,

$$\forall \theta \in \mathbb{R}, \quad |\Phi(\theta)| + |\Psi(\theta)| \geqslant 2c.$$

For $\lambda \in [0,1]$, consider the functions $P_{\delta,\lambda} : \partial\Omega \to \mathbb{R}^2$ defined by

$$P_{\delta,\lambda}(\varphi(\theta_0)) = r_1^\lambda \varphi(\theta_1^\lambda),$$

with (see Lemma 2)

$$\begin{cases} \theta_1^\lambda = \theta_0 + \delta^{1-\beta}[\Phi(\theta_0) + \lambda R_1(\theta_0, \delta)], \\ r_1^\lambda = 1 - \delta^{1-\beta}[\Psi(\theta_0) + \lambda R_2(\theta_0, \delta)]. \end{cases}$$

Let us prove that, for δ sufficiently small, $P_{\delta,\lambda}$ has no fixed points on $\partial\Omega$. Choose $\bar{\delta} > 0$ such that

$$0 < \delta < \bar{\delta} \quad \Rightarrow \quad |R_1(\theta_0, \delta)| < c \quad \text{and} \quad |R_2(\theta_0, \delta)| < c.$$

For such a δ and every $\lambda \in [0, 1]$, since

$$|\Phi(\theta_0)| = \left| \frac{\theta_1^\lambda - \theta_0}{\delta^{1-\beta}} - \lambda R_1(\theta_0, \delta) \right| < \frac{|\theta_1^\lambda - \theta_0|}{\delta^{1-\beta}} + c,$$

$$|\Psi(\theta_0)| = \left| \frac{r_1^\lambda - 1}{\delta^{1-\beta}} - \lambda R_2(\theta_0, \delta) \right| < \frac{|r_1^\lambda - 1|}{\delta^{1-\beta}} + c,$$

we must have $(\theta_1^\lambda, r_1^\lambda) \neq (\theta_0, 1)$.

By the above, for δ sufficiently small, $P_{\delta,\lambda} - \mathrm{Id}$ never vanishes on $\partial\Omega$, for every $\lambda \in [0, 1]$, so that

$$\mathrm{rot}(P_{\delta,1} - \mathrm{Id}, \partial\Omega) = \mathrm{rot}(P_{\delta,0} - \mathrm{Id}, \partial\Omega).$$

Let us concentrate on $P_{\delta,0} : \partial\Omega \to \mathbb{R}^2$, and extend it to a continuous function $P : \bar{\Omega} \to \mathbb{R}^2$, so that

$$P(\varphi(\theta_0)) = R(\theta_0)\varphi(\theta_0 + \Theta(\theta_0)),$$

with $R(\theta_0) = 1 - \delta^{1-\beta}\Psi(\theta_0)$ and $\Theta(\theta_0) = \delta^{1-\beta}\Phi(\theta_0)$. For δ sufficiently small, the assumptions of Theorem 1 are verified, so that

$$\begin{aligned} \deg(P - \mathrm{Id}, \Omega) &= 1 - \deg(\Theta, \,]a, a+\tau[\cap \{R < 1\}) \\ &= 1 - \deg(\Phi, \,]a, a+\tau[\cap \{\Psi > 0\}), \end{aligned}$$

where a is chosen so that $\Phi(a) \neq 0$.

The second formula for the degree simply follows from the excision property. □

Remark. Since the sets Ω_δ are bounded and their union is the whole plane, by excision, we can replace $\deg(\mathcal{P} - \mathrm{Id}, \Omega_\delta)$ in the statement of Theorem 2 by $\deg(\mathcal{P} - \mathrm{Id}, B_R)$, for a sufficiently large disk $B_R = \{x \in \mathbb{R}^2 : \|x\| < R\}$.

We immediately have the following corollaries.

Corollary 1. *Assume that* Φ *never vanishes on* $\{\Psi > 0\}$ *or on* $\{\Psi < 0\}$. *Then, Eq.* (12) *has a T-periodic solution.*

Corollary 2. *Assume that* Φ *never vanishes, or* Ψ *never vanishes. Then, Eq.* (12) *has a T-periodic solution.*

The above corollaries can be viewed as generalizations of the classical Landesman–Lazer situation (see e.g. [4,12]). In both cases, the degree of $\mathcal{P} - \mathrm{Id}$, with respect to large disks centered at the origin, is 1.

Corollary 3. *Assume that* Φ *is differentiable and there exist* $c_1, c_2 \in \mathbb{R}$, *with* $c_1 > 0$, *for which* $\Psi = c_1\Phi' + c_2$, *and*

$$\forall \theta \in \mathbb{R}, \quad |\Phi(\theta)| + |\Psi(\theta)| \neq 0.$$

If Ψ *changes sign more than twice on the zeros of* Φ *in* $[0, \tau[$, *then,* (12) *has a T-periodic solution.*

