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Positively homogeneous hamiltonian systems in the plane ☆

Alessandro Fonda

Dipartimento di Scienze Matematiche, Universitá di Trieste, P. le Europa 1, Italy

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Abstract

I try to give a general description of the dynamics of the solutions for a planar hamiltonian system with positively homogeneous hamiltonian function and periodic forcing term. Most of the results obtained are already known in the special case of a scalar second-order differential equation with asymmetric nonlinearity.

Keywords: Periodic; Bounded and unbounded solutions; Hamiltonian systems

1. Description of the main results

Consider the hamiltonian system

$$J\dot{u} = \nabla H(u) + f(t), \tag{1}$$

where the hamiltonian function $H : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 and the forcing $f : \mathbb{R} \to \mathbb{R}^2$ is continuous and T-periodic. Here, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix.

The hamiltonian function is assumed to be positively homogeneous of degree 2 and positive: for every $u \in \mathbb{R}^2$ and $\lambda > 0$,

$$H(\lambda u) = \lambda^2 H(u), \tag{2}$$

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E-mail address: fondass@univ.trieste.it.

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and

$$\min_{\|u\|=1} H(u) > 0. \tag{3}$$

Moreover, the gradient of H is assumed to be locally Lipschitz continuous (so that, because of (2), ∇H will actually be Lipschitz continuous). In this setting, the solutions of the Cauchy problems associated to (1) are unique and globally defined.

In this situation, the origin is an isochronous center for the autonomous system

$$J\dot{u} = \nabla H(u); \tag{4}$$

all solutions of (4) are periodic with the same minimal period, which will be denoted by τ. The fact that T is an integer multiple of τ or not can be crucial for the existence of periodic solutions to (1), as shown by the two statements below.

Theorem 1. *If*

$$\frac{T}{\tau} \notin \mathbb{N},$$

then (1) *has a T-periodic solution, for every T-periodic forcing f.*

Theorem 2. *If*

$$\frac{T}{\tau} \in \mathbb{N},$$

there exist T-periodic functions f for which all the solutions of (1) *are unbounded: if u is any solution of* (1), *then*

$$\lim_{t\to-\infty} \|u(t)\| = \lim_{t\to+\infty} \|u(t)\| = +\infty.$$

For further investigation, it is convenient to fix a reference solution to the autonomous system (4). Let φ be such a solution for which the hamiltonian function has (constant) value $\frac{1}{2}$: for every $t \in \mathbb{R}$,

$$J\dot{\varphi}(t) = \nabla H(\varphi(t)), \quad H(\varphi(t)) = \frac{1}{2}. \tag{5}$$

Recall that φ has minimal period τ. The function $\Phi : \mathbb{R} \to \mathbb{R}$, defined by

$$\Phi(\theta) = \int_0^T \langle f(t) \mid \varphi(t+\theta) \rangle \, dt,$$

which is continuous and τ-periodic, plays an important role in determining the dynamics of the solutions to (1), as shown by the following two statements. In the first one we need some regularity of the forcing term.

Theorem 3. *Let f be of class C^6. Assume*

$$\frac{T}{\tau} \in \mathbb{N},$$

and that Φ never changes sign. Then, all the solutions of (1) *are bounded: if u is any solution of* (1), *then*

$$\sup_{t \in \mathbb{R}} ||u(t)|| < +\infty.$$

As a consequence of Massera's theorem, assuming that Φ never changes sign implies that (1) has a T-periodic solution. This fact will indeed be proved for continuous f, without further regularity assumptions.

Theorem 4. *Let*

$$\frac{T}{\tau} \in \mathbb{N},$$

and assume that Φ changes sign at least four times in the interval $[0, \tau[$, only having simple zeros. Then (1) *has a T-periodic solution, while all solutions having sufficiently large amplitude are unbounded, either in the past or in the future.*

Being simple, the number of zeros of Φ in $[0, \tau[$ must be even. As will be shown in Section 3, the functions f in Theorem 2 for which (1) has no T-periodic solutions can be constructed in such a way that Φ has exactly two simple zeros in $[0, \tau[$.

Concerning the case

$$\frac{T}{\tau} \in \mathbb{Q},$$

analogues of Theorems 3 and 4 are easily obtained. In fact, if $T = \frac{N}{M}\tau$, for some positive integers M and N, defining $\tilde{T} = MT$ we have that f, being T-periodic, is also $\tilde{T}$-periodic and

$$\frac{\tilde{T}}{\tau} \in \mathbb{N}.$$

Hence, defining

$$\tilde{\Phi}(\theta) = \int_0^{\tilde{T}} \langle f(t) \mid \varphi(t + \theta) \rangle \, dt,$$

one can restate Theorems 3 and 4 in this different situation simply replacing Φ by $\tilde{\Phi}$.

In the complementary case, we have the following.

Theorem 5. *Let f be of class C^6. Assume*

$$\frac{T}{\tau} \notin \mathbb{Q},$$

and that $\int_0^\tau \Phi(\theta)\, d\theta \neq 0$, i.e.

$$\left\langle \int_0^T f \,\middle|\, \int_0^\tau \varphi \right\rangle \neq 0.$$

Then, all the solutions of (1) *are bounded: if u is any solution of* (1), *then*

$$\sup_{t \in \mathbb{R}} \|u(t)\| < +\infty.$$

One could wonder whether, in this setting, the T-periodic solutions are unique or not. Concerning the situation of Theorem 1, we have a partial answer in the negative. Consider system (1) in the form

$$J\dot{u} = \nabla H(u) + v + e(t), \tag{6}$$

where $e : \mathbb{R} \to \mathbb{R}^2$ is continuous and T-periodic and $v \in \mathbb{R}^2$ is a vector to be fixed.

Theorem 6. *Assume H is C^2 at a certain point w_0, with positive definite hessian $H''(w_0)$, and define*

$$\sigma = \frac{2\pi}{\sqrt{\det H''(w_0)}}.$$

Set $v_0 = -\nabla H(w_0)$ and let m and n be integers such that

$$m < \frac{T}{\sigma} < m + 1, \quad n < \frac{T}{\tau} < n + 1. \tag{7}$$

There is a $R_0 > 0$ such that, if $v = Rv_0$ with $R \geqslant R_0$, then the number of T-periodic solutions of (6) *is at least $2|n - m| + 1$.*

All the results stated in this section hold true if we replace $J\dot{u}$ in (1) by $-J\dot{u}$. Equivalently, considering (1), it is possible to assume, instead of (3), that H be negative, i.e.

$$\max_{\|u\|=1} H(u) < 0,$$

by only taking, in Theorem 6, the hessian $H''(w_0)$ to be negative definite, as well, and setting $v_0 = \nabla H(w_0)$ instead of $-\nabla H(w_0)$.

