



Multiple solutions of positively homogeneous equations[☆]

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1. Introduction

We want to study the multiplicity of periodic solutions to a differential system of the form

$$u'' - Au + \mu u^+ - \nu u^- = v + h(t), \quad (1)$$

where $h: \mathbb{R} \rightarrow \mathbb{R}^N$ is a continuous and T -periodic function, v is a vector to be fixed, μ, ν are real numbers and A is a symmetric $N \times N$ matrix. Given $u \in \mathbb{R}^N$, we write $u = u^+ - u^-$, where u^+ denotes the vector whose components are the positive parts of those of u , and similarly for u^- .

This kind of systems appear, e.g. after discretization in space of partial differential equations like the wave or beam equations. They also provide a mathematical model of a system of coupled oscillators with several springs and stops.

Let $\lambda_1, \lambda_2, \dots, \lambda_N$ be the eigenvalues of A , in increasing order, and assume

$$\nu < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N < \mu. \quad (2)$$

This is a generalized Ambrosetti–Prodi type situation [1]. If $N = 1$, adding a further nonresonance assumption, it can easily be proved that the corresponding equation has

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at least two periodic solutions if v is positive and large enough. In [3], the first author and Ortega studied the case $N = 2$ and proved that, in this situation, there is a cone in \mathbb{R}^2 such that, if v belongs to it and has sufficiently large norm, then Eq. (1) has at least four solutions. Consequently, they raised the question whether in the N -dimensional case one can always find 2^N solutions.

We will show with a counterexample that this is not the case, for any dimension N . For instance, if $N = 3$, there are symmetric matrices for which the number of solutions is at most six.

Besides this, we will concentrate on the case $N = 3$ and find some conditions on the matrix A in order to have the existence of the desired eight solutions. This study is based on elementary geometrical considerations on the intersection of cones in \mathbb{R}^3 .

Although our method could theoretically be extended to higher order systems, we loose intuition on what is going on when $N \geq 4$. For this reason, the problem of finding conditions on A in order to have 2^N solutions remains open.

2. A piecewise linear operator in \mathbb{R}^3

In this section, we assume $N = 3$ and focus our attention on the study of the number of (constant) solutions of the equation

$$F(u) := -Au + \mu u^+ - \nu u^- = v, \tag{3}$$

i.e. the equilibria of (1). The function F is a piecewise linear operator in \mathbb{R}^3 . On each octant Q_i of \mathbb{R}^3 , with $i = 1, \dots, 8$, F is a linear operator. The image $F(Q_i)$ of each octant is a cone in \mathbb{R}^3 and the number of solutions of (3) is the number of those cones to which v belongs. Of course, the number of solutions of (3) is at most eight. We would like to find conditions on the matrix A in order that the number of solutions be maximum.

Let $A = (a_{ij})$ be a symmetric matrix of order 3. If the ordered eigenvalues of the matrix are denoted by $\lambda_1, \lambda_2, \lambda_3$, we assume

$$v < \lambda_1 \leq \lambda_2 \leq \lambda_3 < \mu. \tag{4}$$

Let us define the vectors

$$e_{1+} = (1, 0, 0), \quad e_{1-} = (-1, 0, 0),$$

$$e_{2+} = (0, 1, 0), \quad e_{2-} = (0, -1, 0),$$

$$e_{3+} = (0, 0, 1), \quad e_{3-} = (0, 0, -1),$$

and their images $y_{i\pm} = F(e_{i\pm})$, which concretely are

$$y_{1+} = (-a_{11} + \mu, -a_{12}, -a_{13}), \quad y_{1-} = (a_{11} - \nu, a_{12}, a_{13}),$$

$$y_{2+} = (-a_{12}, -a_{22} + \mu, -a_{23}), \quad y_{2-} = (a_{12}, a_{22} - \nu, a_{23}),$$

$$y_{3+} = (-a_{13}, -a_{23}, -a_{33} + \mu), \quad y_{3-} = (a_{13}, a_{23}, a_{33} - \nu).$$

Given any three vectors w_1, w_2, w_3 , we denote with $C(w_1, w_2, w_3)$ the cone defined by them. Hence, the images by F of the eight octants Q_i are precisely the cones $C(y_{1\pm}, y_{2\pm}, y_{3\pm})$. We now see that under some conditions on the coefficients of A , all these cones have nonempty intersection.