Proof. We consider two cases, depending on the sign of c_2. First, assume $c_2 \leqslant 0$. In this case, if $\Psi(\theta) > 0$, then $\Phi'(\theta) > 0$. This means that the zeros of Φ in the set $\{\Psi > 0\}$ are all simple with positive derivative. By assumption, there are at least two of them in $]a, a+\tau[\cap \{\Psi > 0\}$, where a is chosen so that $\Phi(a) \neq 0$. Then,

$$\deg(\mathcal{P} - \mathrm{Id}, \Omega_\delta) = 1 - \deg(\Phi,\ ]a, a+\tau[\cap \{\Psi > 0\}) \leqslant -1.$$

Assume now that $c_2 > 0$. In this case, if $\Psi(\theta) < 0$, then $\Phi'(\theta) < 0$. The zeros of Φ in the set $\{\Psi < 0\}$ are all simple with negative derivative. By assumption, there are at least two of them in $]a, a+\tau[\cap \{\Psi < 0\}$, where a is chosen so that $\Phi(a) \neq 0$. Then,

$$\deg(\mathcal{P} - \mathrm{Id}, \Omega_\delta) = 1 + \deg(\Phi,\ ]a, a+\tau[\cap \{\Psi < 0\}) \leqslant -1,$$

thus concluding the proof. □

Corollary 3 can be applied when $F(t,u)$ is of the form

$$F(t,u) = G(u) + p(t).$$

In this case, we have

$$\Phi(\theta) = \kappa_1 + \int_0^T \langle p(t) \mid \varphi(t+\theta) \rangle \, dt, \quad \Psi(\theta) = \kappa_2 + \int_0^T \langle p(t) \mid \dot\varphi(t+\theta) \rangle \, dt,$$

where

$$\kappa_1 = \int_0^T \langle G(\varphi(t)) \mid \varphi(t)\rangle\, dt, \quad \kappa_2 = \int_0^T \langle G(\varphi(t)) \mid \dot\varphi(t)\rangle\, dt. \tag{20}$$

Then, $\Psi(\theta) = \Phi'(\theta) + \kappa_2$.

Corollary 3 also applies when $F(t, u)$ is the gradient of a function $V(t, u)$, with respect to u. Indeed, by Euler identity,

$$\begin{aligned}
\Phi'(\theta) &= \frac{d}{d\theta}\int_0^T \langle \nabla_u V(t, \varphi(t+\theta)) \mid \varphi(t+\theta)\rangle\, dt \\
&= \frac{d}{d\theta}\int_0^T (\beta+1) V(t, \varphi(t+\theta))\, dt \\
&= (\beta+1)\int_0^T \langle \nabla_u V(t, \varphi(t+\theta)) \mid \dot\varphi(t+\theta)\rangle\, dt \\
&= (\beta+1)\Psi(\theta),
\end{aligned}$$

so that $\Psi(\theta) = (\beta+1)^{-1}\Phi'(\theta)$. In this case, we can be more precise.

Corollary 4. *Assume that Φ is differentiable and there exists $c_1 > 0$, for which $\Psi = c_1\Phi'$, and that Φ only has simple zeros, precisely 2ζ of them, in the interval $[0, \tau[$. Then, for every sufficiently large disk B_R,*

$$\deg(\mathcal{P} - \mathrm{Id}, B_R) = 1 - \zeta.$$

Both situations above appear when $f(t, u)$ is independent of u, a case which has been described in [9]. Corollary 4 extends in various directions an analogous result first obtained in [5] for scalar second order equations.

As a particular case, let $T = \tau = 2\pi$ and $F(t, u) = \cos(nt)\nabla V(u)$, where n is any positive integer. Then,

$$\begin{aligned}
\Phi(\theta) &= \cos(n\theta)\int_0^{2\pi} \cos(nt)\langle \nabla V(\varphi(t)) \mid \varphi(t)\rangle\, dt \\
&\quad + \sin(n\theta)\int_0^{2\pi} \sin(nt)\langle \nabla V(\varphi(t)) \mid \varphi(t)\rangle\, dt,
\end{aligned}$$

and $\Psi(\theta) = (\beta+1)^{-1}\Phi'(\theta)$. So, if Φ does not vanish identically, the degree, in this case, is precisely $1 - n$.

4. Some examples of applications

In order to illustrate our results, let us first consider the scalar second order equation

$$x'' + \mu x^+ - \nu x^- + g(t, x, x') = 0.$$

Here, we take

$$H(x, y) = \frac{1}{2}\left[\mu(x^+)^2 + \nu(x^-)^2 + y^2\right],$$

with $\mu > 0$ and $\nu > 0$. We can choose $\varphi(t) = (\phi(t), \phi'(t))$, where $\phi(t)$ is a nontrivial solution of $x'' + \mu x^+ - \nu x^- = 0$. Such a solution is periodic with minimal period $\tau = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}$, and can be written explicitly:

$$\phi(t) = \begin{cases} \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu} t) & (t \in [0, \frac{\pi}{\sqrt{\mu}}]), \\ -\frac{1}{\sqrt{\nu}} \sin(\sqrt{\nu}(t - \frac{\pi}{\sqrt{\mu}})) & (t \in [\frac{\pi}{\sqrt{\mu}}, \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}]). \end{cases} \tag{21}$$

The function $g(t, x, y)$ is assumed to be continuous, T-periodic in t, and locally Lipschitz continuous in (x, y). We assume that there is an integer N for which $T = N\tau$. Let us illustrate two examples when $\beta = 0$ in (13) and (14).