In Section 3, a complementary situation will be analyzed, where H is neither positive nor negative.

2. Examples and remarks

Examples of hamiltonian functions verifying (2) and (3) are easily found by defining H on the unit circle S^1. Polar coordinates can be useful in this case. For instance, setting $H(e^{i\alpha}) = 1 + \frac{1}{2}\sin(4\alpha)$ gives

$$H(x,y) = x^2 + y^2 + \frac{2xy(x^2 - y^2)}{x^2 + y^2}.$$

This function is of class C^2 except at the origin, where $H(0,0) = 0$. The orbits for the autonomous system (4) are visualized in Fig. 1.

An interesting case appears when H is only piecewise C^2 on S^1. For instance, let

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} = \alpha_1 + 2\pi,$$

and $A_1, A_2, \ldots, A_n$ be symmetric positive definite matrices such that

$$\alpha \in [\alpha_j, \alpha_{j+1}] \quad \Rightarrow \quad H(e^{i\alpha}) = \frac{1}{2} \langle A_j e^{i\alpha} \mid e^{i\alpha} \rangle,$$

for $j = 1, 2, \ldots, n$ (the matrices A_j are chosen in such a way that $\nabla H : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous). In this case, ∇H is piecewise linear and, if u is a solution of the autonomous system (4), writing $u(t) = r(t)e^{i\alpha(t)}$, we have

$$\alpha(t) \in [\alpha_j, \alpha_{j+1}] \quad \Rightarrow \quad -\dot{\alpha}(t) = \langle A_j e^{i\alpha(t)} \mid e^{i\alpha(t)} \rangle.$$

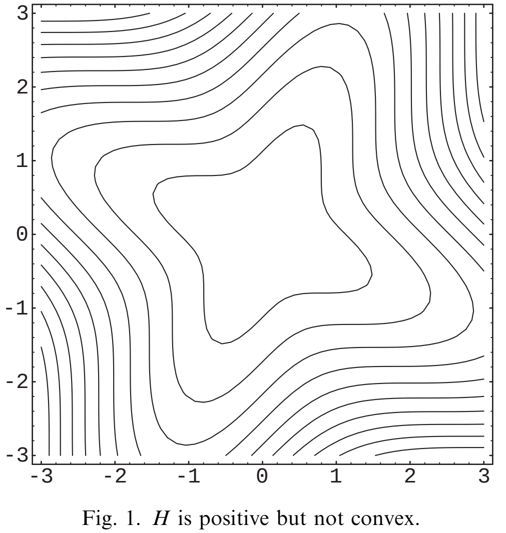

Fig. 1. H is positive but not convex.

Hence,

$$\tau = \sum_{j=1}^{n} \int_{\alpha_j}^{\alpha_{j+1}} \frac{d\alpha}{\langle A_j e^{i\alpha} \mid e^{i\alpha} \rangle}.$$

As a typical special case, let $n = 2$, $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \frac{3\pi}{2}$ and

$$A_1 = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix},$$

with $\mu > 0$, $\nu > 0$. System (1), with $f = (f_1, f_2)$, then reads

$$\begin{cases} -y' = \mu x^+ - \nu x^- + f_1(t), \\ x' = y + f_2(t), \end{cases} \tag{8}$$

where $x^+ = \max\{x, 0\}$ is the positive part of x and $x^- = \max\{-x, 0\}$ is its negative part. Taking $f_2(t) = 0$ and setting $p(t) = -f_1(t)$, system (8) is equivalent to the classical second-order equation

$$x'' + \mu x^+ - \nu x^- = p(t). \tag{9}$$

In this case, one easily computes

$$\tau = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}.$$

Theorem 1 then refers to the pioneering existence results by Dancer [4] and Fučík [11] for Eq. (9).

Concerning the nonexistence of periodic solutions in the setting of Theorem 2, Dancer [5] was the first to find, for Eq. (9), an example of function p, taken as the characteristic function of a small interval, extended periodically. The same kind of idea was developed by Ortega [19] for a second-order equation with an isochronous center. Theorem 2 extends to system (1) those ideas. Different examples of nonexistence of periodic solutions were found for equation (9) by Lazer and McKenna [13] and Wang [20].

Theorem 3 generalizes a result by Liu [14], obtained for Eq. (9), which completed previous work of Ortega [15]. We are in a Landesman–Lazer type of situation (see e.g. [8]). The proof consists in finding arbitrarily large invariant curves for the Poincaré map using a variant of the Small Twist Theorem due to Ortega [16]. The presence of such invariant curves in this case implies the existence of subharmonic solutions and quasi-periodic solutions (see [15]).

Concerning Theorem 4, the existence part was proved for Eq. (9) by Fabry and Fonda [9], showing that the topological degree associated to the problem is related to the number of zeros of the function Φ. In fact, if Φ only has simple zeros and 2ζ is their number in the interval $[0, \tau[$, then the degree on large balls is exactly $1 - \zeta$. The unboundedness part was proved for Eq. (9) by Alonso and Ortega [1].

Theorem 5 generalizes a result by Ortega [18]. As in Theorem 3, the proof consists in finding arbitrarily large invariant curves for the Poincaré map in the phase plane.

The situation in Theorem 6 was first considered by Lazer and McKenna [12], and later by Del Pino et al. [6], for equations like

$$x'' + g(x) = s + p(t),$$

where $g : \mathbb{R} \to \mathbb{R}$ is differentiable and such that

$$\lim_{x\to+\infty} g'(x) = \mu > 0, \quad \lim_{x\to-\infty} g'(x) = \nu > 0.$$

Like in [6], the proof of Theorem 6 uses a version of the Poincaré–Birkhoff Theorem formulated in [14].

As another example of application let $n = 4$, $\alpha_1 = 0$, $\alpha_2 = \frac{\pi}{2}$, $\alpha_3 = \pi$, $\alpha_4 = \frac{3\pi}{2}$ and

$$A_1 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \quad A_2 = \begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix}, \quad A_3 = \begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}, \quad A_4 = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix},$$

with $\mu\nu > 0$. In this case, system (1) can be written as

$$J\dot{u} = \mu u^+ - \nu u^- + f(t),$$

where, being $u = (x, y)$, one has $u^+ = (x^+, y^+)$ and $u^- = (x^-, y^-)$. In this case, one computes

$$\tau = \frac{\pi}{2}\left(\frac{1}{\sqrt{|\mu|}} + \frac{1}{\sqrt{|\nu|}}\right)^2.$$

The results stated in Section 1 can be easily adapted to this situation.