Theorem 1. *Assume (4) and that one of the following conditions holds:*

- (i) $a_{13} = 0$ and $a_{12}a_{23} \geq 0$,
- (ii) $a_{12} = 0$ and $a_{13}a_{23} \geq 0$,
- (iii) $a_{23} = 0$ and $a_{12}a_{13} \geq 0$.

Then, there is an open cone \mathcal{S} such that, if $v \in \mathcal{S}$, Eq. (3) has eight solutions.

Proof. (i) We begin with the easiest cases. First, if $a_{12} = 0 = a_{23}$, then the three equations of the systems are uncoupled. In this case, if all of the components of v are positive, then each one of these equations has two solutions, the combinations of which give us the eight solutions of the system. Hence, in this case, \mathcal{S} is the interior of the first octant.

When $a_{12} = 0$ and $a_{23} > 0$, the first equation of the system is uncoupled and has two solutions whether v has a positive first component. Let A_1 be the matrix obtained eliminating from A the first row and column. As a consequence of the Separation Theorem of the eigenvalues of a matrix (see [6, p. 103] or [5, Theorem 94]) and (4), we have that both eigenvalues of A_1 lie in $]v, \mu[$. So, we are in the situation of [3], where the existence of four solutions was proved for a certain two-dimensional cone. Therefore, in a certain cone \mathcal{S} of \mathbb{R}^3 , the total number of solutions of the whole system is eight. An analogous situation appears when $a_{23} = 0$ and $a_{12} > 0$.

Now, let us assume that $a_{12} > 0$ and $a_{23} > 0$. Again by the Separation Theorem, it is easy to prove that

$$v < a_{ii} < \mu, \quad i = 1, 2, 3$$

and

$$\det(A_2 - \mu I) > 0,$$

where A_2 is the result of the elimination of the second row and column in A . With this information and (4), the sign of the following determinants can be computed:

$$\det(y_{1+}, y_{2+}, y_{3+}) = -\det(A - \mu I) > 0,$$

$$\det(y_{1+}, y_{2+}, y_{2-}) = (\mu - v)[a_{12}a_{13} + a_{23}(\mu - a_{11})] > 0,$$

$$\det(y_{2+}, y_{3+}, y_{2-}) = (\mu - v)[a_{13}a_{23} + a_{12}(\mu - a_{33})] > 0,$$

$$\begin{aligned} \det(y_{1+}, y_{2-}, y_{3+}) &= (a_{22} - v)\det(A_2 - \mu I) + 2a_{12}a_{13}a_{23} \\ &\quad + a_{12}^2(\mu - a_{33}) + a_{23}^2(\mu - a_{11}) > 0. \end{aligned}$$

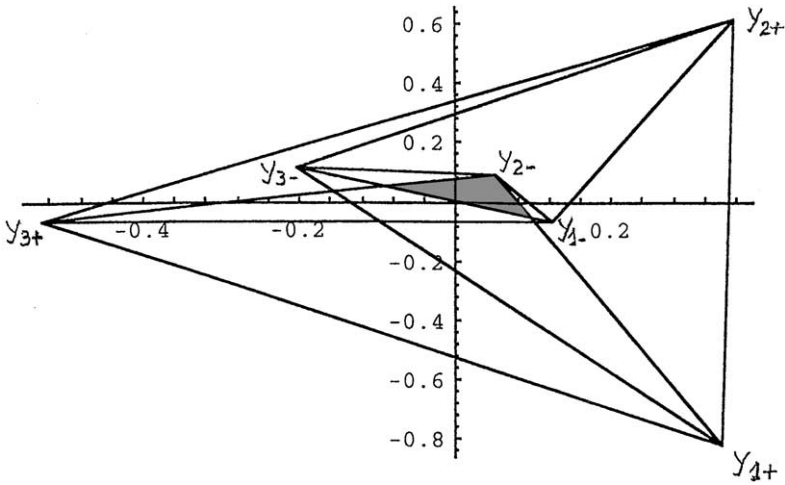


Fig. 1. Intersection of the eight cones.