The first situation deals with the Liénard equation

$$x'' + h_1(x)x' + \mu x^+ - \nu x^- + h_2(x) = e(t). \tag{22}$$

Denoting by H_1 a primitive of h_1, the equation can be written as

$$J\begin{pmatrix} x' \\ y' \end{pmatrix} = \nabla H\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_2(x) \\ H_1(x) \end{pmatrix} - \begin{pmatrix} e(t) \\ 0 \end{pmatrix}.$$

The following follows from Corollary 3 and generalizes a result by Capietto and Wang [2].

Corollary 5. *Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous, and $e : \mathbb{R} \to \mathbb{R}$ be continuous and T-periodic. Let H_1 be a primitive of h_1 and assume*

$$\exists H_1(\pm\infty) \in \mathbb{R} : \quad H_1(\pm\infty) = \lim_{x \to \pm\infty} H_1(x)$$

and

$$\exists h_2(\pm\infty) \in \mathbb{R} : \quad h_2(\pm\infty) = \lim_{x \to \pm\infty} h_2(x).$$

Then,

$$\Phi(\theta) = 2N\left[\frac{h_2(+\infty)}{\mu} - \frac{h_2(-\infty)}{\nu}\right] - \int_0^T e(t)\phi(t+\theta)\,dt,$$

$$\Psi(\theta) = 2N\,[H_1(+\infty) - H_1(-\infty)] - \int_0^T e(t)\phi'(t+\theta)\,dt.$$

Assume

$$\forall\theta \in \mathbb{R}, \quad |\Phi(\theta)| + |\Psi(\theta)| \neq 0.$$

If Ψ *changes sign more than twice on the zeros of* Φ *in* $[0, \tau[$, *then* (22) *has a T-periodic solution.*

We now consider the Rayleigh equation

$$x'' + h_1(x') + \mu x^+ - \nu x^- + h_2(x) = e(t), \tag{23}$$

which can be written as

$$J\begin{pmatrix} x' \\ y' \end{pmatrix} = \nabla H\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1(y) + h_2(x) \\ 0 \end{pmatrix} - \begin{pmatrix} e(t) \\ 0 \end{pmatrix}.$$

Again by Corollary 3, the following generalizes a result by Wang [13].

Corollary 6. *Let* $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz continuous, and* $e : \mathbb{R} \to \mathbb{R}$ *be continuous and T-periodic. Assume*

$$\exists h_1(\pm\infty) \in \mathbb{R} : \quad h_1(\pm\infty) = \lim_{y\to\pm\infty} h_1(y)$$

and

$$\exists h_2(\pm\infty) \in \mathbb{R} : \quad h_2(\pm\infty) = \lim_{x\to\pm\infty} h_2(x).$$

Then,

$$\begin{aligned}\Phi(\theta) &= N\,\frac{\nu-\mu}{\mu\nu}\,(h_1(+\infty) + h_1(-\infty)) \\ &\quad + 2N\left[\frac{h_2(+\infty)}{\mu} - \frac{h_2(-\infty)}{\nu}\right] - \int_0^T e(t)\phi(t+\theta)\,dt, \\ \Psi(\theta) &= N\left(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\right)(h_1(+\infty) - h_1(-\infty)) - \int_0^T e(t)\phi'(t+\theta)\,dt.\end{aligned}$$

Assume

$$\forall\theta \in \mathbb{R}, \quad |\Phi(\theta)| + |\Psi(\theta)| \neq 0.$$

If Ψ *changes sign more than twice on the zeros of* Φ *in* $[0, \tau[$, *then* (23) *has a T-periodic solution.*

As an alternative example, consider now, for $\beta \in [0, 1[$, the equation

$$x'' + \mu x^+ - \nu x^- + e(t)\frac{|x|^\beta x}{|x| + 1} = 0. \tag{24}$$

By Corollary 4, we have the following.

Corollary 7. *Let* $e : \mathbb{R} \to \mathbb{R}$ *be continuous and T-periodic. Then,*

$$\Phi(\theta) = \int_0^T e(t)|\phi(t + \theta)|^{\beta+1}\, dt,$$

and $\Psi = (\beta + 1)^{-1}\Phi'$. *Assume* Φ *only has simple zeros, precisely* 2ζ *of them, in the interval* $[0, \tau[$. *Then, for every sufficiently large disk* B_R,

$$\deg(\mathcal{P} - \mathrm{Id}, B_R) = 1 - \zeta.$$

In particular, if $\zeta \neq 1$, *Eq.* (24) *has a T-periodic solution.*

5. The case $\beta = 1$

In this section, we consider the case when f is asymptotically positively homogeneous of degree 1. It is necessary to assume that f is, in a certain sense, small enough; we will therefore replace Eq. (1) by

$$J\dot{u} = \nabla H(u) + \varepsilon f(t, u), \tag{25}$$

where $\varepsilon \neq 0$ is a small parameter. As in Section 3, we define the set Σ, which we recall is a made of m rays starting from the origin, and we assume that, for some $\alpha > 0$,

$$\|f(t, u)\| \leqslant \alpha(\|u\| + 1),$$

for every $t \in \mathbb{R}$ and $u \in \mathbb{R}^2$, and that there exists a continuous function $F : \mathbb{R} \times (\mathbb{R}^2 \setminus \Sigma) \to \mathbb{R}^2$ such that

$$F(t,u) = \lim_{\lambda \to +\infty} \frac{f(t, \lambda u)}{\lambda},$$

the above limit being uniform with respect to (t, u) when u varies in compact subsets of $\mathbb{R}^2 \setminus \Sigma$.