As a more general situation, we can introduce a symmetric matrix $A = \begin{pmatrix} b & a \\ a & c \end{pmatrix}$ and consider the system

$$J\dot{u} = -Au + \mu u^+ - \nu u^- + f(t), \tag{10}$$

so that now

$$A_1 = \begin{pmatrix} \mu - b & -a \\ -a & \mu - c \end{pmatrix}, \quad A_2 = \begin{pmatrix} \nu - b & -a \\ -a & \mu - c \end{pmatrix},$$
$$A_3 = \begin{pmatrix} \nu - b & -a \\ -a & \nu - c \end{pmatrix}, \quad A_4 = \begin{pmatrix} \mu - b & -a \\ -a & \nu - c \end{pmatrix}.$$

Assume $\det A_j > 0$ for every $j = 1, \ldots, 4$, i.e.

$$a^2 < \min\{(\mu - b)(\mu - c), (\nu - b)(\mu - c), (\mu - b)(\nu - c), (\nu - b)(\nu - c)\}.$$

In this case, because of the symmetries in the homogeneous equation, we can write

$$\tau = \sum_{j=1}^{4} \frac{\pi}{2\sqrt{\det A_j}}.$$

For example, if $a = 0$, one has

$$\tau = \frac{\pi}{2}\left(\frac{1}{\sqrt{|\mu - b|}} + \frac{1}{\sqrt{|\nu - b|}}\right)\left(\frac{1}{\sqrt{|\mu - c|}} + \frac{1}{\sqrt{|\nu - c|}}\right).$$

Again, the results of Section 1 apply. The hamiltonian function is strictly convex. The orbits for the autonomous system (4) are visualized in Fig. 2: they are obtained by glueing together four pieces of ellipses.

3. Multiplicity of periodic solutions: a complementary situation

In the context of system (10), we can deal with a different situation which was studied by Fonda and Ortega [10] for second-order systems: let λ_1, λ_2 be the eigenvalues of A, and assume

$$\nu < \lambda_1 \leqslant \lambda_2 < \mu. \tag{11}$$

In this case, the hamiltonian function,

$$H(u) = \frac{1}{2}(-\langle Au \mid u\rangle + \mu||u^+||^2 + \nu||u^-||^2), \tag{12}$$

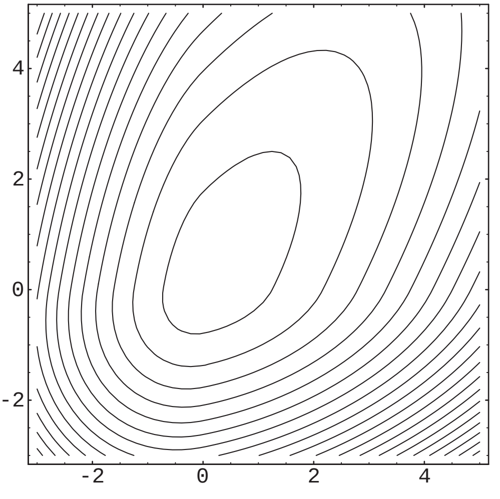

Fig. 2. Here $a = 1$, $b = 2$, $c = 3$, $\mu = 4$, $\nu = 8$.

is neither positive nor negative definite. We call this a situation of Ambrosetti–Prodi type (cf. [2]). For convenience, similarly as for Theorem 6, we write (10) in the form

$$J\dot{u} = -Au + \mu u^+ - \nu u^- + v + e(t), \tag{13}$$

where $e : \mathbb{R} \to \mathbb{R}^2$ is continuous and T-periodic and $v \in \mathbb{R}^2$ is a vector to be fixed.

Recalling that $A = \begin{pmatrix} b & a \\ a & c \end{pmatrix}$, define the cones $\mathscr{S}_1$ and $\mathscr{S}_2$ as follows: if $a = 0$, the two coincide with the first quadrant; otherwise,

$$\mathscr{S}_1 = \left\{ (x, y) \in \mathbb{R}^2 : \frac{a}{b-\mu} x \leqslant y \leqslant \frac{c-\mu}{a} x \right\},$$
$$\mathscr{S}_2 = \left\{ (x, y) \in \mathbb{R}^2 : \frac{a}{b-\nu} x \leqslant y \leqslant \frac{c-\nu}{a} x \right\}.$$

If $a < 0$, then $\mathscr{S}_1 \subseteq \mathscr{S}_2$; on the contrary, if $a > 0$, then $\mathscr{S}_2 \subseteq \mathscr{S}_1$. Assume, for definiteness, $a \leqslant 0$. As shown in [10], if $-v_0$ belongs to $\mathring{\mathscr{S}}_1$, equation

$$-Au + \mu u^+ - \nu u^- + v_0 = 0 \tag{14}$$

has exactly four solutions $w_1, \dots, w_4$, each belonging to a different quadrant of $\mathbb{R}^2$. We have $\det H''(w_j) \neq 0$, for every $j = 1, \dots, 4$. The hessian of H is positive definite at w_1, negative definite at w_3, while at w_2 and at w_4 it is neither positive nor negative definite. If $-v_0 \in \mathring{\mathscr{S}}_2 \backslash \mathscr{S}_1$, Eq. (14) has exactly two solutions, while if $-v_0 \notin \mathscr{S}_2$, it has no solutions at all.

This situation leads to the following result, where each of the solutions of (14) generates a T-periodic solution of (13).

Theorem 7. *Assume*

$$\lambda_2 - \frac{2\pi}{T} < \nu < \lambda_1 \leqslant \lambda_2 < \mu < \lambda_1 + \frac{2\pi}{T}.$$

Taking $-v_0 \in \mathring{\mathscr{S}}_1$ (resp. $-v_0 \in \mathring{\mathscr{S}}_2 \backslash \mathscr{S}_1$ or $-v_0 \notin \mathscr{S}_2$), there is a $R_0 > 0$ such that, if $v = Rv_0$ with $R \geqslant R_0$, then system (13) has exactly four (resp. exactly two or zero) T-periodic solutions.

As an illustration, the orbits for the system

$$J\dot{u} = -Au + \mu u^+ - \nu u^- + v_0,$$

with $-v_0 \in \mathring{\mathscr{S}}_1$, (i.e. (13) with $e(t) \equiv 0$) are visualized in Fig. 3.

When further interaction with the eigenvalues of the differential operator occurs, more periodic solutions can appear.

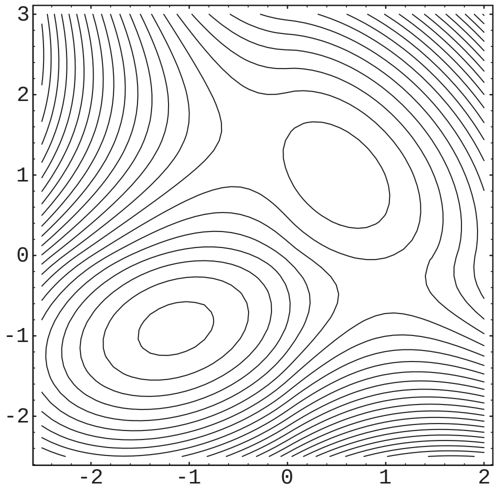

Fig. 3. Here $a = -1$, $b = 2$, $c = 3$, $\mu = 5$, $\nu = -1$ and $v_0 = (-5, -5)$.