Thus, the four basis formed by the previous sets of vectors are positively oriented, and moreover $y_{2-} \in C(y_{1+}, y_{2+}, y_{3+})$. In the same way,

$$\det(y_{1+}, y_{2+}, y_{1-}) = (\mu - \nu)[a_{12}a_{23} + a_{13}(\mu - a_{22})] > 0,$$

$$\det(y_{2+}, y_{2-}, y_{1-}) = (\mu - \nu)[a_{23}(a_{11} - \nu) - a_{12}a_{13}] > 0,$$

$$\det(y_{1+}, y_{1-}, y_{2-}) = (\mu - \nu)[a_{12}a_{23} + a_{13}(a_{22} - \nu)] > 0,$$

so $y_{1-} \in C(y_{1+}, y_{2+}, y_{2-})$. Furthermore,

$$\det(y_{2+}, y_{3+}, y_{3-}) = (\mu - \nu)[a_{12}a_{23} + a_{13}(\mu - a_{22})] > 0,$$

$$\det(y_{3+}, y_{2-}, y_{3-}) = (\mu - \nu)[a_{13}(\nu - a_{22}) + a_{12}a_{23}] > 0,$$

$$\det(y_{2+}, y_{3-}, y_{2-}) = (\mu - \nu)[a_{12}(a_{33} - \nu) - a_{13}a_{23}] > 0$$

so that $y_{3-} \in C(y_{2+}, y_{3+}, y_{2-})$. Finally, being

$$\det(y_{2+}, y_{3-}, y_{2-}) > 0,$$

$$\det(y_{2+}, y_{2-}, y_{1-}) > 0,$$

$$\det(y_{1-}, y_{2-}, y_{3-}) = \det(A - \nu I) > 0$$

we have $y_{2-} \in C(y_{2+}, y_{1-}, y_{3-})$. This is sufficient to determine exactly the relative position of the vectors. The situation is clarified if we draw a section of the cones by a common transversal plane (see Fig. 1). In this case, the open cone \mathcal{S} is the interior of

$$C(y_{1+}, y_{2-}, y_{3+}) \cap C(y_{1-}, y_{2-}, y_{3-}).$$

Finally, if $a_{12} < 0$ and $a_{23} < 0$, we define the operator

$$G(u) := F(-u) = -Bu - vu^+ + \mu u^-,$$

where $B = -A$, so that $\sigma(B) = \{-\lambda_1, -\lambda_2, -\lambda_3\}$ and

$$-\mu < -\lambda_3 \leq -\lambda_2 \leq -\lambda_1 < -v.$$

Here $b_{13} = 0$, $b_{12} = -a_{12} > 0$, $b_{23} = -a_{23} > 0$, so we are in the previous situation, but obviously the number of solutions of the equation $G(u) = v$ is the same of the equation $F(u) = v$.

(ii) If $u = (u_1, u_2, u_3)$, we only have to make the change $\tilde{u}_1 = u_1$, $\tilde{u}_2 = u_3$, $\tilde{u}_3 = u_2$ and we are back to case (i).

(iii) As before, the change $\tilde{u}_1 = u_2$, $\tilde{u}_2 = u_1$, $\tilde{u}_3 = u_3$ reduces this case to case (i). \square

Remark. Theorem 1 applies, for instance, to the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix},$$

which comes from a discretization in space of the wave equation (this matrix has been used to produce Fig. 1 with $\mu = -v = 4$).

3. A counterexample

In this section, we will give an answer to a problem posed in [3] showing that the result in Theorem 1 is not true for an arbitrary symmetric matrix A under the sole assumption (4).

Let us consider the symmetric matrix

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{5}$$

The eigenvalues of this matrix are $-\sqrt{2}$, 0 and $\sqrt{2}$. We assume

$$v < -\sqrt{2} < \sqrt{2} < \mu$$

and we impose moreover that $v = -\mu$. We are going to study the maximum number of solutions of Eq. (3) depending on the parameter μ .