Let us define the functions Φ and Ψ as in Section 3, and denote by $\mathcal{P}_\varepsilon$ the Poincaré map associated to system (25) for the period T.

Theorem 3. *Assume that*

$$\forall \theta \in \mathbb{R}, \quad |\Phi(\theta)| + |\Psi(\theta)| \neq 0. \tag{26}$$

If δ and ε are both sufficiently small, then

$$\begin{aligned} \deg(\mathcal{P}_\varepsilon - \mathrm{Id}, \Omega_\delta) &= 1 - \deg(\Phi, \;]a, a+\tau[\cap \{\Psi > 0\}) \\ &= 1 + \deg(\Phi, \;]a, a+\tau[\cap \{\Psi < 0\}), \end{aligned}$$

where a is chosen so that $\Phi(a) \neq 0$.

The proof follows the same lines as that of Theorem 2 and will be omitted here, for brevity. The same remark made for Theorem 2 applies here: the set Ω_δ can be replaced by any sufficiently large disk B_R. Let us state the analogues of Corollaries 1–4 in this situation.

Corollary 8. *Assume that Φ never vanishes on $\{\Psi > 0\}$ or on $\{\Psi < 0\}$. Then, Eq.* (25) *has a T-periodic solution, provided that ε is small enough.*

Corollary 9. *Assume that Φ never vanishes, or Ψ never vanishes. Then, Eq.* (25) *has a T-periodic solution, provided that ε is small enough.*

Corollary 10. *Assume that Φ is differentiable and there exist $c_1, c_2 \in \mathbb{R}$, with $c_1 > 0$, for which $\Psi = c_1 \Phi' + c_2$, and*

$$\forall \theta \in \mathbb{R}, \quad |\Phi(\theta)| + |\Psi(\theta)| \neq 0.$$

If Ψ changes sign more than twice on the zeros of Φ in $[0, \tau[$, then, (25) *has a T-periodic solution, provided that ε is small enough.*

Corollary 11. *Assume that Φ is differentiable and there exists $c_1 > 0$, for which $\Psi = c_1 \Phi'$, and that Φ only has simple zeros, precisely 2ζ of them, in the interval*

$[0, \tau[$. *If* δ *and* ε *are both sufficiently small, then*

$$\deg(\mathcal{P}_\varepsilon - \mathrm{Id}, \Omega_\delta) = 1 - \zeta.$$

As a simple example, we may consider a linear perturbation, by taking $f(t, u) = \mathbb{B}(t)u$, with $\mathbb{B}(t)$ a continuous and T-periodic 2×2 matrix. Then,

$$\Phi(\theta) = \int_0^T \langle \mathbb{B}(t)\varphi(t+\theta) \mid \varphi(t+\theta)\rangle \, dt,$$

$$\Psi(\theta) = \int_0^T \langle \mathbb{B}(t)\varphi(t+\theta) \mid \dot\varphi(t+\theta)\rangle \, dt.$$

Assuming (26), Theorem 3 tells us that the degree is well defined: by the positive homogeneity of the problem, $\deg(\mathcal{P}_\varepsilon - \mathrm{Id}, \Omega_\delta)$ does not depend on δ, and the only possible T-periodic solution of

$$J\dot u = \nabla H(u) + \varepsilon \mathbb{B}(t)u$$

is the trivial one, when ε is small enough.

Writing $\mathbb{B}(t) = \mathbb{A}(t) + \gamma(t)J$, with $\mathbb{A}(t)$ symmetric, we have

$$\Psi(\theta) = \frac{1}{2}\,\Phi'(\theta) - \int_0^T \gamma(t)\, dt,$$

so that Corollaries 8–10 easily apply. When $\gamma \equiv 0$, the symmetric case, by Corollary 11 we have the following.

Corollary 12. *If* $f(t, u) = \mathbb{A}(t)u$, *where* $\mathbb{A}(t)$ *is a symmetric* 2×2 *matrix, then* $\Psi = \frac{1}{2}\Phi'$. *Let* Φ *only have simple zeros, precisely* 2ζ *of them, in the interval* $[0, \tau[$. *Then, for* ε *small enough,* $\deg(\mathcal{P}_\varepsilon - \mathrm{Id}, \Omega) = 1 - \zeta$.