Theorem 8. *Assume* (11) *and take* $-v_0 \in \mathring{\mathscr{S}}_1$. *Define, for* $j = 1$ *and* 3,

$$\sigma_j = \frac{2\pi}{\sqrt{\det H''(w_j)}},$$

and let m_j *be integers such that*

$$m_j < \frac{T}{\sigma_j} < m_j + 1. \tag{15}$$

There is a R_0 *such that, if* $v = Rv_0$ *with* $R \geqslant R_0$, *the number of* T*-periodic solutions of* (13) *is at least* $2(m_1 + m_3) + 4$.

4. Proofs

In this section we prove the results stated in Sections 1 and 3.

4.1. Preliminary remarks and Theorem 1

Since H is positively homogeneous of degree 2, by Euler's Identity, for every $u \in \mathbb{R}^2$,

$$\langle \nabla H(u) \mid u \rangle = 2H(u).$$

Moreover, H being positive, the degree of ∇H on any ball $B_R = \{u \in \mathbb{R}^2 : ||u|| < R\}$ is

$$\deg(\nabla H, B_R, 0) = 1.$$

The matrix J is invertible and $J^{-1} = J^T = -J$. Theorem 1 is thus a direct consequence of Corollary 6 in [3].

The solutions of the autonomous system (4) have star-shaped orbits which surround the origin and rotate clockwise. Their orbits can be distinguished by the energy H which, by (2) and (3), is coercive:

$$\lim_{||u|| \to \infty} H(u) = +\infty .$$

Choosing φ satisfying (5), all the solutions of the autonomous equation (4) can be written as $u(t) = \bar{\rho}\varphi(t + \bar{\theta})$, for some $\bar{\rho} \geqslant 0$ and $\bar{\theta} \in \mathbb{R}$.

For any solution u of (1), if $u(t) \neq (0,0)$, we can define $(\rho(t), \theta(t))$, with $\rho(t) > 0$, such that

$$u(t) = \rho(t)\varphi(t + \theta(t)).$$

Being the map $(\rho, \Theta) \mapsto \rho\varphi(\Theta)$ a diffeomorphism from the half-plane $\{\rho > 0\}$ to $\mathbb{R}^2 \backslash \{(0,0)\}$, the functions ρ, θ are of class C^1, as far as $u(t)$ does not cross the origin, and substitution into (1) gives

$$\dot{\rho} J\varphi(t + \theta) + \rho\dot{\theta} J\dot{\varphi}(t + \theta) = f(t). \tag{16}$$

By (5) and Euler's Identity, a scalar product in (16) with $\varphi(t + \theta)$ yields

$$\rho\dot{\theta} = \langle f(t) \mid \varphi(t + \theta) \rangle,$$

while a scalar product with $\dot{\varphi}(t + \theta)$ yields

$$\dot{\rho} = -\langle f(t) \mid \dot{\varphi}(t + \theta) \rangle.$$

The search of solutions which never pass through the origin will thus lead to the system

$$\begin{cases} \dot{\theta} = \dfrac{1}{\rho} \langle f(t) \mid \varphi(t + \theta) \rangle, \\ \dot{\rho} = -\langle f(t) \mid \dot{\varphi}(t + \theta) \rangle. \end{cases} \tag{$\odot$}$$

4.2. Unbounded solutions: Theorem 2

Fix $v \in \mathbb{R}^2$ with $||v|| = 1$ and consider the impulse differential system

$$J\dot{u} = \nabla H(u) + \delta_\sharp(t) Jv, \tag{17}$$

where $\delta_\sharp$ is the distribution defined as

$$\langle \delta_\sharp, \phi \rangle = \sum_{n \in \mathbb{Z}} \phi(nT),$$

for every C^∞-function ϕ with compact support. A solution of (17), in the sense of distributions, is a function $u \in L^1_{\text{loc}}$ which is C^2 on $\mathbb{R} \backslash \{nT : n \in \mathbb{Z}\}$, and such that

$$\begin{cases} J\dot{u} = \nabla H(u), & \text{for every} \quad t \in \mathbb{R} \backslash \{nT : n \in \mathbb{Z}\}, \\ u(nT+) = u(nT-) + v. \end{cases} \tag{18}$$

Let $F : \mathbb{R} \to \mathbb{R}$ be defined as

$$F(t) = n - \frac{t}{T}, \quad \text{when} \quad t \in]nT, (n+1)T],$$

for any $n \in \mathbb{Z}$, so that $F'(t) = \delta_\sharp(t) - \frac{1}{T}$. The change of variable $w = u - F(t)v$ transforms (17) into

$$J\dot{w} = \nabla H(w + F(t)v) + \frac{1}{T} Jv, \tag{19}$$

which is a Carathéodory type equation. In order to deal with a classical equation, let $G : \mathbb{R} \to \mathbb{R}$ be a smooth T-periodic function such that

$$\int_0^T |F(t) - G(t)| \, dt \leqslant \frac{1}{2L} e^{-LT},$$

where $L > 0$ is the Lipschitz constant for ∇H. Let z be a solution of

$$J\dot{z} = \nabla H(z + G(t)v) + \frac{1}{T} Jv, \tag{20}$$

and w be a solution of (19) such that $w(0) = z(0)$. Then, for $t > 0$,

$$\begin{aligned} ||z(t) - w(t)|| &= \left|\left| \int_0^t (\dot{z}(s) - \dot{w}(s)) \, ds \right|\right| \\ &\leqslant \int_0^t ||\dot{z}(s) - \dot{w}(s)|| \, ds \\ &= \int_0^t ||\nabla H(z(s) + G(s)v) - \nabla H(w(s) + F(s)v)|| \, ds \\ &\leqslant L \int_0^t (||z(s) - w(s)|| + |F(s) - G(s)|) \, ds \\ &\leqslant L \int_0^t ||z(s) - w(s)|| \, ds + \frac{1}{2} e^{-LT}. \end{aligned}$$