Theorem 2. *Let $v = -\mu$. If $\sqrt{2} < \mu \leq 2$, Eq. (3) has at most six solutions. On the other hand, if $\mu > 2$, there is an open cone \mathcal{S} such that, if $v \in \mathcal{S}$, Eq. (3) has eight solutions.*

Proof. In this particular case, we have

$$\begin{aligned}
 y_{1+} &= (\mu, -1, 0), & y_{1-} &= (\mu, 1, 0), \\
 y_{2+} &= (-1, \mu, 1), & y_{2-} &= (1, \mu, -1), \\
 y_{3+} &= (0, 1, \mu), & y_{3-} &= (0, -1, \mu).
 \end{aligned}$$

The problem is to find the maximum number of cones $C(y_{1\pm}, y_{2\pm}, y_{3\pm})$ with nonempty intersection. Unfortunately, an analysis of determinants as in Theorem 1 is not sufficient. Our first step will be to change the problem into two dimensions, in the following way: we intersect the lines defined by the previous vectors with the plane $x + y + z = 1$, and the result is moved to the plane $z = 0$ by a translation and a rotation. After some computations on these linear transformations (which will be avoided here), we obtain the following points in \mathbb{R}^2 :

$$\begin{aligned}
 \hat{y}_{1+} &= \left(\frac{1}{\sqrt{6}}, \frac{1 + \mu}{\sqrt{2}(1 - \mu)} \right), & \hat{y}_{1-} &= \left(\frac{1}{\sqrt{6}}, \frac{1 - \mu}{\sqrt{2}(1 + \mu)} \right), \\
 \hat{y}_{2+} &= \left(\frac{\mu - 3}{\sqrt{6}\mu}, \frac{\mu + 1}{\sqrt{2}\mu} \right), & \hat{y}_{2-} &= \left(\frac{\mu + 3}{\sqrt{6}\mu}, \frac{\mu - 1}{\sqrt{2}\mu} \right), \\
 \hat{y}_{3+} &= \left(\frac{1 - 2\mu}{\sqrt{6}(1 + \mu)}, \frac{1}{\sqrt{2}(1 + \mu)} \right), & \hat{y}_{3-} &= \left(\frac{1 + 2\mu}{\sqrt{6}(1 - \mu)}, \frac{1}{\sqrt{2}(1 - \mu)} \right).
 \end{aligned}$$

Now, the problem is reduced to study the intersections of the triangles $T(\hat{y}_{1\pm}, \hat{y}_{2\pm}, \hat{y}_{3\pm})$. Our aim is to prove that if $\sqrt{2} < \mu \leq 2$ and $v = -\mu$ then

$$T(\hat{y}_{1+}, \hat{y}_{2+}, \hat{y}_{3+}) \cap T(\hat{y}_{1-}, \hat{y}_{2-}, \hat{y}_{3-}) \cap T(\hat{y}_{1-}, \hat{y}_{2-}, \hat{y}_{3+}) = \emptyset. \tag{6}$$

Let us write the equation of the line \mathcal{L}_1 through \hat{y}_{1+} and \hat{y}_{2+} , which is

$$y = -\frac{2\mu^2 + \mu - 1}{\sqrt{3}(\mu - 1)}x + \frac{\sqrt{2}}{3} \frac{\mu^2 - \mu - 2}{\mu - 1}. \tag{7}$$

On the other hand, the equation of the line \mathcal{L}_2 through \hat{y}_{2-} and \hat{y}_{3-} is

$$y = \frac{\mu^2 - \mu - 2}{\sqrt{3}(\mu^2 + \mu - 1)}x + \frac{\sqrt{2}}{3} \frac{\mu^2 - \mu - 2}{\mu^2 + \mu - 1} \tag{8}$$

and the intersection point is

$$P = \left(\frac{\mu^2 - \mu - 2}{\sqrt{6}(\mu^2 + 2\mu - 2)}, \frac{\mu^2 - \mu - 2}{\sqrt{2}(\mu^2 + 2\mu - 2)} \right).$$