Example. Consider the scalar equation

$$x'' + \mu x^+ - \nu x^- + \varepsilon e(t)x = 0,$$

with $\mu > 0$, $\nu > 0$, for which

$$\Phi(\theta) = \int_0^T e(t)|\phi(t+\theta)|^2 \, dt,$$

with ϕ defined in (21), with minimal period $\tau = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}$. If $T = \tau = 2\pi$ and $e(t) = \cos(nt)$, n being a positive integer, then

$$\Phi(\theta) = \cos(n\theta)\int_0^{2\pi} \cos(nt)\phi^2(t)\, dt + \sin(n\theta)\int_0^{2\pi} \sin(nt)\phi^2(t)\, dt.$$

When $\mu \neq n^2/4$ and $\nu \neq n^2/4$, we compute

$$\int_0^{2\pi} \cos(nt)\phi^2(t)\,dt = \frac{8}{n}\,\sin\left(\frac{n\pi}{\sqrt{\mu}}\right)\frac{\nu-\mu}{(4\mu-n^2)(4\nu-n^2)}$$

and

$$\int_0^{2\pi} \sin(nt)\phi^2(t)\,dt = \frac{8}{n}\left(1-\cos\left(\frac{n\pi}{\sqrt{\mu}}\right)\right)\frac{\nu-\mu}{(4\mu-n^2)(4\nu-n^2)}.$$

If $\mu \neq \nu$, and $\frac{n}{2\sqrt{\mu}} \notin \mathbb{N}$, then Φ has exactly $2n$ simple zeros in the interval $[0, 2\pi[$, and $\deg(\mathcal{P}_\varepsilon - \mathrm{Id}, \Omega) = 1 - n$, for ε small enough. (The condition $\frac{n}{2\sqrt{\mu}} \notin \mathbb{N}$ is certainly verified when $n = 1$, or $n = 2$ and $\mu \neq \nu$.)

Note that, in the linear case $\mu = \nu = 1$, one has $\phi(t) = \sin t$ and Φ is identically zero unless $n = 2$, in which case it has four zeros in $[0, 2\pi[$, leading to $\deg(\mathcal{P}_\varepsilon - \mathrm{Id}, \Omega) = -1$.

Let us briefly mention that, when $f(t, u) = \mathbb{B}u$, $\mathbb{B}$ being a constant 2×2 matrix, both Φ and Ψ are constant, given by

$$\Phi(\theta) = \int_0^T \langle \mathbb{A}\varphi(t) \mid \varphi(t)\rangle\,dt, \qquad \Psi(\theta) = -\gamma T.$$

Unless they are both zero, Corollary 9 can thus be applied. Note that, if $\gamma < 0$, the system is actually dissipative, the function $H(u) + \frac{1}{2}\langle \mathbb{A}u, u\rangle$ being decreasing along trajectories.

6. The second-order approximation of the Poincaré map

For simplicity, in this section we consider system (1) with f independent of u, i.e.

$$J\dot{u} = \nabla H(u) + f(t), \tag{27}$$

with the same assumptions on H as in Section 3. The function $f : \mathbb{R} \to \mathbb{R}^2$ is assumed to be continuous and T-periodic. Following the approach developed there, the change of variable $v = \delta u$ gives

$$J\dot{v} = \nabla H(v) + \delta f(t). \tag{28}$$

Using the same notations, we denote by $\mathcal{P}$ and $\tilde{\mathcal{P}}_\delta$ the Poincaré maps associated to (27) and (28), respectively, and we write, for any $\theta_0 \in [0, \tau]$,

$$\tilde{\mathcal{P}}_\delta(\varphi(\theta_0) = r_1\varphi(\theta_1).$$

By the further change of variable $v(t) = r(t)\varphi(t+\theta(t))$, we have

$$\begin{cases} \theta' = \dfrac{\delta}{r}\langle f(t) \mid \varphi(t+\theta)\rangle, \\ r' = -\delta\langle f(t) \mid \dot\varphi(t+\theta)\rangle, \end{cases} \tag{29}$$

as long as $r(t) > 0$. Let $\chi_0 = (\theta_0, 1)$, with $\theta_0 \in [0, \tau]$, and denote by $\chi(t;\theta_0;\delta) = (\theta(t;\theta_0;\delta), r(t;\theta_0;\delta))$ the solution of (29) with starting point $\chi(0;\theta_0;\delta) = \chi_0 = (\theta_0, 1)$:

$$\theta(0;\theta_0;\delta) = \theta_0 \in [0,\tau], \qquad r(0;\theta_0;\delta) = 1.$$

We use the following notation: for a function $\mathcal{G}(t;\theta_0;\delta)$, we write

$$\mathcal{G}(t;\theta_0;\delta) = o(\delta^n),$$

if

$$\lim_{\delta\to 0^+} \frac{\mathcal{G}(t;\theta_0;\delta)}{\delta^n} = 0,$$

uniformly in $t \in [0, T]$ and $\theta_0 \in [0, \tau]$.