By Gronwall's inequality,

$$||z(t) - w(t)|| \leqslant \frac{1}{2} e^{L(t-T)}. \tag{21}$$

Compare the Poincaré map associated to (17), (19) and (20). Since $T = N\tau$ for some integer N, in the interval of time $[0, T]$ a solution of (17) rotates N times around the origin and then "jumps" in the direction of v (see (18)). So, the Poincaré map for (17) is just a translation by v. Since

$$w(T) = u(T) - F(T)v = (u(0) + v) + v = w(0) + v,$$

the Poincaré map for (19) is the same as for (17). On the other hand, by (21),

$$\begin{aligned}\langle z(T) \,|\, v \rangle &= \langle w(T) \,|\, v \rangle + \langle z(T) - w(T) \,|\, v \rangle \\ &\geqslant \langle w(0) \,|\, v \rangle + 1 - \|z(T) - w(T)\| \\ &\geqslant \langle z(0) \,|\, v \rangle + \frac{1}{2}.\end{aligned}$$

This implies, by iteration,

$$\|z(nT)\| \geqslant \langle z(nT) \,|\, v \rangle \geqslant \langle z(0) \,|\, v \rangle + \frac{n}{2},$$

and hence, being ∇H Lipschitz continuous,

$$\lim_{t \to +\infty} \|z(t)\| = +\infty.$$

Setting $u(t) = z(t) + G(t)v$ we are back to system (1) with $f(t) = G'(t) + \frac{1}{T}$, and we have

$$\lim_{t \to +\infty} \|u(t)\| = +\infty.$$

Similarly one proves the unboundedness in the past, and the proof is thus completed.

Remark. For system (17) we formally have

$$\begin{aligned}\Phi(\theta) &= \int_0^T \langle \delta_\sharp(t) Jv \,|\, \varphi(t + \theta) \rangle \, dt \\ &= \int_{-T/2}^{T/2} \delta_\sharp(t) \langle Jv \,|\, \varphi(t + \theta) \rangle \, dt \\ &= \langle Jv \,|\, \varphi(\theta) \rangle,\end{aligned}$$

so that, being the orbit of φ star-shaped, the function Φ has exactly two simple zeros in $[0, \tau[$. The same will thus be true for our "smoothed" system.

4.3. Bounded solutions: Theorem 3

We want to prove that the Poincaré map associated to $(\odot)$ has an infinite sequence of arbitrarily large invariant curves (cf. [14–17]).

Let us first transform $(\odot)$ into a hamiltonian system by taking $r(t) = (\rho(t))^2$ and $s(t) = t + \theta(t)$. We have

$$\begin{cases} \dot{s} = 1 + \dfrac{1}{\sqrt{r}} \langle f(t) \mid \varphi(s) \rangle = \dfrac{\partial h}{\partial r}(t, r, s), \\ \dot{r} = -2\sqrt{r} \langle f(t) \mid \dot{\varphi}(s) \rangle = -\dfrac{\partial h}{\partial s}(t, r, s), \end{cases} \tag{22}$$

with

$$h(t, r, s) = r + 2\sqrt{r} \langle f(t) \mid \varphi(s) \rangle.$$

It is useful to consider the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(t, s) = \langle f(t) \mid \varphi(s) \rangle.$$

Let $\delta > 0$ be a small parameter, to be precised later. Set

$$\mathscr{H}(\sigma, p, q) = \left[\sqrt{\delta^2 F(q, \sigma)^2 + p} - \delta F(q, \sigma) \right]^2,$$

and consider the hamiltonian system

$$\begin{cases} \dot{q} = \dfrac{\partial \mathscr{H}}{\partial p}(\sigma, p, q) = 1 - \delta \dfrac{F(q, \sigma)}{\sqrt{\delta^2 F(q, \sigma)^2 + p}}, \\ \dot{p} = -\dfrac{\partial \mathscr{H}}{\partial q}(\sigma, p, q) = 2\delta \partial_1 F(q, \sigma) \dfrac{\left[\sqrt{\delta^2 F(q, \sigma)^2 + p} - \delta F(q, \sigma) \right]^2}{\sqrt{\delta^2 F(q, \sigma)^2 + p}}. \end{cases} \tag{23}$$

Notice that, for δ small enough, if $(q(\sigma), p(\sigma))$ is a solution of (23), then q is invertible on $[0, T]$. Letting s to be the inverse of q, i.e.

$$s(t) = \sigma \quad \Leftrightarrow \quad q(\sigma) = t,$$

and defining r as

$$r(t) = \frac{1}{\delta^2} \mathscr{H}(s(t), p(s(t)), t), \tag{24}$$

computation shows that $(s(t), r(t))$ is a solution of (22). Thus, if we are able to find an invariant curve for the Poincaré map of (23), for δ small enough, we also have an invariant curve for the Poincaré map of (22), and consequently for $(\odot)$. As a final change of variable, let $z(\sigma) = 1/p(\sigma)$. Then, (23) becomes

$$\begin{cases} \dot{q} = 1 - \delta \dfrac{F(q,\sigma)}{\sqrt{\delta^2 F(q,\sigma)^2 + 1/z}}, \\ \dot{z} = -2\delta \partial_1 F(q,\sigma) z^2 \dfrac{\left[\sqrt{\delta^2 F(q,\sigma)^2 + 1/z} - \delta F(q,\sigma)\right]^2}{\sqrt{\delta^2 F(q,\sigma)^2 + 1/z}}. \end{cases} \tag{25}$$

Even if (25) is not a hamiltonian system any more, it still has the intersection property (cf. [16]). The assumption that Φ never changes sign will permit us to use a variant of the Small Twist Theorem due to Ortega [16]. Let $0 < m := \min|\Phi| \leqslant \max|\Phi| := M$, and define the following numbers:

$$a = 1, \quad \tilde{a} = \left(\frac{2M}{m}\right)^2, \quad \tilde{b} = \left(\frac{2M}{m}\right)^4, \quad b = \left(\frac{2M}{m}\right)^6.$$

Denote by $\chi(\sigma; q_0, z_0; \delta) = (q(\sigma; q_0, z_0; \delta), z(\sigma; q_0, z_0; \delta))$ the solution of (23) with starting point $\chi(0; q_0, z_0; \delta) = (q_0, z_0)$ in $[0, T] \times [a, b]$:

$$q(0; q_0, z_0; \delta) = q_0 \in [0, T], \quad z(0; q_0, z_0; \delta) = z_0 \in [a, b].$$

Moreover, let us use the following notation: for a function $g(\sigma; q_0, z_0; \delta)$ we write

$$g(\sigma; q_0, z_0; \delta) = o_n(\delta),$$

if it is of class C^n in (q_0, z_0) and, for $j + k \leqslant n$,

$$\lim_{\delta \to 0} \frac{1}{\delta} \left(\frac{\partial^{j+k} g}{\partial q_0^j \partial z_0^k} (\sigma; q_0, z_0; \delta) \right) = 0,$$

uniformly in $\sigma, q_0 \in [0, T]$ and $z_0 \in [a, b]$.