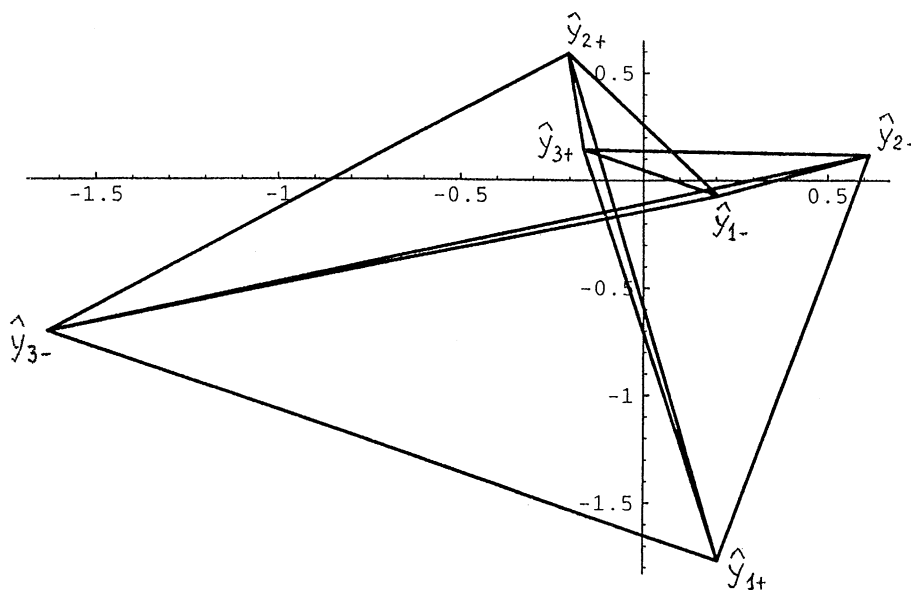


Fig. 2. Section of the cones for $\mu = 1.5$.

Being

$$\det(y_{1+}, y_{2+}, y_{3+}) = -\det(A - \mu I) > 0$$

and

$$\det(y_{1-}, y_{2-}, y_{3-}) = \det(A + \mu I) > 0,$$

the intersection of the triangles $T(\hat{y}_{1+}, \hat{y}_{2+}, \hat{y}_{3+})$ and $T(\hat{y}_{1-}, \hat{y}_{2-}, \hat{y}_{3-})$ lies entirely in the region

$$\{(x, y) \in \mathbb{R}^2: y \leq \min\{L_1(x), L_2(x)\}\},$$

where $L_1(x)$ and $L_2(x)$ are the right hand sides of (7) and (8), respectively.

On the other hand, the equation of the line \mathcal{L}_3 through \hat{y}_{1-} and \hat{y}_{3+} is

$$y = -\frac{1}{\sqrt{3}}x + \frac{\sqrt{2}}{3} \frac{2 - \mu}{1 + \mu},$$

and it is easy to verify that its slope is between the slope of \mathcal{L}_1 and \mathcal{L}_2 . As

$$\det(y_{1-}, y_{2-}, y_{3+}) = \mu^3 > 0,$$

the question depends on the situation of the point P with respect to the line \mathcal{L}_3 . Again, a drawing helps to have a better understanding of the situation. Figs. 2, 3 and 4 show different situations depending on the value of μ .