Let us write (29) as

$$\dot\chi = \delta\mathcal{F}(t, \chi).$$

Then $\chi(t;\chi_0;0) = \chi_0$, for every $t \in [0, T]$, and

$$\lim_{\delta\to 0} \frac{\chi(t;\chi_0;\delta) - \chi_0}{\delta} = \lim_{\delta\to 0} \frac{1}{\delta}\int_0^t \delta\mathcal{F}(s, \chi(s;\chi_0;\delta))\,ds = \int_0^t \mathcal{F}(s,\chi_0)\,ds,$$

uniformly in $t \in [0, T]$ and $\chi_0 = (\theta_0, 1)$, with $\theta_0 \in [0, \tau]$. Hence,

$$\chi(t;\chi_0;\delta) = \chi_0 + \delta\int_0^t \mathcal{F}(s,\chi_0)\,ds + o(\delta),$$

so that, as already proved in Lemma 2 (notice that, here, $\beta = 0$),

$$\begin{cases} \theta_1 = \theta_0 + \delta\Phi(\theta_0) + o(\delta), \\ r_1 = 1 - \delta\Psi(\theta_0) + o(\delta). \end{cases} \tag{30}$$

We would like to extend formulas (30) to the second order. We have

$$\begin{aligned}\lim_{\delta\to 0}\frac{\chi(t;\chi_0;\delta)-\chi_0-\delta\int_0^t\mathcal{F}(s,\chi_0)\,ds}{\delta^2} &= \lim_{\delta\to 0}\frac{\int_0^t\mathcal{F}(s,\chi(s;\chi_0;\delta))\,ds-\int_0^t\mathcal{F}(s,\chi_0)\,ds}{\delta}\\ &= \int_0^t d\mathcal{F}(s,\chi_0)\left(\int_0^s\mathcal{F}(\sigma,\chi_0)\,d\sigma\right)ds,\end{aligned}$$

uniformly in $t\in[0,T]$ and $\chi_0=(\theta_0,1)$, with $\theta_0\in[0,\tau]$, where $d\mathcal{F}$ stands for the differential of $\mathcal{F}$ in the second variable. So, we can write

$$\chi(t;\chi_0;\delta)=\chi_0+\delta\int_0^t\mathcal{F}(s,\chi_0)\,ds+\delta^2\int_0^t d\mathcal{F}(s,\chi_0)\left(\int_0^s\mathcal{F}(\sigma,\chi_0)\,d\sigma\right)ds+o(\delta^2).$$

Recall now that, since ∇H is Lipschitz continuous, φ is twice differentiable almost everywhere. For $h=(h_1,h_2)$, writing $\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2)$, we have

$$\begin{aligned}d\mathcal{F}(s,\chi_0)h &= \left(\frac{\partial\mathcal{F}_1}{\partial\theta}(s,\chi_0)h_1+\frac{\partial\mathcal{F}_1}{\partial r}(s,\chi_0)h_2,\ \frac{\partial\mathcal{F}_2}{\partial\theta}(s,\chi_0)h_1+\frac{\partial\mathcal{F}_2}{\partial r}(s,\chi_0)h_2\right)\\ &= \big(\langle f(s)\mid\dot\varphi(s+\theta_0)\rangle h_1-\langle f(s)\mid\varphi(s+\theta_0)\rangle h_2,\ -\langle f(s)\mid\ddot\varphi(s+\theta_0)\rangle h_1\big),\end{aligned}$$

for almost every $s\in[0,T]$. Hence, with $h=\int_0^s\mathcal{F}(\sigma,\chi_0)\,d\sigma$, i.e.

$$h_1=\int_0^s\langle f(\sigma)\mid\varphi(\sigma+\theta_0\rangle\,d\sigma,\qquad h_2=-\int_0^s\langle f(\sigma)\mid\dot\varphi(\sigma+\theta_0\rangle\,d\sigma,$$

we find

$$\begin{aligned}\theta(t;\theta_0;\delta) &= \theta_0+\delta\int_0^t\langle f(s)\mid\varphi(s+\theta_0\rangle\,ds\\ &\quad+\delta^2\int_0^t\Bigg[\langle f(s)\mid\dot\varphi(s+\theta_0)\rangle\left(\int_0^s\langle f(\sigma)\mid\varphi(\sigma+\theta_0)\rangle\,d\sigma\right)\\ &\quad+\langle f(s)\mid\varphi(s+\theta_0)\rangle\left(\int_0^s\langle f(\sigma)\mid\dot\varphi(\sigma+\theta_0)\rangle\,d\sigma\right)\Bigg]ds+o(\delta^2)\\ &= \theta_0+\delta\int_0^t\langle f(s)\mid\varphi(s+\theta_0\rangle\,ds\\ &\quad+\delta^2\int_0^t\frac{d}{ds}\Bigg[\int_0^s\langle f(\sigma)\mid\dot\varphi(\sigma+\theta_0)\rangle\,d\sigma\\ &\quad\times\int_0^s\langle f(\sigma)\mid\varphi(\sigma+\theta_0)\rangle\,d\sigma\Bigg]ds+o(\delta^2)\end{aligned}$$

$$= \theta_0 + \delta \int_0^t \langle f(s) \mid \varphi(s+\theta_0)\rangle \, ds$$
$$+\delta^2 \int_0^t \langle f(s) \mid \dot\varphi(s+\theta_0)\rangle \, ds \cdot \int_0^t \langle f(s) \mid \varphi(s+\theta_0)\rangle \, ds + o(\delta^2)$$

and

$$r(t;\theta_0;\delta) = 1 - \delta \int_0^t \langle f(s) \mid \dot\varphi(s+\theta_0)\rangle \, ds$$
$$-\delta^2 \int_0^t \langle f(s) \mid \ddot\varphi(s+\theta_0)\rangle \left(\int_0^s \langle f(\sigma) \mid \varphi(\sigma+\theta_0)\rangle \, d\sigma \right) ds + o(\delta^2).$$