Since f is of class C^6, from (25) we have

$$\chi(\sigma; q_0, z_0; \delta) = \chi(\sigma; q_0, z_0; 0) + \delta \frac{\partial \chi}{\partial \delta}(\sigma; q_0, z_0; 0) + o_5(\delta) \tag{26}$$

(cf. Proposition 11 in [17]). From (25),

$$\dot{q}(\sigma; q_0, z_0; 0) = 1, \quad \dot{z}(\sigma; q_0, z_0; 0) = 0,$$

so that

$$q(\sigma; q_0, z_0; 0) = q_0 + \sigma, \quad z(\sigma; q_0, z_0; 0) = z_0. \tag{27}$$

Let $\frac{\partial\chi}{\partial\delta}(\sigma; q_0, z_0; 0) = (\xi(\sigma; q_0, z_0), \eta(\sigma; q_0, z_0))$. Being

$$\frac{\partial\chi}{\partial\delta}(\sigma; q_0, z_0; 0) = \lim_{\delta\to 0} \frac{1}{\delta}(\chi(\sigma; q_0, z_0; \delta) - \chi(\sigma; q_0, z_0; 0)),$$

from (25) and (27) we have

$$\begin{aligned}\dot{\xi}(\sigma; q_0, z_0) &= \lim_{\delta\to 0} \frac{1}{\delta}\left(1 - \delta \frac{F(q(\sigma; q_0, z_0; \delta), \sigma)}{\sqrt{\delta^2 F(q(\sigma; q_0, z_0; \delta), \sigma)^2 + 1/z(\sigma; q_0, z_0; \delta)}} - 1\right) \\ &= -F(q_0 + \sigma, \sigma)\sqrt{z_0}.\end{aligned}$$

Similarly,

$$\dot{\eta}(\sigma; q_0, z_0) = -2\partial_1 F(q_0 + \sigma, \sigma)\sqrt{z_0^3}.$$

Since $\chi(0; q_0, z_0; \delta)$ is constant with respect to δ, we have

$$\xi(0; q_0, z_0) = 0, \quad \eta(0; q_0, z_0) = 0.$$

Integrating,

$$\begin{aligned}\xi(\sigma; q_0, z_0) &= -\sqrt{z_0}\int_0^\sigma F(q_0 + s, s)\, ds, \\ \eta(\sigma; q_0, z_0) &= -2\sqrt{z_0^3}\int_0^\sigma \partial_1 F(q_0 + s, s)\, ds.\end{aligned}$$

By (26) we then have

$$q(T; q_0, z_0; \delta) = q_0 + T + \delta\ell_1(q_0, z_0) + o_5(\delta), \tag{28}$$

$$z(T; q_0, z_0; \delta) = z_0 + \delta\ell_2(q_0, z_0) + o_5(\delta), \tag{29}$$

where

$$\ell_1(q_0, z_0) = -\sqrt{z_0}\Phi(-q_0), \quad \ell_2(q_0, z_0) = 2\sqrt{z_0^3}\Phi'(-q_0).$$

Setting

$$I(q_0, z_0) = \frac{\sqrt{z_0}}{|\Phi(-q_0)|},$$

one has that ℓ_1 and I are of class C^6, ℓ_2 is of class C^5, they are all τ-periodic in q_0, $T = N\tau$ for some integer N,

$$\ell_1(q_0, z_0)\frac{\partial\ell_1}{\partial z_0}(q_0, z_0) > 0, \quad \frac{\partial I}{\partial z_0}(q_0, z_0) > 0,$$

and

$$\ell_1(q_0,z_0)\frac{\partial I}{\partial q_0}(q_0,z_0)+\ell_2(q_0,z_0)\frac{\partial I}{\partial z_0}(q_0,z_0)=0.$$

Defining

$$\bar{I}(z_0):=\max_{q_0\in\mathbb{R}} I(q_0,z_0)=\frac{\sqrt{z_0}}{m},\quad \underline{I}(z_0):=\min_{q_0\in\mathbb{R}} I(q_0,z_0)=\frac{\sqrt{z_0}}{M},$$

one has

$$a<\tilde{a}<\tilde{b}<b,\quad \bar{I}(a)<\underline{I}(\tilde{a})\leqslant\bar{I}(\tilde{a})<\underline{I}(\tilde{b})\leqslant\bar{I}(\tilde{b})<\underline{I}(b),$$

and Ortega's Theorem applies: For every δ small enough we have an invariant curve for the Poincaré map

$$(q_0,z_0)\mapsto\chi(T;q_0,z_0;\delta),$$

which lies in $T^1\times[a,b]$, where $T^1=\mathbb{R}/T\mathbb{Z}$, which is homotopic to the circle $z_0=$ constant.

Consider the sequence $(\delta_n)_n$ with $\delta_n=\frac{1}{n}$. For every sufficiently large n, we have an invariant curve of the Poincaré map above with $\delta=\delta_n$ and, correspondingly, an invariant curve for the Poincaré map of (22), as above.

So, we have a sequence of invariant curves $\mathscr{C}_n$ for the Poincaré map of (1), as well. Notice that, even if some of these curves may coincide, one sees from (24) that their amplitudes necessarily go to infinity with n. Each curve $\mathscr{C}_n$ is then the base of a time-periodic and flow-invariant cylinder in the extended phase space $(t,u)\in\mathbb{R}\times\mathbb{R}^2$, which confines the solutions in its interior.

4.4. Coexistence of periodic and unbounded solutions: Theorem 4

Let us introduce $\delta>0$, a small parameter, and make the change of variable $r(t)=\delta\rho(t)$ in system ($\odot$). We then have

$$\begin{cases}\dot{\theta}=\dfrac{\delta}{r}\langle f(t)\mid\varphi(t+\theta)\rangle,\\ \dot{r}=-\delta\langle f(t)\mid\dot{\varphi}(t+\theta)\rangle.\end{cases}\tag{30}$$

Let $\chi_0=(\theta_0,r_0)$ in $[0,T]\times[1,2]$ and denote by $\chi(t;\chi_0;\delta)=(\theta(t;\theta_0,r_0;\delta),r(t;\theta_0,r_0;\delta))$ the solution of (30) with starting point $\chi(0;\chi_0;\delta)=\chi_0$:

$$\theta(0;\theta_0,r_0;\delta)=\theta_0\in[0,T],\quad r(0;\theta_0,r_0;\delta)=r_0\in[1,2].$$

Moreover, let us use the following notation: for a function $g(t;\theta_0,r_0;\delta)$ we write

$$g(t;\theta_0,r_0;\delta)=o_0(\delta),$$

if it is continuous in (θ_0, r_0) and

$$\lim_{\delta\to 0} \frac{1}{\delta} g(t;\theta_0, r_0;\delta) = 0,$$

uniformly in $t, \theta_0 \in [0, T]$ and $r_0 \in [1, 2]$.