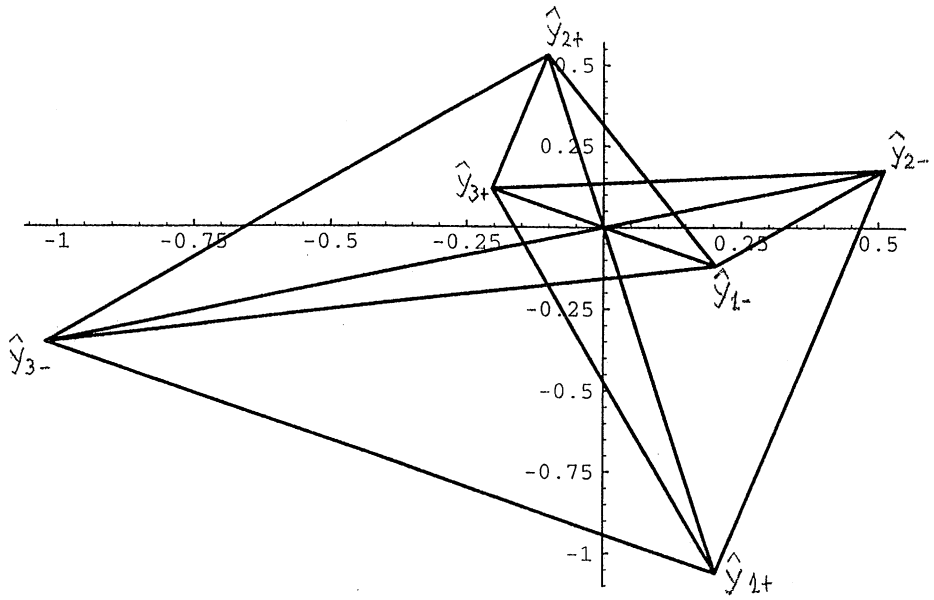


Fig. 3. Section of the cones for $\mu=2$.

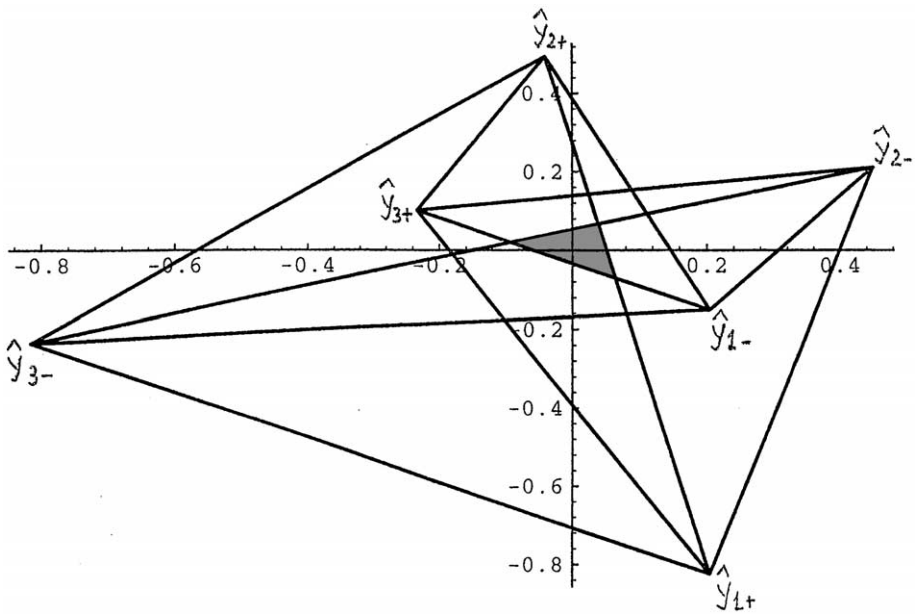


Fig. 4. Section of the cones for $\mu=2.5$.

In fact, (6) holds if and only if

$$P \in \left\{ (x, y) \in \mathbb{R}^2: y \leq -\frac{1}{\sqrt{3}}x + \frac{\sqrt{2}}{3} \frac{2 - \mu}{1 + \mu} \right\}$$

and this happens exactly when $\mu \leq 2$.

A symmetric argument with the triangle $T(\hat{y}_{1-}, \hat{y}_{2+}, \hat{y}_{3+})$ proves that, if $\mu \leq 2$, then

$$T(\hat{y}_{1+}, \hat{y}_{2+}, \hat{y}_{3+}) \cap T(\hat{y}_{1-}, \hat{y}_{2-}, \hat{y}_{3-}) \cap T(\hat{y}_{1-}, \hat{y}_{2+}, \hat{y}_{3+}) = \emptyset.$$

In conclusion, if $\mu \leq 2$, Eq. (3) has at most six solutions.