Let us define the new function

$$\Xi(\theta) = \int_0^T \langle f(s) \mid \ddot\varphi(s+\theta)\rangle \left(\int_0^s \langle f(\sigma) \mid \varphi(\sigma+\theta)\rangle \, d\sigma \right) ds,$$

so that we can finally write

$$\begin{cases} \theta_1 = \theta_0 + \delta\Phi(\theta_0) + \delta^2\Phi(\theta_0)\Phi'(\theta_0) + o(\delta^2), \\ r_1 = 1 - \delta\Phi'(\theta_0) - \delta^2\Xi(\theta_0) + o(\delta^2). \end{cases} \tag{31}$$

Theorem 4. *Assume that*

$$\forall\theta \in [0,\tau], \quad |\Phi(\theta)| + |\Phi'(\theta)| + |\Xi(\theta)| \neq 0. \tag{32}$$

Let Φ *change sign only a finite number of times in the interval* $[0,\tau]$, *precisely* 2ζ *times. Consider the set*

$$Z = \{\theta \in \mathbb{R} : \Phi(\theta) = \Phi'(\theta) = 0\},$$

and assume there exists a $\gamma > 0$ *such that*

$$0 < d(\theta, Z) < \gamma \quad \Rightarrow \quad \Phi'(\theta)\,\Xi(\theta) > 0. \tag{33}$$

Then, for every sufficiently small δ,

$$\deg(\mathcal{P} - \mathrm{Id}, \Omega_\delta) = 1 - \zeta.$$

Remarks. (1) For any $\alpha < \beta$, the set $Z \cap [\alpha,\beta]$ is made of a finite union of closed disjoint intervals, each of them possibly reduced to a point.

(2) Assumption (33) excludes the possibility for Φ to have a strict local maximum or minimum with value 0.

(3) The same remark made for Theorem 2 applies here, as well: the set Ω_δ can be replaced by any sufficiently large disk B_R.

Proof. First of all, we claim that there exists a $C > 0$ and a $\bar{\delta}_1 > 0$ such that

$$0 < \delta < \bar{\delta}_1 \quad \Rightarrow \quad \forall \theta \in \mathbb{R}, \quad |\Phi(\theta)| + |\Phi'(\theta) + \delta \Xi(\theta)| \geqslant C\delta. \tag{34}$$

By contradiction, let $(\delta_n)_n$ and $(\theta_n)_n$ be such that $\delta_n > 0$, $\delta_n \to 0$, $\theta_n \in [0, \tau]$ and

$$|\Phi(\theta_n)| + |\Phi'(\theta_n) + \delta_n \Xi(\theta_n)| \leqslant \frac{\delta_n}{n}.$$

For a subsequence, we can assume that $\theta_n \to \bar{\theta} \in [0, \tau]$. Then, $\Phi(\bar{\theta}) = \Phi'(\bar{\theta}) = 0$, i.e. $\bar{\theta} \in Z$. On the other hand, for n sufficiently large, $d(\theta_n, Z) < \gamma$, so that, by (33), either $\theta_n \in Z$, or $\Phi'(\theta_n)$ and $\Xi(\theta_n)$ have the same sign. In any case, $|\Xi(\theta_n)| \leqslant 1/n$, and hence $\Xi(\bar{\theta}) = 0$, a contradiction with (32) which proves our claim.

Recalling (31), let us write

$$\begin{cases} \theta_1 = \theta_0 + \delta\Phi(\theta_0) + \delta^2\Phi(\theta_0)\Phi'(\theta_0) + R_1(\theta_0, \delta), \\ r_1 = 1 - \delta\Psi(\theta_0) - \delta^2\Xi(\theta_0) + R_2(\theta_0, \delta), \end{cases}$$

where R_1 and R_2 are such that

$$\lim_{\delta \to 0^+} \frac{R_1(\theta_0, \delta)}{\delta^2} = \lim_{\delta \to 0^+} \frac{R_2(\theta_0, \delta)}{\delta^2} = 0,$$

uniformly for $\theta_0 \in [0, \tau]$. To compute the degree of $\tilde{\mathcal{P}}_\delta - \mathrm{Id}$ on the set Ω, we study its rotation number on the curve $\partial\Omega$.