Let us write (30) as

$$\dot{\chi} = \delta \mathscr{F}(t, \chi).$$

Then $\chi(t;\chi_0;0) = \chi_0$, for every $t \in [0, T]$, and

$$\lim_{\delta\to 0} \frac{\chi(t;\chi_0;\delta) - \chi_0}{\delta} = \lim_{\delta\to 0} \frac{1}{\delta} \int_0^t \delta \mathscr{F}(s, \chi(s;\chi_0;\delta))\, ds = \int_0^t \mathscr{F}(s, \chi_0)\, ds,$$

uniformly in $t \in [0, T]$ and $\chi_0 = (\theta_0, r_0) \in [0, T] \times [1, 2]$. Hence,

$$\chi(t;\chi_0;\delta) = \chi_0 + \delta \int_0^t \mathscr{F}(s, \chi_0)\, ds + o_0(\delta),$$

an analogue of (26), from which we get

$$\theta(T;\theta_0, r_0;\delta) = \theta_0 + \frac{\delta}{r_0} \Phi(\theta_0) + o_0(\delta), \tag{31}$$

$$r(T;\theta_0, r_0;\delta) = r_0 - \delta \Phi'(\theta_0) + o_0(\delta), \tag{32}$$

the analogues of (28) and (29). Consider the Poincaré map $\mathscr{P}$ for (1), which associates to every point $u_0 = \frac{1}{\delta} r_0 \varphi(\theta_0)$ the point

$$\mathscr{P}(u_0) = \frac{1}{\delta} r(T;\theta_0, r_0;\delta) \varphi(\theta(T;\theta_0, r_0;\delta)).$$

We want to compute the degree of $\mathscr{P} - \mathrm{Id}$, where Id denotes the identity map in $\mathbb{R}^2$. Fix $r_0 \in [1, 2]$ and define the corresponding function

$$\mathscr{V}_\delta(\theta_0) = r(T;\theta_0, r_0;\delta) \varphi(\theta(T;\theta_0, r_0;\delta)) - r_0 \varphi(\theta_0).$$

From (31) and (32) we can see that

$$\begin{aligned}
\mathscr{V}_\delta(\theta_0) &= (r_0 - \delta \Phi'(\theta_0)) \varphi\left(\theta_0 + \frac{\delta}{r_0} \Phi(\theta_0)\right) - r_0 \varphi(\theta_0) + o_0(\delta) \\
&= (r_0 - \delta \Phi'(\theta_0)) \left(\varphi(\theta_0) + \frac{\delta}{r_0} \Phi(\theta_0) \dot{\varphi}(\theta_0)\right) - r_0 \varphi(\theta_0) + o_0(\delta) \\
&= \delta [\Phi(\theta_0) \dot{\varphi}(\theta_0) - \Phi'(\theta_0) \varphi(\theta_0)] + o_0(\delta).
\end{aligned}$$

Define the comparison function

$$\mathscr{W}(\theta_0) = \Phi(\theta_0)\dot{\varphi}(\theta_0) - \Phi'(\theta_0)\varphi(\theta_0),$$

for which we have seen that

$$\mathscr{V}_\delta(\theta_0) = \delta\mathscr{W}(\theta_0) + o_0(\delta). \tag{33}$$

Notice that

$$\langle \mathscr{W}(\theta_0) \mid J\varphi(\theta_0) \rangle = -\Phi(\theta_0), \tag{34}$$

$$\langle \mathscr{W}(\theta_0) \mid J\dot{\varphi}(\theta_0) \rangle = -\Phi'(\theta_0). \tag{35}$$

As a consequence of the fact that Φ only has simple zeros, we have that $\mathscr{W}(\theta_0) \neq (0,0)$, for every $\theta_0 \in [0, T]$. Thus, we can compute the number of rotations of $\mathscr{W}$ while θ_0 varies in $[0, T]$. This can be done visually. Imagine standing at the origin and looking in the direction of $\varphi(\theta_0)$. Then, $J\varphi(\theta_0)$ is directed to our left. If $\Phi(\theta_0) > 0$, then by (34) we have $\mathscr{W}(\theta_0)$ to the right. Increasing θ_0, it can happen that, at a certain point, $\Phi(\theta_0) = 0$, and consequently $\Phi'(\theta_0) < 0$. Then, since $\langle J\dot{\varphi}(\theta_0) \mid \varphi(\theta_0) \rangle = 1$, by (5) and Euler's Identity we have that $J\dot{\varphi}(\theta_0)$ stays in our front region and hence, by (35), $\mathscr{W}(\theta_0)$ is exactly in front of us. Increasing some more θ_0 we have that $\mathscr{W}(\theta_0)$ passes to the left. In this way, if Φ has 2ζ zeros in the interval $[0, \tau[$, in this time $\mathscr{W}$ will turn around us exactly ζ times counterclockwise. Since in the same time we ourselves turn clockwise exactly once, we may conclude that the number of clockwise rotations of $\mathscr{W}$ while θ_0 varies in $[0, T]$ is exactly $1 - \zeta$.

By (33), Rouché's Theorem can be applied so that, taking δ small enough, $\mathscr{V}_\delta$ rotates the same number of times as $\mathscr{W}$ while θ_0 varies in $[0, T]$, for every $r_0 \in [1, 2]$. We conclude that the topological degree of $\mathscr{P} - \mathrm{Id}$ is $1 - \zeta$. Since $\zeta \neq 1$, this proves the existence part of Theorem 4.

Notice that, if $\zeta = 0$ (the Landesman–Lazer type of situation), the degree is 1. This shows that, in the situation of Theorem 3, (1) has a T-periodic solution, even if f is only assumed to be continuous.

Concerning the unboundedness part, let $s(t) = t + \theta(t)$ in $(\odot)$, so that

$$\begin{cases} \dot{s} = 1 + \dfrac{1}{\rho} \langle f(t) \mid \varphi(s) \rangle, \\ \dot{\rho} = -\langle f(t) \mid \dot{\varphi}(s) \rangle. \end{cases} \tag{36}$$

Denote by $(s(t; s_0, \rho_0; \delta), \rho(t; s_0, \rho_0; \delta))$ the solution of (36) with starting point $s(0; s_0, \rho_0; \delta) = s_0$, $\rho(0; s_0, \rho_0; \delta) = \rho_0$. By (31) and (32),

$$s(T; s_0, \rho_0; \delta) = s_0 + T + \frac{1}{\rho_0} \Phi(s_0) + o_0(\delta),$$
$$\rho(T; s_0, \rho_0; \delta) = \rho_0 - \Phi'(s_0) + o_0(\delta),$$

so that Proposition 3.1 in [1] directly applies. (Notice that, here, we could also admit the function Φ to have 2 simple zeros in the interval $[0, \tau[$.)