On the other hand, if $\mu > 2$, then all the triangles have nonempty intersection. In this case we have eight solutions if v belongs to the interior of

$$C(y_{1+}, y_{2+}, y_{3+}) \cap C(y_{1-}, y_{2-}, y_{3+}) \cap C(y_{1-}, y_{2-}, y_{3-}),$$

which is our open cone \mathcal{S} . \square

Remark. It would be interesting to know whether it is possible to give an example of a matrix A such that the maximum number of solutions of (3) is four. This could be related to a counterexample by Dancer [2] concerning a conjecture by Lazer and McKenna [4].

From this result it is easy to construct a counterexample for an arbitrary dimension N .

Corollary 1. *For any $N \geq 3$, there exists a symmetric $N \times N$ matrix A and positive numbers μ, v satisfying (2) such that the maximum number of solutions of Eq. (3) is less than 2^N .*

Proof. The matrix A is constructed, e.g., as being block diagonal:

$$A = \begin{pmatrix} A_3 & O \\ O & O \end{pmatrix} \tag{9}$$

where A_3 is defined in (5). The eigenvalues of this matrix are exactly the same of those of A_3 with a higher multiplicity for 0. Then, Eq. (3) is made of two separate systems. The first of these has at most six solutions when $\sqrt{2} < \mu \leq 2$ and $v = -\mu$. Besides, the zeroes of A suppose separate $(N - 3)$ scalar equations with at most two new solutions for each one. The combination of all these solutions leads to at most $6 \times 2^{N-3}$ solutions of the whole system. \square

On the other hand, we always have 2^N solutions if μ is large enough and $v = -\mu$, as shown in the following:

Theorem 3. *Given a symmetric $N \times N$ matrix A and a vector v with positive components, for any $k > 0$ and $C > 0$ there is a $M > 0$ such that, if $|k\mu + v| \leq C$ and $\mu \geq M$, then Eq. (3) has 2^N solutions.*

Proof. Eq. (3) can be written equivalently as

$$-\frac{2k}{k\mu - v}Au + \frac{k\mu + v}{k\mu - v}\hat{u} + \check{u} = \tilde{v} := \frac{2k}{k\mu - v}v, \tag{10}$$

where $\hat{u} = u^+ - ku^-$ and $\check{u} = u^+ + ku^-$. Since v has positive components, the equation $\check{u} = v$ has eight solutions, and the same is true for $\check{u} = \tilde{v}$, by the positive homogeneity. Hence, if $1/(k\mu - v)$ and $(k\mu + v)/(k\mu - v)$ are small enough, Eq. (10) also has eight solutions. \square

4. Periodic solutions

We begin with the following lemma, the proof of which can be found in [3, Theorems 4.1, 5.1, Step 1].

Lemma 1. *Let A be a symmetric $N \times N$ matrix and assume*

$$\max\{v, \mu\} < \lambda_1 + \left(\frac{2\pi}{T}\right)^2. \tag{11}$$

Then, there is a $M > 0$ such that, for every v with $\|v\| \geq M$, the number of periodic solutions of the differential Eq. (1) is the same as the number of its equilibria.

By the above lemma, the number of periodic solutions to (1), when v has a sufficiently large norm, is the same as the number of solutions to Eq. (3). As an illustration, we consider the situation of Corollary 1.

Let A be as in (9), where A_3 is defined in (5). Recall that the eigenvalues of this matrix are $-\sqrt{2}$, 0 and $\sqrt{2}$.

Corollary 2. *Assume $T < 2\pi/\sqrt[4]{8}$. If $v = -\mu$ and*

$$\sqrt{2} < \mu < \min \left\{ 2, \left(\frac{2\pi}{T}\right)^2 - \sqrt{2} \right\},$$

there is a $M > 0$ such that, for every v with $\|v\| \geq M$, Eq. (1) has less than 2^N periodic solutions.

Remark. In the line of [3, Theorem 4.1] we can as well prove the existence of at least 2^N periodic solutions in the situations of Theorems 1 or 3.

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