Let $\lambda \in [0, 1]$ and consider the functions $P_{\delta,\lambda} : \partial\Omega \to \mathbb{R}^2$ such that

$$P_{\delta,\lambda}(\varphi_\delta(\theta_0)) = r_1^\lambda \varphi_\delta(\theta_1^\lambda),$$

with

$$\begin{cases} \theta_1^\lambda = \theta_0 + \delta\Phi(\theta_0) + \lambda[\delta^2\Phi(\theta_0)\Phi'(\theta_0) + R_1(\theta_0, \delta)], \\ r_1^\lambda = 1 - \delta\Phi'(\theta_0) - \delta^2\Xi(\theta_0) + \lambda R_2(\theta_0, \delta). \end{cases}$$

Let us prove that, for δ sufficiently small, $P_{\delta,\lambda}$ has no fixed points in $\partial\Omega$. By contradiction, let $(\delta_n)_n$ and $(\theta_{0,n})_n$ be such that $\delta_n > 0$, $\delta_n \to 0$, $\theta_{0,n} \in [0, \tau]$ and

$$\begin{cases} \delta_n \Phi(\theta_{0,n}) + \lambda_n [\delta_n^2 \Phi(\theta_{0,n}) \Phi'(\theta_{0,n}) + R_1(\theta_{0,n}, \delta_n)] = 0, \\ -\delta_n \Phi'(\theta_{0,n}) - \delta_n^2 \Xi(\theta_{0,n}) + \lambda_n R_2(\theta_{0,n}, \delta_n) = 0. \end{cases}$$

Then, for n large enough, $\delta_n |\Phi'(\theta_{0,n})| \leqslant \frac{1}{2}$, so that, by (34),

$$\begin{aligned} 2|R_1(\theta_{0,n}, \delta_n)| + |R_2(\theta_{0,n}, \delta_n)| &\geqslant 2|\lambda_n R_1(\theta_{0,n}, \delta_n)| + |\lambda_n R_2(\theta_{0,n}, \delta_n)| \\ &\geqslant \delta_n (|\Phi(\theta_{0,n})| + |\Phi'(\theta_{0,n}) + \delta_n \Xi(\theta_{0,n})|) \\ &\geqslant C\delta_n^2, \end{aligned}$$

in contradiction with the above. Hence, there is a $\bar{\delta}_2 \in]0, \bar{\delta}_1]$ such that

$$0 < \delta < \bar{\delta}_2 \quad \Rightarrow \quad \mathrm{rot}(P_{\delta,1} - \mathrm{Id}, \partial\Omega) = \mathrm{rot}(P_{\delta,0} - \mathrm{Id}, \partial\Omega).$$

We concentrate now on $P_{\delta,0} : \partial\Omega \to \mathbb{R}^2$, and extend it to a continuous function $P : \bar{\Omega} \to \mathbb{R}^2$, so that

$$P(\varphi(\theta_0)) = R(\theta_0)\varphi(\theta_0 + \Theta(\theta_0)),$$

with $R(\theta_0) = 1 - \delta\Phi'(\theta_0) - \delta^2 \Xi(\theta_0)$ and $\Theta(\theta_0) = \delta\Phi(\theta_0)$. There is a $\bar{\delta}_3 \in]0, \bar{\delta}_2]$ such that, for $\delta \leqslant \bar{\delta}_3$, the assumptions of Theorem 1 are verified, so that

$$\begin{aligned} \deg(P - \mathrm{Id}, \Omega) &= 1 - \deg(\Theta, \ ]a, a + \tau[\cap \{R < 1\}) \\ &= 1 - \deg(\Phi, \ ]a, a + \tau[\cap \{\Phi' + \delta\Xi > 0\}), \end{aligned}$$

where a is chosen so that $\Phi(a) \neq 0$.

After noticing that the simple zeros of Φ are in finite number, let $\bar{\delta}_4 \in]0, \bar{\delta}_3]$ be such that, for every such simple zero $\bar{\theta}$,

$$\bar{\delta}_4 \left| \Xi(\bar{\theta}) \right| < |\Phi'(\bar{\theta})|.$$

For $\delta \leqslant \bar{\delta}_4$, each time Φ attains a zero $\hat{\theta}$ in the set $\{\Phi' + \delta\Xi > 0\}$, either it is a simple zero, and then $\Phi'(\hat{\theta}) > 0$, or $\hat{\theta} \in Z$ and $\Xi(\hat{\theta}) > 0$. In any case, because of (33), we can find $\hat{\theta}_1$ and $\hat{\theta}_2$ such that $\hat{\theta}_1 \leqslant \hat{\theta} \leqslant \hat{\theta}_2$ so that, if $\theta \in [\hat{\theta}_1, \hat{\theta}_2]$, then $\Phi(\theta) = 0$, and, for some $\gamma > 0$,

$$\begin{aligned} \theta \in]\hat{\theta}_1 - \gamma, \hat{\theta}_1[\quad &\Rightarrow \quad \Phi(\theta) < 0, \\ \theta \in]\hat{\theta}_2, \hat{\theta}_2 + \gamma[\quad &\Rightarrow \quad \Phi(\theta) > 0. \end{aligned}$$

Conversely, it can be shown that, each time Φ goes from negative to positive values, it has a zero contained in the set $\{\Phi' + \delta\Xi > 0\}$. Hence, $\deg(\Phi, \,]a, a+\tau[\cap \{\Phi' + \delta\Xi > 0\})$ is exactly the number of times that Φ changes sign from negative to positive in the interval $]a, a+\tau[$, i.e. precisely ζ, thus concluding the proof. $\square$

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