4.5. The irrational case: Theorem 5

The proof goes exactly as the one of Theorem 3, so to obtain (28) and (29). At this point, since $T \notin \tau\mathbb{Q}$ and

$$\int_0^\tau \frac{\partial \ell_1}{\partial z_0}(q_0, z_0)\, dq_0 = -\frac{1}{2\sqrt{z_0}} \int_0^\tau \Phi(-q_0)\, dq_0 \neq 0,$$

for every $z_0 \in [a, b]$, we can apply directly Theorem 1 in [18] to get an invariant curve for the Poincaré map in $T^1 \times [a, b]$, homotopic to a circle, when δ is sufficiently small. The conclusion then follows as for Theorem 3.

4.6. Multiplicity of periodic solutions: Theorem 6

In the case $m = n$, one simply applies Theorem 1 to get the existence of at least one T-periodic solution to (6).

Let $v = Rv_0$ and set $\varepsilon = \frac{1}{R}$; by the change of variable $z(t) = \varepsilon u(t)$, Eq. (6) becomes

$$J\dot{z} = \nabla H(z) + v_0 + \varepsilon e(t). \tag{37}$$

Notice that w_0 is an equilibrium point for (37) when $\varepsilon = 0$. The solutions of the linearized equation,

$$J\dot{u} = H''(w_0)u,$$

are periodic with minimal period σ. Since T is not a multiple of σ, equation (37) has, for ε small enough, a T-periodic solution $z_\varepsilon(t)$ whose orbit lies near w_0.

With the further change of variable $\xi(t) = z(t) - z_\varepsilon(t)$, (37) becomes

$$J\dot{\xi} = \nabla H(\xi + z_\varepsilon(t)) - \nabla H(z_\varepsilon(t)). \tag{38}$$

Now the origin is an equilibrium point and the linearized equation is

$$J\dot{u} = H''(z_\varepsilon(t))u.$$

By (7), being $z_\varepsilon(t)$ near w_0, in the time T the solutions of (38) starting near the origin rotate clockwise around it more than m times, and less than $m+1$ times (by uniqueness, the solutions starting away from the origin never cross it).

Going back to (37), setting $z(t) = \rho(t)\varphi(t + \theta(t))$ leads to ($\odot$), with $f(t) = v_0 + \varepsilon e(t)$. The further change of variable $r(t) = \delta\rho(t)$ leads to (30), so that, for δ small enough, $\theta(t)$ and $r(t)$ are nearly constant. By (7) we conclude that, in the time T, the solutions of (37) starting sufficiently far away from the origin rotate clockwise around it less than $n+1$ times, and more than n times. The same is true for the solutions of (38), for δ small enough, $z_\varepsilon(t)$ being bounded.

Assume $n \geqslant m+1$. The Poincaré–Birkhoff Theorem as in [7] can thus be applied: for any $k \in \{m+1, m+2, \dots, n\}$ there are two T-periodic solutions of (38) which, in the time T, rotate clockwise around the origin exactly k times. These give us $2(n-m)$ supplementary T-periodic solutions of (37) to be added to the one, $z_\varepsilon(t)$, we already found.

In case $m \geqslant n+1$, we have the same type of conclusion: for any $k \in \{n+1, n+2, \dots, m\}$ there are two T-periodic solutions of (38) which, in the time T, rotate clockwise around the origin exactly k times. Hence, in both cases, the number of T-periodic solutions is at least $2|n-m|+1$.

4.7. The Ambrosetti–Prodi situation: Theorems 7 and 8

The proof of Theorem 7 is a simple adaptation of the one in [10] (see Theorems 4.1 and 5.1).

In order to prove Theorem 8, we proceed as in the previous section. First write (13) in the form (37), with H as in (12). Recall that the points $w_1, \dots, w_4$ are equilibria when $\varepsilon = 0$, and that $H''(w_1)$ is positive definite, $H''(w_3)$ is negative definite, while $H''(w_2)$ and $H''(w_4)$ are neither positive nor negative definite.

Consider first w_1, find a T-periodic solution $z_\varepsilon^1(t)$ near w_1 and make the change of variable $\xi(t) = z(t) - z_\varepsilon^1(t)$, to obtain (38), with z_ε replaced by z_ε^1. By (15), the solutions of (38) starting near the origin rotate clockwise around it more than m_1 times, and less than m_1+1 times.

The difference from the previous section is that now the solutions of (37) starting sufficiently far from the origin cannot turn around it, neither clockwise nor counterclockwise. Indeed, the homogeneous equation

$$J\dot{u} = -Au + \mu u^+ - \nu u^-,$$

with $A = \begin{pmatrix} b & a \\ a & c \end{pmatrix}$ and $u = \begin{pmatrix} x \\ y \end{pmatrix}$, has two half-lines as separatrices:

$$y = \frac{a - \sqrt{a^2 - (\nu - b)(\mu - c)}}{\mu - c} x, \quad \text{for } x < 0,$$
$$y = \frac{a + \sqrt{a^2 - (\mu - b)(\nu - c)}}{\nu - c} x, \quad \text{for } x > 0,$$

both with negative slope. Going back to (37), setting $z(t) = \frac{1}{\delta} r(t)\varphi(t + \theta(t))$ leads to (30), so that, for δ small enough, $\theta(t)$ and $r(t)$ are nearly constant. Hence, in the time T, the solutions of (37) starting sufficiently far away from the origin will not be able to rotate around it, in either direction. The same is true for the solutions of (38), for δ small enough, being $z_\varepsilon^1(t)$ bounded.

The Poincaré–Birkhoff Theorem as in [14] can thus be applied: for any $k \in \{1, 2, \ldots, m_1\}$ there are two T-periodic solutions of (38) which, in the time T, rotate clockwise around the origin exactly k times. These give us $2m_1$ supplementary T-periodic solutions of (37), to be added to the one, $z_\varepsilon^1(t)$, we already found.

Consider now w_3 and proceed in the same way as above: find a T-periodic solution $z_\varepsilon^3(t)$ near w_3 and make the change of variable $\xi(t) = z(t) - z_\varepsilon^3(t)$, to obtain (38), with z_ε replaced by z_ε^3. By (15), the solutions of (38) starting near the origin rotate around it more than m_3 times, and less than $m_3 + 1$ times, but now counterclockwise instead of clockwise.

The Poincaré–Birkhoff Theorem then gives us $2m_3$ supplementary T-periodic solutions of (37), to be added to the one, $z_\varepsilon^3(t)$, we already found.

Concerning w_2 and w_4, we still have the perturbed T-periodic solutions near them, but no more, in general. So, in the total, the number of T-periodic solutions is at least $2m_1 + 2m_3 + 4$.

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